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ON A FIXED POINT THEOREM FOR A CYCLICAL KANNAN-TYPE MAPPING *

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Abstract. This paper deals with an extension of a recent result by the authors generalizing Kannan's fixed point theorem based on a theorem of Vittorino Pata. The generalization takes place via a cyclical condition.

1. Introduction

Somewhat in parallel with the renowned Banach contraction principle (see, for instance, [3]), Kannan's fixed point theorem has carved out a niche for itself in fixed point theory since its inception in 1969 [4]. Let (X, d) be a metric space. If we define $T\colon X \to X$ to be a *Kannan mapping* provided there exists some $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \frac{\lambda}{2}[d(x, Tx) + d(y, Ty)] \tag{1.1}$$

for each $x, y \in X$, then Kannan's theorem essentially states that:

Every Kannan mapping in a complete metric space has a unique fixed point.

To see that the two results are independent of each other, one can turn to [14], e.g., and Subrahmanyam has shown in [17] that Kannan's theorem characterizes metric completeness, i.e.: if every Kannan mapping on a metric has a fixed point, then that space must necessarily be complete.

Kirk et al. [6] introduced the so-called cyclical contractive conditions to generalize Banach's fixed point theorem and some other fundamental results in fixed point theory. Further works in this aspect, viz. the cyclic representation of a complete metric space with respect to a map, have been carried out in [5, 9, 16]. For the treatment of cyclic contractions yielding fixed points, see [7, 8].

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Pata in [10], however, extended Banach's result in a totally different direction and ended up proving that if (X, d) is a complete metric space and $T\colon X \to X$ a map such that there exist fixed constants $\Lambda \geq 0$, $\alpha \geqslant 1$, and $\beta \in [0, \alpha]$ with

$$d(Tx, Ty) \leq (1-\varepsilon)d(x, y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + d(x, x_0) + d(y, x_0)]^{\beta} \tag{1.2}$$

for every $\varepsilon \in [0, 1]$ and every $x, y \in X$ (where $\psi\colon [0, 1] \to [0, \infty)$ is an increasing function that vanishes with continuity at zero, and $x_0 \in X$ is an arbitrarily chosen point), then T has a unique fixed point in X. Combining Pata's theorem and the cyclical framework, Alghamdi et al. have next come up with a theorem of their own [1].

On the one hand, proofs of cyclic versions of Kannan's theorem (as well as that of various other important contractive conditions in metric fixed point theory, including some best proximity point results) were given in [11, 12, 13, 16]; the present authors, on the other hand, have already established an analogue of Pata's result that generalizes Kannan's theorem instead [2]. Letting everything else denote the same as in [10] except for fixing a slightly more general $\beta \geq 0$, we have actually shown the following:

Theorem 1.1. *[2] If the inequality*

(1.3)

$$\begin{aligned} d(Tx, Ty) \quad \leq \quad & \frac{1-\varepsilon}{2}[d(x, Tx) + d(y, Ty] \\ & + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + d(x, x_0) + d(Tx, x_0) + d(y, x_0) + d(Ty, x_0)]^{\beta} \end{aligned}$$

is satisfied $\forall \varepsilon \in [0, 1]$ *and* $\forall x, y \in X$, *then* T *possesses a unique fixed point*

$$x^* = Tx^* \, (x^* \in X).$$

In this article, we want to utilize Theorem 1.1 to bridge the gap by providing the only remaining missing link, viz. a fixed point theorem for cyclical contractive mappings in the sense of both Kannan and Pata.

2. The Main Result

Let us start by recalling a definition which has its roots in [6]; we shall make use of a succinct version of this as furnished in [5]:

Definition 2.1. [5] Let X be a non-empty set, $m \in \mathbb{N}$, and $T\colon X \to X$ a map. Then we say that $\bigcup_{i=1}^{m} A_i$ (where $\emptyset \neq A_i \subset X \, \forall i \in \{1, 2, \ldots, m\}$) is a *cyclic representation* of X with respect to T *iff* the following two conditions hold.

