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SOME PROPERTIES OF FUNCTIONAL BANACH ALGEBRA

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Abstract. Suppose that A is a non-zero Banach space and $A^{(n)}$ is the n'th topological dual of A. In this paper we define two products on $A^{(n)}$ and prove that with each of them $A^{(n)}$ is a Banach algebra. Then we investigate some properties of these Banach algebras.

1. Introduction and preliminaries

Throughout this paper A is a non-empty Banach space.

We start our work with this question: is the dual of A; A^* a Banach algebra?

In the first step everyone considers the pointwise product on A^* to be an algebraic product on A^*, but this product is not well defined on A^*. So we must define another product on this space to make A^* into a Banach algebra.

First we recall briefly the concept of amenable and weakly amenable Banach algebras from [4] and n-weak amenability from [2].

Suppose that A is a Banach algebra and X is a Banach A-bimodule. A linear map $D : A \longrightarrow X$ is a derivation if for all $a, b \in A$,

$$D(ab) = D(a).b + a.D(b). \tag{1.1}$$

For $x \in X$, $D_x : A \to X$ defined by, $D_x(a) = a.x - x.a$ for each $a \in A$, is a derivation that called inner derivation.

If X is a Banach A-bimodule, X^* is also a Banach A-bimodule with the module actions,

$$< a.x^*, b >=< x^*, ba >, \quad < x^*.a, b >=< x^*, ab > \quad (a, b \in A, x^* \in X^*). \tag{1.2}$$

The Banach algebra A is amenable if every bounded derivation $D : A \longrightarrow X^*$ for every Banach A-bimodule X is inner and weakly amenable if every bounded derivation $D : A \longrightarrow A^*$ is inner. Also for each positive integers n, the Banach algebra A is n-weak amenable if every bounded derivation $D : A \longrightarrow A^{(n)}$ is inner.

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We recall the Arens regularity from [1]. Let A be a Banach algebra. For $a, b \in A$, $\lambda \in A^*$ and $\Phi \in A^{**}$, we define $a.\lambda, \lambda.a, \lambda.\Phi, \Phi.\lambda \in A^*$ by

$$< a.\lambda, b >=< \lambda, ba > \quad , \quad < \lambda.a, b >=< \lambda, ab >$$
$$< \lambda.\Phi, a >=< \Phi, a.\lambda > \quad , \quad < \Phi.\lambda, a >=< \Phi, \lambda.a > \cdot$$

Define two products $\Box$ and $\Diamond$ on A^{**} by

$$< \Phi\Box\Psi, \lambda >=< \Phi, \Psi.\lambda >, \quad < \Phi\Diamond\Psi, \lambda >=< \Psi, \lambda.\Phi >, \qquad (\Phi, \Psi \in A^{**}, \lambda \in A^*)\cdot$$

Then $(A^{**}, \Box)$ and $(A^{**}, \Diamond)$ are Banach algebras. We say that A is Arens regular if for all

$$\Phi, \Psi \in A^{**}, \ \ \Phi\Box\Psi = \Phi\Diamond\Psi.$$

This is equivalent to the following double limit criterion. For each $\Phi, \Psi \in A^{**}$ and nets $\{a_\alpha\}, \{b_\beta\}$ in A, where converges in w^*-topology to Φ and Ψ respectively

$$\lim_\alpha \lim_\beta < \lambda, a_\alpha b_\beta >= \lim_\beta \lim_\alpha < \lambda, a_\alpha b_\beta >, \qquad (\lambda \in A^*) \tag{1.3}$$

whenever both iterated limits exist.

This paper is organized as follows. In Section 2, for positive integer n we define two products on $B = A^{(n)}$ that each of them, make B into a Banach algebra. Then investigate some of the algebraic properties of these Banach algebras. In Section 3, we investigate the weak amenability, amenability and Arens regularity of these spaces as Banach algebras.

2. Banach algebras B^F and B_f

For each $n \in \mathbb{N}$, let $B = A^{(n)}$ be the n'th topological dual of A and $0 \neq f \in B_*$ the predual of B with $\|f\| \leq 1$ and $F \in B^*$ with $\|F\| \leq 1$.

We define two products on B and prove that with each of them B is a Banach algebra and call them the *functional Banach algebras*.

