FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. Vol. 28, No 2 (2013), 133–150

NEW RESULTS FOR A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we present new results for a coupled system of fractional differential equations. Applying Banach contraction principle and Schaefer fixed point theorem, new existence and uniqueness results are obtained. To illustrate our results, some examples are also presented.

Keywords: Caputo derivative, Banach contraction principle, coupled system, fixed point.

1. Introduction

Differential equations of arbitrary order have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as: electrochemistry, physics, chemistry, visco-elasticity, control, image and signal processing. For more details, we refer the reader to [3, 5, 7, 10, 11, 12, 15, 18]. There has been a significant progress in the investigation of these equations in recent years, see [6, 8, 13]. More recently, a basic theory for the initial boundary value problems of fractional differential equations has been discussed in [2, 13, 14, 15]. On the other hand, existence and uniqueness of solutions to boundary value problems for fractional differential equations has attracted the attention of many authors, see for example, [9, 15, 16, 17, 19] and the references therein. Moreover, the study of coupled systems of fractional order is also important in various problems of applied nature [4, 14, 22, 23]. Recently, many people have established the existence and uniqueness for solutions of some fractional systems, see [1, 20, 21, 23] and the reference therein.

This paper deals with the existence and uniqueness of solutions to the following coupled system of fractional differential equations:

Received April 25, 2013. Accepted June 18, 2013.
2010 *Mathematics Subject Classification.* 34A34, 34B10.

$$
\text{(1.1)} \qquad \begin{cases} D^{\alpha}x(t) + f_1\left(t, y(t), D^{\delta}y(t)\right) = 0, t \in J, \\ D^{\beta}y(t) + f_2(t, x(t), D^{\sigma}x(t)) = 0, t \in J, \\ x(0) = x_0^*, y(0) = y_0^*, \\ \left|x'(0)\right| + \left|x''(0)\right| + \left|y'(0)\right| + \left|y''(0)\right| = 0, \\ x'''(0) = J^p x(\eta), y'''(0) = J^q y(\xi), \end{cases}
$$

where $\alpha, \beta \in]3,4]$, $\delta \leq \alpha - 1$, $\sigma \leq \beta - 1$, $\xi, \eta \in]0,1[$, J^p, J^q are the Riemann-Liouville fractional integrals, $D^{\alpha}, D^{\beta}, D^{\delta}, D^{\sigma}$ are the Caputo fractional derivatives, $J = [0,1]$, x_0^*, y_0^* are real constants, and f_1 and f_2 are two functions which will be specified later.

The rest of this paper is organized as follows: In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of problem (1.1). In section 4, some examples are treated to illustrate our results.

2. Preliminaries

The following notations, definitions and preliminary facts will be used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as:

$$
\text{(2.1)} \qquad J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0,
$$

$$
\text{(2.2)} \qquad J^0 f(t) = f(t),
$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2. The fractional derivative of $f \in C^n([0, \infty[)$ in the Caputo's sense is defined as:

$$
\text{(2.3)} \qquad D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, n-1 < \alpha, n \in N^*.
$$

For more details about fractional calculus, we refer the reader to [15, 18].

The following lemmas give some properties of Riemann-Liouville fractional integral and Caputo fractional derivative [12, 13].

Lemma 2.1. *Let* $r, s > 0$, $f \in L^1([a,b])$. *Then* $J^r J^s f(t) = J^{r+s} f(t), D^s J^s f(t) = f(t), t \in [a,b]$.

Lemma 2.2. *Let* $s > r > 0$, $f \in L^1([a,b])$. *Then* $D^r J^s f(t) = J^{s-r} f(t), t \in [a,b]$.

Let us now introduce the spaces $X = \{x : x \in C([0,1]), D^\sigma x \in C([0,1])\}$ and $Y = \{y : y \in C([0,1]), D^\delta y \in C([0,1])\}$ endowed respectively with the norms

$$\| x \|_X = \| x \| + \| D^\sigma x \|; \| x \| = \sup_{t\in J} |x(t)|, \| D^\sigma x \| = \sup_{t\in J} |D^\sigma x(t)|$$

and

$$\| y \|_Y = \| y \| + \| D^\delta y \|; \| y \| = \sup_{t\in J} \left|y(t)\right|, \| D^\delta y \| = \sup_{t\in J} \left|D^\delta y(t)\right|.$$

Obviously, $(X, \| . \|_X)$ and $(Y, \| . \|_Y)$ are two Banach spaces. The product space $\left(X \times Y, \left\|(x,y)\right\|_{X\times Y}\right)$ is also a Banach space with norm $\left\|(x,y)\right\|_{X\times Y} = \|x\|_X + \left\|y\right\|_Y$.