1. $X = \bigcup_{i=1}^{m} A_i$;

2. $T(A_i) \subset A_{i+1}$ for $1 \leq i \leq m-1$, and $T(A_m) \subset A_1$.

Now let (X, d) be a complete metric space. We have to first assign $\psi\colon [0, 1] \to [0, \infty)$ to be an increasing function that vanishes with continuity at zero. With this, we are ready to formulate our main result, viz.:

Theorem 2.1. *Let $\Lambda \geq 0$, $\alpha \geqslant 1$, and $\beta \geq 0$ be fixed constants. If $A_1, \ldots, A_m$ are non-empty closed subsets of X with $Y = \bigcup_{i=1}^{m} A_i$, and if $T\colon Y \to Y$ is such a map that $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, then, provided the inequality*

(2.1)

$$\begin{aligned} d(Tx, Ty) \quad &\leqslant \quad \frac{1-\varepsilon}{2}[d(x, Tx) + d(y, Ty)] \\ &\quad +\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + d(x, x_1) + d(Tx, x_1) + d(y, x_1) + d(Ty, x_1)]^{\beta} \end{aligned}$$

is satisfied $\forall \varepsilon \in [0, 1]$ and $\forall x \in A_i$, $y \in A_{i+1}$ (where $A_{m+1} = A_1$ and, as in [10], $x_1 \in Y$ is arbitrarily chosen — to serve as a sort of “zero” of the space Y), T has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$.

Remark 2.1. Since we can always redefine Λ to keep (2.1) valid no matter what initial $x_1 \in X$ we choose, we are in no way restricting ourselves by choosing that x_1 as our “zero” instead of a generic $x \in X$ [10].

Proof

For the sake of brevity and clarity both, we shall henceforth exploit the following notation when $j > m$:

$$A_j \colon = A_i,$$

where $i \equiv j \pmod{m}$ and $1 \leq i \leq m$.

Let’s begin by choosing our zero from A_1, i.e., we fix $x_1 \in A_1$. Starting from x_1, we then introduce the sequence of Picard iterates

$$x_n = Tx_{n-1} = T^{n-1}x_1 \quad (n \geqslant 2).$$

Also, let

$$c_n \colon = d(x_n, x_1) \quad (n \in \mathbb{N}).$$

With the assumption that $x_n \neq x_{n+1}, \forall n \in \mathbb{N}$, (2.1) gives us

$$\begin{aligned} d(x_{n+1}, x_n) \quad &= \quad d(Tx_n, Tx_{n-1}) \\ &\leqslant \quad \frac{1}{2}[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \end{aligned}$$

if we consider the case where $\varepsilon = 0$. But this means that

$$\begin{aligned} 0 &\leq d(x_{n+1}, x_n) \\ &\leqslant d(x_n, x_{n-1}) \\ &\leq \cdots \\ &\leqslant d(x_2, x_1) \\ &= c_2, \end{aligned} \tag{2.2}$$

whence the next result, i.e., our first lemma, is delivered:

Lemma 2.1. *$\{c_n\}$ is bounded.*

Proof. Let $n \in \mathbb{N}$. We assume that

$$n \equiv k \pmod m \, (1 \leq k \leq m).$$

Since $x_{k-1} \in A_{k-1}$ and $x_{k-2} \in A_{k-2}$, we have, using (2.1) with $\varepsilon = 0$,

$$\begin{aligned} c_n &= d(x_n, x_1) \\ &= [d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-2}, x_{k-1})] + d(x_{k-1}, x_n) \\ &\leq (k-2)c_2 + d(Tx_{k-2}, Tx_{k-1}) \\ &\leq (k-2)c_2 + \frac{1}{2}[d(x_{k-1}, x_{k-2}) + d(x_n, x_{n-1})] \\ &\leq (k-2)c_2 + \frac{1}{2}(c_2 + c_2) \\ &= (k-1)c_2. \end{aligned}$$