Let

$$\circ_f : B \times B \longrightarrow B$$

and

$$\circ^F : B \times B \longrightarrow B$$

be the mappings defined by

$$b_1 \circ_f b_2 = b_1(f)b_2, \quad b_1 \circ^F b_2 = F(b_1)b_2 \quad (b_1, b_2 \in B) \tag{2.1}$$

Proposition 2.1. *For each $f \in B_*, F \in B^*$ with property*

$$\|f\| \leq 1, \|F\| \leq 1, \ \ (B, \circ_f), \ \ (B, \circ^F)$$

are Banach algebras with operator norm.

Proof. The proof is easy and left as an exercise for the reader. □

From now on, we use these terminologies

$$(B, \circ_f) = B_f, \quad (B, \circ^F) = B^F.$$

If $F = \widehat{f}$ for some $f \in B_*$, then obviously $B^F = B_f$, So when $dimB_* < \infty$, $B^F = B_f$ for all $F \in U_{B^*}$ where $U_{B^*} = \{F \in B^* : \|F\| \leq 1\}$, since in this case B_* is reflexive. Note that $\widehat{f} : B \to \mathbb{C}$ is a linear functional defined by $\widehat{f}(b) = b(f)$ for each $b \in B$.

In general Goldstine theorem ([4], Theorem A.3.29) links this algebras as follow.

Let $F \in U_{B^*}$, then there exists a bounded net $(f_\alpha) \in B_*$, such that $\|f_\alpha\| \leq \|F\|$ and $F = w^* - \lim_\alpha \widehat{f_\alpha}$. So,

$$b_1 \circ^F b_2 = F(b_1)b_2 = \lim_\alpha \widehat{f_\alpha}(b_1)b_2 = \lim_\alpha (b_1 \circ_{f_\alpha} b_2) \quad (b_1, b_2 \in B). \tag{2.2}$$

Hence with a bad use of symbols we write, $B^F = \lim_\alpha B_{f_\alpha}$.
If for each α, B_{f_α} is commutative and unital, so is B^F.

Now we investigate when this Banach algebras are unital and when they are commutative.

Let $suppF$ be the support of the functional F and take

$$E = \{b/F(b) : b \in suppF\}.$$

Let $E_F = conv(E)$ be the convex hull of the set E.

Also let

$$E_f = \{b \in B : b(f) = 1\}.$$

Clearly E_f is a non-empty set, since if $b(f) = 0$ for all $b \in B$, then from the Hahn-Banach theorem we get, $f = 0$ and this is a contradiction. So there exists $b \in B$ and $0 \neq \alpha \in \mathbb{C}$ such that, $b(f) = \alpha$, hence $b/\alpha \in E_f$.

For each $e_F \in E_F$ and $e_f \in E_f$ we have,

$$e_F \circ^F b = b, \quad e_f \circ_f b = b \quad (b \in B)\cdot \tag{2.3}$$

So the Banach algebras, B_f and B^F have many left units.

Proposition 2.2. *The Banach algebras, B_f and B^F are commutative and unital if and only if B is a one dimensional vector space.*

Proof. Here we prove this statement just for B_f. The similar proof works for B^F.

Let $B =< b >$ such that $\|b\| = 1$, i.e., the Banach algebra generated by b, then from Hahn-Banach theorem $B_* =< f >$ such that $\|f\| = 1$ and $b(\alpha f) = \alpha$.

So b is a unit for $(B, \circ_f)$. On the other hand for all $b_1, b_2 \in B$, there exists $\alpha_{b_1}, \alpha_{b_2} \in \mathbb{C}$ such that, $b_1 = \alpha_{b_1} b, b_2 = \alpha_{b_2} b$. Hence,

$$b_1 \circ_f b_2 = b_2 \circ_f b_1 = \alpha_{b_1}\alpha_{b_2} b\cdot \tag{2.4}$$

So the Banach algebra B_f is unital and commutative.

Now let B_f be unital. So there exists $b \in B_f$ such that, $\|b\| = 1$ and

$$b^{'} \circ_f b = b \circ_f b^{'} = b^{'} \quad (b^{'} \in B)\cdot \tag{2.5}$$

If for one $0 \neq b^{'} \in B$, $b^{'}(f) = 0$, the above relations failed. So for all $0 \neq b^{'} \in B$, there exists $0 \neq \alpha_{b'} \in \mathbb{C}$ such that, $b^{'}(f) = \alpha_{b'}$.

Therefore, $b^{'} = \alpha_{b'} b$, for all $b^{'} \in B$. So B is a one dimensional vector space. □

3. Some properties of B^F and B_f

In this section we investigate some cohomological properties of these algebras, such as weak amenability and amenability, then turn our attention to Arens regularity of these algebras and some nice relation between them.