We give the following lemmas [11]:

Lemma 2.3. *For* $\alpha > 0$, *the general solution of the fractional differential equation* $D^\alpha x(t) = 0$ *is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1$.

Lemma 2.4. *Let* $\alpha > 0$. *Then*

$$J^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \tag{2.5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1$.

We need also the following auxiliary result:

Lemma 2.5. *Let* $g \in C([0,1])$. *The solution of the equation*

$$D^\alpha x(t) + g(t) = 0, t \in J,\ 3 < \alpha \leq 4 \tag{2.6}$$

subject to the conditions

$$\begin{array}{ll} x(0) & = x_0^*, \\ \left|x'(0)\right| + \left|x''(0)\right| & = 0, \\ x'''(0) & = J^p x(\eta), \end{array} \tag{2.7}$$

is given by:

$$\begin{aligned} x(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds + x_0^* \\ & + \frac{\Gamma(p+4) t^3}{\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)} \left(\frac{1}{\Gamma(\alpha+p)} \int_0^\eta (\eta-s)^{\alpha+p-1} g(s)\, ds - x_0^* \frac{\eta^p}{\Gamma(p+1)} \right). \end{aligned} \tag{2.8}$$

Proof. For $c_i \in \mathbb{R}$, $i = 0, 1, 2, 3$, and by lemma 2.3 and Lemma 2.4, the general solution of (1.1) is given by

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3. \tag{2.9}$$

Thanks to Lemma 2.1, we get

$$J^p x(t) = \frac{1}{\Gamma(p+\alpha)} \int_0^t (t-s)^{p+\alpha-1} g(s)\, ds + \frac{x_0^* t^p}{\Gamma(p+1)} - \frac{\Gamma(4)\, t^{p+3}}{\Gamma(p+4)}. \tag{2.10}$$

Using the conditions (2.7), we get $c_1 = c_2 = 0$ and $c_0 = -x_0^*$.
For c_3, we have

$$c_3 = \frac{\Gamma(p+4)}{\Gamma(4)\,\eta^{p+3} - 6\Gamma(p+4)} \left(x_0^* \frac{\eta^p}{\Gamma(p+1)} - \frac{1}{\Gamma(p+\alpha)} \int_0^\eta (\eta - s)^{p+\alpha-1} g(s)\, ds \right). \tag{2.11}$$

Substituting the values of c_0 and c_3 in (2.9), we get (2.8). □

3. Main Results

Let us introduce the quantities:

$$\begin{aligned}
N_1 &= \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(p+4)\,\eta^{p+\alpha}}{\left|\Gamma(4)\,\eta^{p+3} - 6\Gamma(p+4)\right| \Gamma(p+\alpha+1)},\\
N_2 &= \frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+\alpha+1)\Gamma(4-\sigma)},\\
N_3 &= \frac{1}{\Gamma(\beta+1)} + \frac{\Gamma(q+4)\xi^{q+\beta}}{\left|\Gamma(4)\xi^{q+3} - 6\Gamma(q+4)\right|\Gamma(q+\beta+1)},\\
N_4 &= \frac{1}{\Gamma(\beta-\delta+1)} + \frac{\Gamma(q+4)\xi^{q+\beta}}{\left|\Gamma(4)\xi^{q+3} - 6\Gamma(q+4)\right|\Gamma(q+\beta+1)\Gamma(4-\delta)},\\
M_1 &= \left|x_0^*\right| + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+1)} + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)},\\
M_2 &= \left|y_0^*\right| + \frac{\left|y_0^*\right|\xi^q\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3} - 6\Gamma(q+4)\right|\Gamma(q+1)} + \frac{\left|y_0^*\right|\xi^q\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3} - 6\Gamma(q+4)\right|\Gamma(q+1)\Gamma(4-\delta)}.
\end{aligned} \tag{3.1}$$

We list also the following hypotheses:

$(H1)$: The functions $f_1, f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