And hence we have our proof. □

Remark 2.2. One finds in [1] an attempt to prove the boundedness of an analogous sequence c_n (the notations in play there and in the present article are virtually the same) using the cyclic contractive condition from its main theorem (vide inequality (2.1) from Theorem 2.4 in [1]) on two points $x_1 \, (\in A_1)$ and $x_n \, (\in A_n)$. But this inequality as well as our own (2.1) can only be applied to points that are members of *consecutive* sets A_i and A_{i+1} for some $i \in \{1, \ldots, m\}$ according to their respective applicative restrictions, both of which stem from the very definition of cyclical conditions given in [6]. x_n being the general n-th term of the sequence $\{x_n\}$ is in a *general* set A_n, and, following the notational convention agreed upon in both [1] and this article, $A_n = A_l$, where $l \equiv n \pmod m$ and $1 \leq l \leq m$. Since the index l need not either be succeeding or be preceding the index 1 in general, $x_1 \, (\in A_1)$ and $x_n \, (\in A_l)$ need not necessarily be members of consecutive sets as well. Hence the justifiability of using the cyclic criterion on them is lost, and suitable adjustments have to be made in the structure of the proving argument. This is precisely what we have endeavoured to do in our proof above.

To return to our central domain of discourse, next we need another:

Lemma 2.2. $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$

Proof. (2.2) assures that we end up with a sequence, viz. $\{d(x_{n+1}, x_n)\}$, that is both monotonically decreasing and bounded below, and, therefore,

$$\begin{aligned}\lim_{n\to\infty} d(x_{n+1}, x_n) &= \inf_{n\in\mathbb{N}} d(x_{n+1}, x_n)\\ &= r\,(\text{say})\\ &\geq 0.\end{aligned}$$

But, a quick use of (2.1) shows that,

$$\begin{aligned}r &\leq d(x_{n+1}, x_n)\\ &= d(Tx_n, Tx_{n-1})\\ &\leq \frac{1-\varepsilon}{2}[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)(1 + c_{n+1} + 2c_n + c_{n-1})^{\beta}\\ &\leq \frac{1-\varepsilon}{2}[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] + K\varepsilon\psi(\varepsilon)\end{aligned}$$

for some $K \geq 0$. (By virtue of Lemma 2.1, it is ensured that K does not depend on n.) Letting $n \to \infty$,

$$\begin{aligned} r &\leq \frac{1-\varepsilon}{2}(r + r) + K\varepsilon\psi(\varepsilon)\\ \implies r &\leq K\psi(\varepsilon),\ \forall\varepsilon \in (0, 1]\\ \implies r &= 0.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \quad \square$$

With this, we are now in a position to derive:

Lemma 2.3. *$\{x_n\}$ is a Cauchy sequence.*

Proof. This proof has the same generic character as the one given in [6]. We suppose, first, that $\exists\rho > 0$ such that, given any $N \in \mathbb{N}$, $\exists n > p \geq N$ with $n - p \equiv 1 \pmod{m}$ and

$$d(x_n, x_p) \geq \rho > 0.$$

Clearly, x_{n-1} and x_{p-1} lie in different but consecutively labelled sets A_i and A_{i+1} for some $i \in \{1, \dots, m\}$. Then, from (2.1), $\forall\varepsilon \in [0, 1]$,

$$\begin{aligned}d(x_n, x_p) &\leq \frac{1-\varepsilon}{2}[d(x_n, x_{n-1}) + d(x_p, x_{p-1})]\\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)(1 + c_n + c_{n-1} + c_p + c_{p-1})^{\beta}\\ &\leq \frac{1-\varepsilon}{2}[d(x_n, x_{n-1}) + d(x_p, x_{p-1})] + C\varepsilon^{\alpha}\psi(\varepsilon),\end{aligned}$$

where, to be precise, $C = \sup_{j\in\mathbb{N}} \Lambda(1 + 4c_j)^{\beta} < \infty$ (on account of Lemma 2.1 again). If we let $n, p \to \infty$ with $n - p \equiv 1 \pmod{m}$, then Lemma 2.2 gives us that

$$0 < \rho \leq d(x_n, x_p) \to 0$$

as $\varepsilon \to 0+$, which is, clearly contrary to what we had supposed earlier.