Recall that the character space of a Banach algebra A, denoted by $\Delta(A)$ and defined, the space of all non-zero homomorphism from A into $\mathbb{C}$.

Theorem 3.1. *The Banach algebras B_f and B^F for all $f \in B_*, F \in B^*$ such that $\|f\| = 1, \|F\| = 1$, are $(2m+1)$-weakly amenable for every $m \in \mathbb{Z}^+$.*

Proof. First we show that for all $b \in B$ and $b^{\{2m\}} \in B^{\{2m\}}$,

$$b.b^{\{2m\}} = b(f)b^{\{2m\}}. \tag{3.1}$$

For $m = 0$ the above relation holds, since the module action is the product of B_f. If $m = 1$, for $b^* \in B^*$, we have

$$b^*.b(c) = b^*(b \circ_f c) = b(f)b^*(c) \quad (c \in B). \tag{3.2}$$

So $b^*.b = b(f)b^*$. Therefore, for $b^{**} \in B^{**}$,

$$b.b^{**}(b^*) = b^{**}(b^*.b) = b^{**}(b(f)b^*) \quad (b^* \in B^*) \tag{3.3}$$

Hence, $b.b^{**} = b(f)b^{**}$. Now let for $m = k$ the relation (3.1) hold, we show that it also holds for $m = k+1$. We have,

$$b.b^{\{2k+2\}}(b^{\{2k+1\}}) = b^{\{2k+2\}}(b^{\{2k+1\}}.b) \quad (b^{\{2k+1\}} \in B^{\{2k+1\}}). \tag{3.4}$$

Similar to the relation (3.2) using the hypothesis, we have

$$b^{\{2k+1\}}.b = b(f)b^{\{2k+1\}}.$$

Hence, the relation (3.1) is proved.

Now let $D : B_f \longrightarrow B^{\{2m+1\}}$ be a continuous derivation. For the evaluation homomorphism, $\varepsilon_f : B_f \longrightarrow \mathbb{C}$, in view of (3.1) take $b^{\{2m+1\}} = (D')^*(\varepsilon_f)$, where

$$D' = D^*_{|B^{\{2m\}}} : B^{\{2m\}} \longrightarrow (B_f)^*.$$

Then similar to the proof of theorem (1.1) of [10], we have, $D = D_{b^{\{2m+1\}}}$, where $D_{b^{\{2m+1\}}}$ is an inner derivation on B_f that implemented by $b^{\{2m+1\}}$, since the Hahn-Banach theorem yields there exists a ε_f-mean on $(B_f)^*$.

So B_f is (2m+1)-weakly amenable.

For the Banach algebra B^F, take $\varphi \in \Delta(B^F)$ that defined by, $\varphi(b) = F(b)$. Similarly with the preceding case, the Hahn-Banach theorem yields that there exists a φ-mean on B^*. On the other hand easily verified that,

$$b.b^{\{2m\}} = F(b)b^{\{2m\}} \quad (b \in B^F). \tag{3.5}$$

So the Banach algebra B^F is also (2m+1)-weakly amenable. □

Theorem 3.2. *For any $f \in B_*$ with $\|f\| \leq 1$, the Banach algebra B_f is amenable if and only if $dimB_* = 1$.*

Proof. If $dimB_* = 1$, so $dim(B_*)^* = dim(B) = 1$ and so proposition (2.2) implies that B_f is unital, so it is amenable.

Let $dimB_* > 1$. We know that every amenable Banach algebra has a bounded approximate identity. Let (b_α) be a bounded approximate identity for B_f. So,

$$\|b_\alpha \circ_f b - b\|, \|b \circ_f b_\alpha - b\| \to 0 \quad (b \in B). \tag{3.6}$$

Let $M =< f, f' >$ be the subspace of B_* generated by f, f', then $b' : M \longrightarrow \mathbb{C}$ defined by,

$$b'(\gamma_1 f + \gamma_2 f') = \gamma_2 \|f'\| \quad (\gamma_1, \gamma_2 \in \mathbb{C}), \tag{3.7}$$

is a bounded linear functional on M, so the Hahn-Banach theorem implies that there exists a functional $b \in B$, such that, $b_{|M} = b'$ and $\|b\| = \|b'\|$. Hence $0 \neq b \in B$ and $b(f) = 0$. Therefore, the relation (3.6) does not hold for every $b \in B$.