$(H2)$: There exist non-negative continuous functions $a_i, b_i \in C([0,1])$, $i = 1, 2$, such that for all $t \in [0,1]$ and all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} \left|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)\right| &\leq a_1(t)|x_1 - x_2| + b_1(t)\left|y_1 - y_2\right|, \\ \left|f_2(t, x_1, y_1) - f_2(t, x_2, y_2)\right| &\leq a_2(t)|x_1 - x_2| + b_2(t)\left|y_1 - y_2\right|, \end{aligned} \tag{3.2}$$

with

$$\omega_1 = \sup_{t\in J} a_1(t), \omega_2 = \sup_{t\in J} b_1(t), \varpi_1 = \sup_{t\in J} a_2(t), \varpi_2 = \sup_{t\in J} b_2(t).$$

$(H3)$: There exist non-negative continuous functions l_1 and l_2, such that

$$\left|f_1(t, x, y)\right| \leq l_1(t), \left|f_2(t, x, y)\right| \leq l_2(t) \text{ for each } t \in J \text{ and all } x, y \in \mathbb{R},$$

with

$$\theta_1 = \sup_{t\in J} l_1(t), \theta_2 = \sup_{t\in J} l_2(t).$$

Our first result is based on Banach contraction principle. We have:

Theorem 3.1. *Suppose that* $\eta^{p+3} \neq \frac{6\Gamma(p+4)}{\Gamma(4)}$, $\xi^{q+3} \neq \frac{6\Gamma(q+4)}{\Gamma(4)}$ *and assume that the hypothesis (H2) holds.*
If

$$(N_1 + N_2)(\omega_1 + \omega_2) + (N_3 + N_4)(\varpi_1 + \varpi_2) < 1, \tag{3.3}$$

then the boundary value problem (1.1) has a unique solution on J.

Proof. Consider the operator $\phi : X \times Y \to X \times Y$ defined by:

$$\phi(x, y)(t) := \left(\phi_1 y(t), \phi_2 x(t)\right), \tag{3.4}$$

where

$$\phi_1 y(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1\left(s, y(s), D^\delta y(s)\right) ds + x_0^*$$
$$\frac{\Gamma(p+4)t^3}{\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)} \left(\frac{1}{\Gamma(p+\alpha)} \int_0^\eta (\eta-s)^{p+\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds - x_0^* \frac{\eta^p}{\Gamma(p+1)}\right),$$

and

$$\phi_2 x(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), D^\sigma x(s)) ds + y_0^*$$
$$+ \frac{\Gamma(q+4)t^3}{\Gamma(4)\xi^{q+3} - 6\Gamma(q+4)} \left(\frac{1}{\Gamma(q+\beta)} \int_0^\eta (\eta-s)^{q+\beta-1} f\left(s, y(s), D^\delta y(s)\right) ds - y_0^* \frac{\xi^q}{\Gamma(q+1)}\right).$$

We shall prove that ϕ is contraction mapping :

Let $(x,y),(x_1,y_1)\in X\times Y$. Then, for each $t\in J$, we have:

$$\begin{aligned}&\left|\phi_1 y(t)-\phi_1 y_1(t)\right|\\ &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds\\ &+\tfrac{\Gamma(p+4)t^3}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha)}\\ &\times\int_0^{\eta}(\eta-s)^{p+\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds.\end{aligned}$$

Thanks to $(H2)$, we obtain

$$\begin{aligned}\left|\phi_1 y(t)-\phi_1 y_1(t)\right| &\leq \frac{\omega_1\left\|y-y_1\right\|+\omega_2\left\|D^{\delta}y-D^{\delta}y_1\right\|}{\Gamma(\alpha+1)}\\ &+\frac{\Gamma(p+4)\eta^{p+\alpha}\left(\omega_1\left\|y-y_1\right\|+\omega_2\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)}.\end{aligned}\tag{3.5}$$

Consequently,

$$\begin{aligned}&\left|\phi_1 y(t)-\phi_1 y_1(t)\right|\\ &\leq\left[\tfrac{(\omega_1+\omega_2)}{\Gamma(\alpha+1)}+\tfrac{\Gamma(p+4)\eta^{p+\alpha}(\omega_1+\omega_2)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)}\right]\left(\left\|y-y_1\right\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right),\end{aligned}\tag{3.6}$$

which implies that

$$\left\|\phi_1(y)-\phi_1(y_1)\right\|\leq N_1(\omega_1+\omega_2)\left(\left\|y-y_1\right\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right),\tag{3.7}$$

and

$$\begin{aligned}&\left|D^{\sigma}\phi_1 y(t)-D^{\sigma}\phi_1 y_1(t)\right|\\ &\leq\tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^t(t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds\\ &+\tfrac{\Gamma(p+4)t^{3-\sigma}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha)\Gamma(4-\sigma)}\\ &\times\int_0^{\eta}(\eta-s)^{p+\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds.\end{aligned}$$