Thus we can safely state that, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$d(x_n, x_p) \leq \frac{\varepsilon}{m} \tag{2.3}$$

whenever $n, p \geq N$ and $n - p \equiv 1 \pmod m$.

Again, by Lemma 2.2 it is possible to choose $M \in \mathbb{N}$ so that

$$d(x_{n+1}, x_n) \leq \frac{\varepsilon}{m}$$

if $n \geq M$. If we now let $n, p \geq \max\{N, M\}$ with $n > p$, then $\exists r \in \{1, 2, \ldots, m\}$ such that

$$n - p \equiv r \pmod m.$$

Thus

$$n - p + i \equiv 1 \pmod m,$$

where $i = m - r + 1$. And, bringing into play (2.3),

$$\begin{aligned} d(x_n, x_p) &\leq d(_p, x_{n+i}) + [d(x_{n+i}, x_{n+i-1}) + \cdots + d(x_{n+1}, x_n)] \\ &\leq \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is Cauchy. □

Now, looking at $Y = \bigcup_i A_i$, a complete metric space on its own, we can conclude straightaway that $\{x_n\}$, a Cauchy sequence in it, converges to a point $y \in Y$.

But $\{x_n\}$ has infinitely many terms in each A_i, $i \in \{1, \ldots, m\}$, and each A_i is a closed subset of Y. Therefore,

$$\begin{aligned} & y \in A_i \,\forall i \\ \Longrightarrow \quad & y \in \bigcap_{i=1}^{m} A_i \\ \Longrightarrow \quad & \bigcap_{i=1}^{m} A_i \neq \emptyset. \end{aligned}$$

Moreover, $\bigcap_{i=1}^{m} A_i$ is, just as well, a complete metric space *per se*. Thus, considering the restricted mapping

$$U: = T\restriction_{\bigcap A_i}: \bigcap A_i \to \bigcap A_i,$$

we notice that it satisfies the criterion to be a Kannan-type generalized map already proven by us to have a unique fixed point $x^* \in \bigcap A_i$ by virtue of Theorem 1.1. □

Remark 2.3. We have to minutely peruse a certain nuance here for rigour's sake: the moment we know that $\bigcap A_i \neq \emptyset$, we can choose an arbitrary $y_1 \in \bigcap A_i$ to serve as its zero, and the restriction of T to $\bigcap A_i$ can still be made to satisfy (a modified form of) (2.1) insofar as Λ can be appropriately revised as per remark Remark 2.1; this renders the employment of Theorem 1.1 in the above proof vindicated.

3. Some Conclusions

Having proved Theorem 2.1, we can now, following the terminology of [15], actually show something more, viz.:

Corollary 3.1. *T is a Picard operator, i.e., T has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$, and the sequence of Picard iterates $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^* irrespective of our initial choice of $x \in Y$.*

Proof. So far, we have already shown that for a fixed $x_1 \in A_1$ *one and only one* fixed point x^* of T exists. To complete the proof, let's first observe that the decision to let $x_1 \in A_1$ at the beginning of the main proof was based partly on mere convention and partly on an intention to develop our argument thenceforth more or less analogously to the proof given in [1]; if we would have chosen any generic $x \in Y$ instead, then, seeing as how $Y = \bigcup A_i$, that x would have belonged to A_j for some $j \in \{1, \dots, m\}$, and our discussion thereupon would have differed only in some labellings, *not* in its conclusion: i.e., we would have, eventually, inferred the existence of a unique fixed point of T in $\bigcap A_i$.