So the Banach algebra B_f is not amenable. □

Corollary 3.1. *If $dimB_* > 1$, then for all $F \in \widehat{U_{B_*}}$, B^F is not amenable.*

If $F \in U_{B^*} \setminus \widehat{U_{B_*}}$ what we say about the amenability of B^F?

Hence we must discus on the case that B_* is an infinite dimensional vector space.

Theorem 3.3. *Let $dimB_* = \infty$, then for all $F \in U_{B^*}$, the Banach algebra B^F is not amenable.*

Proof. Let B^F is an amenable Banach algebra. So it has a bounded approximate identity (b_α). Hence,

$$\|b_\alpha \circ^F b - b\|, \|b \circ^F b_\alpha - b\| \to 0 \quad (b \in B). \tag{3.8}$$

If for one $0 \neq b \in B$, $F(b) = 0$, we reach to a contradiction and the proof is complete. Otherwise we show that the existence of this bounded approximate identity contradict the infinite dimensionality of B_*.

Let (c_γ) be a net in $C = \{c \in B : \|c\| = 1\}$. from the relation (3.8), we get, $c_\gamma = \lim_\alpha(F(c_\gamma)b_\alpha)$.

So $\lim_\alpha(b_\alpha)$ exists and it is equal to $c_\gamma/F(c_\gamma)$. Now since $(F(c_\gamma))$ is a bounded net in $\mathbb{C}$, the Bolzano-Weierstrass's theorem imply that it has a convergent subnet, $(F(c_{\gamma_w}))$.

Consider the subnet (c_{γ_w}) of (c_γ). This subnet convergent in C. Therefore, from the Heine-Borel's theorem we conclude that B is finite dimensional and so is B_*. □

Now we discuss the Arens regularity of these algebras.

Theorem 3.4. *The Banach algebras B^F and B_f are Arens regular.*

Proof. Let $b_1^{**}, b_2^{**} \in (B^F)^{**}$. So there exists nets $(b_\alpha^1), (b_\beta^2)$ in B^F such that, $b_1^{**} = w^* - \lim_\alpha \widehat{b_\alpha^1}$, and $b_2^{**} = w^* - \lim_\beta \widehat{b_\beta^2}$. Hence we have

$$\begin{aligned} b_1^{**} \square b_2^{**} &= w^* - \lim_\alpha w^* - \lim_\beta \widehat{b_\alpha^1 \circ^F b_\beta^2} \\ &= \lim_\alpha w^* - \lim_\beta F(b_\alpha^1)\widehat{b_\beta^2} \\ &= b_1^{**}(F)(w^* - \lim_\beta \widehat{b_\beta^2}) \\ &= b_1^{**}(F)b_2^{**}. \end{aligned}$$

On the other hand

$$\begin{aligned} b_1^{**} \diamond b_2^{**} &= w^* - \lim_\beta w^* - \lim_\alpha \widehat{b_\alpha^1 \circ^F b_\beta^2} \\ &= w^* - \lim_\beta \lim_\alpha F(b_\alpha^1)\widehat{b_\beta^2} \\ &= b_1^{**}(F)(w^* - \lim_\beta \widehat{b_\beta^2}) \\ &= b_1^{**}(F)b_2^{**}. \end{aligned}$$

So,

$$b_1^{**} \square b_2^{**} = b_1^{**} \diamond b_2^{**}.$$

Therefore, B^F is Arens regular.

For the Arens regularity of B_f, we show that for all $g \in (B_f)^*$ the mapping, $T_g : B_f \longrightarrow (B_f)^*$ that defined by, $T_g(b) = g\Box b$ is weakly compact.

For all $c \in B_f$ we have,

$$(g\Box b)(c) = g(b \circ_f c) = g(b(f)c) = b(f)g(c).$$

So $T_g(b) = b(f)g$. Let (b_α) be a bounded net in B_f. If we show that $(T_g(b_\alpha))$ has a weakly convergent subnet, the proof will be complete.
But the net $(b_\alpha(f))$ is bounded in $\mathbb{C}$, so by Bolzano-Weierstrass's theorem we get a convergent subnet of that like, $(b_{\alpha_w}(f))$.

Clearly the subnet $(T_g(b_{\alpha_w}))$ is weakly convergent in $(B_f)^*$. □

4. Conclusion

For every natural number n, the Banach algebras $B^F = (A^{(n)}, \circ^F)$ and $B_f = (A^{(n)}, \circ_f)$ are not amenable if $dim(A) > 1$, are $(2m + 1)$-weakly amenable for every positive integer m and both are Arens regular Banach algebras.

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