By $(H2)$, we have

$$\left|D^{\sigma}\phi_1 y(t) - D^{\sigma}\phi_1 y_1(t)\right| \leq \frac{(\omega_1+\omega_2)\left(\left\|y-y_1\right\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right)}{\Gamma(\alpha-\sigma+1)}$$
$$+\frac{\Gamma(p+4)\eta^{p+\alpha}(\omega_1+\omega_2)\left(\left\|y-y_1\right\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)\Gamma(4-\sigma)}.$$

Hence,

$$\left|D^{\sigma}\phi_1 y(t) - D^{\sigma}\phi_1 y_1(t)\right|$$
$$\leq \left[\frac{(\omega_1+\omega_2)}{\Gamma(\alpha-\sigma+1)} + \frac{\Gamma(p+4)\eta^{p+\alpha}(\omega_1+\omega_2)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)\Gamma(4-\sigma)}\right] \tag{3.8}$$
$$\times\left(\left\|y-y_1\right\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right).$$

Therefore,

$$\left|D^{\sigma}\phi_1 y(t) - D^{\sigma}\phi_1 y_1(t)\right| \leq N_2(\omega_1+\omega_2)\left(\left\|y-y_1\right\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.9}$$

And consequently,

$$\left\|D^{\sigma}\phi_1(y) - D^{\sigma}\phi_1(y_1)\right\| \leq N_2(\omega_1+\omega_2)\left(\left\|y-y_1\right\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.10}$$

By (3.7) and (3.10), we can write

$$\left\|\phi_1(y) - \phi_1(y_1)\right\|_X$$
$$\leq (N_1+N_2)(\omega_1+\omega_2)\left(\left\|y-y_1\right\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.11}$$

With the same arguments as before, we have

$$\left\|\phi_2(x) - \phi_2(x_1)\right\|_Y$$
$$\leq (N_3+N_4)(\varpi_1+\varpi_2)(\|x-x_1\| + \|D^{\sigma}x - D^{\sigma}x_1\|). \tag{3.12}$$

And by (3.11) and (3.12), we obtain

$$\left\|\phi(x,y) - \phi(x_1,y_1)\right\|_{X\times Y} \leq \left[\begin{array}{c}(N_1+N_2)(\omega_1+\omega_2)\\ +(N_3+N_4)(\varpi_1+\varpi_2)\end{array}\right]\left\|(x-x_1, y-y_1)\right\|_{X\times Y}. \tag{3.13}$$

Tanks to (3.3), we conclude that ϕ is contraction. As a consequence of Banach fixed point theorem, we deduce that ϕ has a fixed point which is a solution of the problem (1.1). □

The second main result is the following theorem:

Theorem 3.2. *Suppose that* $\eta^{p+3} \neq \frac{6\Gamma(p+4)}{\Gamma(4)}$, $\xi^{q+3} \neq \frac{6\Gamma(q+4)}{\Gamma(4)}$ *and assume that the hypotheses (H1)and (H3) are satisfied. Then, the coupled system* (1.1) *has at least a solution on J.*

Proof. We shall use Scheafer's fixed point theorem to prove that ϕ has at least a fixed point on $X \times Y$. It is to note that ϕ is continuous on $X \times Y$ in view of the continuity of f_1 and f_2 (hypothesis $(H1)$).

(1∗ :) : We shall prove that ϕ maps bounded sets into bounded sets in $X \times Y$: Taking $\rho > 0$, and $(x, y) \in B_\rho := \{(x, y) \in X \times Y; \left\|(x, y)\right\|_{X\times Y} \leq \rho\}$, then for each $t \in J$, we have:

$$\begin{aligned}
&\left|\phi_1 y(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds + \left|x_0^*\right| \\
&+ \frac{\Gamma(p+4)t^3}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+\alpha)} \int_0^\eta (\eta-s)^{p+\alpha-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
&+ \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)t^3}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+1)}.
\end{aligned}$$

Tanks to $(H3)$, we can write

$$\begin{aligned}
\left|\phi_1 y(t)\right| &\leq \frac{\sup_{t\in J} l_1(t)}{\Gamma(\alpha+1)} + \frac{\sup_{t\in J} l_1(t)\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+\alpha+1)} \\
&+ \left|x_0^*\right| + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+1)} \\
&\leq \sup_{t\in J} l_1(t)\left[\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+\alpha+1)}\right] \\
&+ \left|x_0^*\right| + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+1)}.
\end{aligned}$$