Next we want to demonstrate that the limit of the Picard iterates turns out to be a fixed point of T. To this end we recall that $\{T^n x_1\}_{n\in\mathbb{N}} = \{x_{n+1}\}_{n\in\mathbb{N}}$ converges to $y \in \bigcap A_i$. Our claim is that this y itself is a fixed point of T. This we can verify summarily:

As $x_n \in A_k$ for some $k \in \{1, \dots, m\}$ and as $y \in \bigcap_{i=1}^m A_i \subset A_{k+1}$,

$$\begin{aligned} d(y, Ty) &\leq d(y, x_{n+1}) + d(x_{n+1}, Ty) \\ &= d(y, x_{n+1}) + d(Tx_n, Ty) \\ &\leq d(y, x_{n+1}) + \frac{1}{2}[d(x_{n+1}, x_n) + d(y, Ty)] \end{aligned}$$

for every $n \in \mathbb{N}$ (using (2.1) with $\varepsilon = 0$ again), and, from that,

$$\frac{1}{2}d(y, Ty) \leq d(y, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_n)$$

for all n. Letting $n \to \infty$,

$$d(y, Ty) = 0$$

as $x_{n+1} \to y$ and $d(x_{n+1}, x_n) \to 0$. Therefore,

$$y = Ty.$$

As observed, the choice of the starting point x_1 is irrelevant, and we already know that $x^* \in \bigcap A_i$ is *the* unique fixed point of T. So obviously,

$$x^* = y,$$

i.e., $T^n x_1 \to x^*$ as $n \to \infty$.

This completes the proof. □

Remark 3.1. Trying to show their map f (corresponding to the T in the present article) is a Picard operator, the authors in [1] have set out to prove that $x_n \to x^*$ as $n \to \infty$ too. (Again, apart from denoting the operator differently, we haven't really changed much of their notation so as to make a comparison fairly self-explanatory.) The argument they have used in this follows from a technique given in [10], and they, too, have ended up stating, to quote a portion of the concerned reasoning in [1] verbatim,

$$d(x_n, x^*) = \lim_{p\to\infty} d(x_n, x_{n+p}).$$

This is where a problem arises.

In [10] the convergence of the Cauchy sequence x_n is established directly to its limit x_*, and, therefore, the utilization of an equality like

$$d(x_*, x_n) = \lim_{m\to\infty} d(x_{n+m}, x_n)$$

(quoted as it is from [10] this time) is perfectly justified. The cyclical setting in both this article and [1], however, only ensures initially that $x_n \to y$ for some $y \in \bigcap A_i$ as $n \to \infty$, and, as a consequence, guarantees next the existence of a unique fixed point x^* for the operator. The fact that this y turns out to be a fixed point for the operator as well (thereby rendering it equal to the unique x^*) is something that needs to be *actually proved* in a separate treatment, which we believe is exactly the task that we've accomplished here. [1], though, overlooks this distinction and assumes the very fact (viz. $x_n \to x^*$) it wants to prove in the proof itself, committing the fallacy of *petitio principii*.

As a final note, let us remind ourselves of the fact that (1.3) is weaker than (1.1) (see [2]), and, in light of this we also have, as another corollary to our Theorem 2.1 the following:

Corollary 3.2. *[16, 12] Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space X. Suppose that*

$$T\colon \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$$

is a cyclic map, i.e., it satisfies $T(A_i) \subset A_{i+1}$ for every $i \in \{1, \dots, p\}$ (with $A_{p+1} = A_1$), such that

$$d(Tx, Ty) \le \frac{\alpha}{2}[d(x, Tx) + d(y, Ty)]$$

for all $x \in A_i$, $y \in A_{i+1}$ $(1 \le i \le p)$, where $\alpha \in (0, 1)$ is a constant. Then T has a unique fixed point x^* in $\bigcap_{i=1}^p A_i$ and is a Picard operator.

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