Therefore,

$$\left|\phi_1 y(t)\right| \leq \theta_1 N_1 + \left|x_0^*\right| + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+1)}. \tag{3.14}$$

Hence, we have

$$\left\|\phi_1(y)\right\| \leq \theta_1 N_1 + \left|x_0^*\right| + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3} - 6\Gamma(p+4)\right|\Gamma(p+1)}. \tag{3.15}$$

On the other hand,

$$\left|D^{\sigma}\phi_1 y(t)\right| \le \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds + \left|x_0^*\right|$$
$$+\frac{\Gamma(p+4)t^{3-\sigma}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha)\Gamma(4-\sigma)}\int_0^{\eta}(\eta-s)^{p+\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds$$
$$+\frac{\left|x_0^*\right|\eta^p\Gamma(p+4)t^{3-\sigma}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}.$$

By $(H3)$, we have,

$$\left|D^{\sigma}\phi_1 y(t)\right| \le \sup_{t\in J} l_1(t)\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha)\Gamma(4-\sigma)}\right]$$
$$+\frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}.$$

Consequently we obtain,

$$\left|D^{\sigma}\phi_1 y(t)\right| \le \theta_1 N_2 + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}. \tag{3.16}$$

Therefore,

$$\left\|D^{\sigma}\phi_1(y)\right\| \le \theta_1 N_2 + \frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}. \tag{3.17}$$

Combining (3.15) and (3.17), yields

$$\begin{aligned}&\left\|\phi_1(y)\right\|_X\\ &\le \theta_1(N_1+N_2)+\left|x_0^*\right|+\frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)}+\frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}.\end{aligned} \tag{3.18}$$

Similarly, it can be shown that,

$$\begin{aligned}&\left\|\phi_2(x)\right\|_Y\\ &\le \theta_2(N_3+N_4)+\left|y_0^*\right|+\frac{\left|y_0^*\right|\xi^q\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+1)}+\frac{\left|y_0^*\right|\xi^q\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+1)\Gamma(4-\delta)}.\end{aligned} \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\left\|\phi(x,y)\right\|_{X\times Y} \le \theta_1(N_1+N_2)+\theta_2(N_3+N_4)+M_1+M_2. \tag{3.20}$$

Consequently,

$$\left\|\phi\left(x,y\right)\right\|_{X\times Y}<\infty. \tag{3.21}$$

(2∗ :) : Now, we will prove that ϕ is equicontinuous on J : For $(x,y)\in B_{\rho}$, and $t_1,t_2\in J$, such that $t_1<t_2$. We have:

$$\begin{aligned}\left|\phi_1 y\left(t_2\right)-\phi_1 y\left(t_1\right)\right| \leq\ & \frac{1}{\Gamma\left(\alpha\right)}\int_0^{t_1}\left(\left(t_1-s\right)^{\alpha-1}-\left(t_2-s\right)^{\alpha-1}\right)\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\ &+\frac{1}{\Gamma\left(\alpha\right)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\ &\frac{\Gamma\left(p+4\right)\left(t_2^3-t_1^3\right)}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+\alpha\right)}\int_0^{\eta}\left(\eta-s\right)^{p+\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\ &+\frac{\left|x_0^*\right|\eta^{p}\Gamma\left(p+4\right)\left(t_1^3-t_2^3\right)}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+1\right)}.\end{aligned} \tag{3.22}$$

Thus,

$$\begin{aligned}\left|\phi_1 y\left(t_2\right)-\phi_1 y\left(t_1\right)\right| \leq\ & \frac{\theta_1}{\Gamma\left(\alpha+1\right)}\left(t_1^{\alpha}-t_2^{\alpha}\right)+\frac{2\theta_1}{\Gamma\left(\alpha+1\right)}\left(t_2-t_1\right)^{\alpha}\\ &+\frac{L_1\Gamma\left(p+4\right)\eta^{p+\alpha}}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+\alpha+1\right)}\left(t_2^3-t_1^3\right)\\ &+\frac{\left|x_0^*\right|\eta^{p}\Gamma\left(p+4\right)}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+1\right)}\left(t_1^3-t_2^3\right),\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&\left|D^{\sigma}\phi_1 y\left(t_2\right)-D^{\sigma}\phi_1 y\left(t_1\right)\right|\\ &\leq \tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^{t_1}\left(\left(t_1-s\right)^{\alpha-\sigma-1}-\left(t_2-s\right)^{\alpha-\sigma-1}\right)\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\ &+\tfrac{1}{\Gamma(\alpha-\sigma)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\alpha-\sigma-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\ &+\tfrac{\Gamma(p+4)\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha)\Gamma(4-\sigma)}\int_0^{\eta}\left(\eta-s\right)^{p+\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds\\ &\tfrac{\left|x_0^*\right|\eta^{p}\Gamma(p+4)\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)},\end{aligned}$$

Using $(H3)$, we obtain:

$$
\begin{aligned}
\left|D^{\sigma}\phi_1 y(t_2)-D^{\sigma}\phi_1 y(t_1)\right| \leq\ & \frac{\theta_1}{\Gamma(\alpha-\sigma+1)}\left(t_1^{\alpha-\sigma}-t_2^{\alpha-\sigma}\right)+\frac{2\theta_1}{\Gamma(\alpha-\sigma+1)}(t_2-t_1)^{\alpha-\sigma} \\
&+\frac{\theta_1\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right) \\
&+\frac{\left|x_0^*\right|\eta^{p}\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right). \qquad (3.24)
\end{aligned}
$$

Hence, by (3.23) and (3.24), we can write

$$
\begin{aligned}
\left\|\phi_1 y(t_2)-\phi_1 y(t_1)\right\|_X \leq\ & \frac{\theta_1}{\Gamma(\alpha+1)}\left(t_1^{\alpha}-t_2^{\alpha}\right)+\frac{2\theta_1}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \\
&+\frac{\theta_1\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)}\left(t_2^{3}-t_1^{3}\right) \qquad (3.25) \\
&+\frac{\left|x_0^*\right|\eta^{p}\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)}\left(t_1^{3}-t_2^{3}\right) \\
&+\frac{\theta_1}{\Gamma(\alpha-\sigma+1)}\left(t_1^{\alpha-\sigma}-t_2^{\alpha-\sigma}\right)+\frac{2\theta_1}{\Gamma(\alpha-\sigma+1)}(t_2-t_1)^{\alpha-\sigma} \\
&+\frac{\theta_1\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right) \\
&+\frac{\left|x_0^*\right|\eta^{p}\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).
\end{aligned}
$$

With the same arguments as before, we get

$$
\begin{aligned}
\left\|\phi_1 x(t_2)-\phi_1 x(t_1)\right\|_Y \leq\ & \frac{\theta_2}{\Gamma(\beta+1)}\left(t_1^{\beta}-t_2^{\beta}\right)+\frac{2\theta_2}{\Gamma(\beta+1)}(t_2-t_1)^{\beta} \\
&+\frac{\theta_2\Gamma(q+4)\eta^{q+\beta}}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+\beta+1)}\left(t_2^{3}-t_1^{3}\right) \qquad (3.26) \\
&+\frac{\left|y_0^*\right|\xi^{q}\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+1)}\left(t_1^{3}-t_2^{3}\right) \\
&+\frac{\theta_2}{\Gamma(\beta-\delta+1)}\left(t_1^{\beta-\delta}-t_2^{\beta-\delta}\right)+\frac{2\theta_2}{\Gamma(\beta-\delta+1)}(t_2-t_1)^{\beta-\delta} \\
&+\frac{\theta_2\Gamma(q+4)\xi^{q+\beta}}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+\beta+1)\Gamma(4-\delta)}\left(t_2^{3-\delta}-t_1^{3-\delta}\right) \\
&+\frac{\left|y_0^*\right|\xi^{q}\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+1)\Gamma(4-\delta)}\left(t_1^{3-\delta}-t_2^{3-\delta}\right).
\end{aligned}
$$

Thanks to (3.25) and (3.26), we can state that $\left\|\phi\left(x,y\right)\left(t_2\right)-\phi\left(x,y\right)\left(t_1\right)\right\|_{X\times Y}\to 0$ as $t_2\to t_1$. Combining (1∗) and (2∗) and using Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

(3∗ :) : Finally, we shall show that the set Ω defined by

$$\Omega=\left\{(x,y)\in X\times Y,(x,y)=\mu\phi\left(x,y\right),0<\mu<1\right\}, \tag{3.27}$$

is bounded:

Let $(x,y)\in\Omega$, then $(x,y)=\mu\phi\left(x,y\right)$, for some $0<\mu<1$. Thus, for each $t\in J$, we have:

$$y\left(t\right)=\mu\phi_1 y\left(t\right),x\left(t\right)=\mu\phi_2 x\left(t\right). \tag{3.28}$$

Then

$$\begin{aligned}
&\frac{1}{\mu}\left|y\left(t\right)\right| \\
&\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}\left(s\right)\right)\right|ds+\left|x_0^*\right| \\
&+\tfrac{\Gamma(p+4)t^3}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha)}\int_0^{\eta}\left(\eta-s\right)^{p+\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds \\
&+\tfrac{\left|x_0^*\right|\eta^p\Gamma(p+4)t^3}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)}.
\end{aligned}$$

Thanks to $(H3)$, we can write

$$\begin{aligned}
\frac{1}{\mu}\left|y\left(t\right)\right| &\leq \frac{\sup_{t\in J} l_1\left(t\right)}{\Gamma\left(\alpha+1\right)}+\frac{\sup_{t\in J} l_1\left(t\right)\Gamma\left(p+4\right)\eta^{p+\alpha}}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+\alpha+1\right)} \\
&+\left|x_0^*\right|+\frac{\left|x_0^*\right|\eta^p\Gamma\left(p+4\right)}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+1\right)}.
\end{aligned} \tag{3.29}$$

Therefore,

$$\begin{aligned}
\left|y\left(t\right)\right| &\leq \mu\theta_1\left[\frac{1}{\Gamma\left(\alpha+1\right)}+\frac{\Gamma\left(p+4\right)\eta^{p+\alpha}}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+\alpha+1\right)}\right] \\
&+\mu\left(\left|x_0^*\right|+\frac{\left|x_0^*\right|\eta^p\Gamma\left(p+4\right)}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+1\right)}\right).
\end{aligned} \tag{3.30}$$

Hence,

$$\left|y\left(t\right)\right|\leq\mu\left(\theta_1 N_1+\left|x_0^*\right|+\frac{\left|x_0^*\right|\eta^p\Gamma\left(p+4\right)}{\left|\Gamma\left(4\right)\eta^{p+3}-6\Gamma\left(p+4\right)\right|\Gamma\left(p+1\right)}\right), \tag{3.31}$$

On the other hand,

$$\frac{1}{\mu}\left|D^{\sigma}y(t)\right|$$

$$\leq \tfrac{1}{\Gamma(\alpha-\delta)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds$$

$$+\tfrac{\Gamma(p+4)t^{3-\sigma}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha)\Gamma(4-\sigma)}\int_0^{\eta}(\eta-s)^{p+\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds$$

$$+\tfrac{\left|x_0^*\right|\eta^p\Gamma(p+4)t^{3-\sigma}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}.$$

By ($H3$), we have

$$\frac{1}{\mu}\left|D^{\sigma}y(t)\right|$$

$$\leq \sup_{t\in J} l_1(t)\left[\tfrac{1}{\Gamma(\alpha-\sigma+1)}+\tfrac{\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)\Gamma(4-\sigma)}\right]$$

$$+\tfrac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}.$$

Therefore,

$$\left|D^{\sigma}y(t)\right|$$

$$\leq \mu\theta_1\left[\tfrac{1}{\Gamma(\alpha-\sigma+1)}+\tfrac{\Gamma(p+4)\eta^{p+\alpha}}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+\alpha+1)\Gamma(4-\sigma)}\right]$$

$$+\mu\tfrac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}.$$

Thus,

$$\left|D^{\beta}y(t)\right|\leq\mu\left(\theta_1N_2+\frac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}\right). \tag{3.32}$$

From (3.31) and (3.32), we get

$$\left\|y\right\|_X\leq\mu\left[\begin{array}{c}\theta_1(N_1+N_2)+\left|x_0^*\right|+\tfrac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)}\\ +\tfrac{\left|x_0^*\right|\eta^p\Gamma(p+4)}{\left|\Gamma(4)\eta^{p+3}-6\Gamma(p+4)\right|\Gamma(p+1)\Gamma(4-\sigma)}\end{array}\right]. \tag{3.33}$$

Analogously, we can obtain

$$\|x\|_Y\leq\mu\left[\begin{array}{c}\theta_2(N_3+N_4)+\left|y_0^*\right|+\tfrac{\left|y_0^*\right|\eta^q\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+1)}\\ +\tfrac{\left|y_0^*\right|\eta^q\Gamma(q+4)}{\left|\Gamma(4)\xi^{q+3}-6\Gamma(q+4)\right|\Gamma(q+1)\Gamma(4-\delta)}\end{array}\right]. \tag{3.34}$$

It follows from (3.33) and (3.34) that

$$\left\|(x,y)\right\|_{X\times Y} \leq \mu\left[\theta_1\left(N_1+N_2\right)+\theta_2\left(N_3+N_4\right)+M_1+M_2\right]. \tag{3.35}$$

Hence,

$$\left\|\phi\left(x,y\right)\right\|_{X\times Y} < \infty. \tag{3.36}$$

This shows that the set Ω is bounded.

Thanks to $(1*)$, $(2*)$ and $(3*)$, we deduce that ϕ has at least one fixed point, which is a solution of the problem (1.1). □

Corollary 3.1. *Assume that* $(H1)$ *holds and* $\eta^{p+3} \neq \frac{6\Gamma(p+4)}{\Gamma(4)}$, $\xi^{q+3} \neq \frac{6\Gamma(q+4)}{\Gamma(4)}$. *If there exist* $L_1>0, L_2>0$, *such that* $f_1 \leq L_1$, $f_2 \leq L_2$ *on* $J\times\mathbb{R}^2$, *then the coupled system* (1.1) *has at least a solution on* J.

4. Examples

To illustrate our results, we will present three examples:

Example 4.1. Let us consider the following coupled system:

$$\begin{cases} D^{\frac{7}{2}}x(t)+\frac{\left|y(t)\right|+\left|D^{\frac{1}{2}}y(t)\right|}{(32+t^2)\left(e^{-t}+\left|y(t)\right|+\left|D^{\frac{1}{2}}y(t)\right|\right)}+\arctan(1+t)=0, t\in[0,1], \\ D^{\frac{7}{2}}y(t)+\frac{1}{20\pi+t^2}\left(\sin\left|x(t)\right|+\sin\left|D^{\frac{1}{3}}x(t)\right|\right)+\ln\left(2+t^2\right)=0, t\in[0,1], \\ x(0)=\sqrt{2}, y(0)=\sqrt{3}, \\ \left|x'(0)\right|+\left|x''(0)\right|+\left|y'(0)\right|+\left|y''(0)\right|=0, \\ x'''(0)=J^2x\left(\frac{3}{4}\right), y'''(0)=J^{\frac{3}{2}}y\left(\frac{3}{5}\right). \end{cases} \tag{4.1}$$

For this example, we have

$$\begin{aligned} f_1(t,x,y) &= \frac{|x|+|y|}{(32+t^2)\left(e^{-t}+|x|+|y|\right)}+\arctan(1+t), t\in[0,1], x,y\in\mathbb{R}, \\ f_2(t,x,y) &= \frac{1}{20\pi+t^2}\left(\sin|x|+\sin|y|\right)+\ln\left(2+t^2\right),, t\in[0,1], x,y\in\mathbb{R}. \end{aligned}$$

Taking $x, y, x_1, y_1 \in \mathbb{R}, t\in[0,1]$, then:

$$\begin{aligned} \left|f_1(t,x,y)-f_1(t,x_1,y_1)\right| &\leq \frac{1}{32+t^2}|x-x_1|+\frac{1}{32+t^2}|y-y_1|, \\ \left|f_2(t,x,y)-f_2(t,x_1,y_1)\right| &\leq \frac{1}{20\pi+t^2}|x-x_1|+\frac{1}{20\pi+t^2}|y-y_1|. \end{aligned}$$

So, we can take

$$a_1(t)=b_1(t)=\frac{1}{32+t^2}, a_2(t)=b_2(t)=\frac{1}{20\pi+t^2}. \tag{4.2}$$

It follows then that

$$\begin{aligned}
\omega_1 &= \frac{1}{32} = \omega_2, \\
\varpi_1 &= \frac{1}{20\pi} = \varpi_2, \\
N_1 &= 0,0860791,\, N_2 = 0,4264507,\, N_3 = 0,0860755,\, N_4 = 0,1666720.
\end{aligned}$$

And then,

$$(N_1 + N_2)(\omega_1 + \omega_2) + (N_3 + N_4)(\varpi_1 + \varpi_2) = 0,0360577 < 1. \tag{4.3}$$

Hence by Theorem 3.1, the system (4.1) has a unique solution on $[0, 1]$.

Example 4.2. The second example is the following:

$$\left\{\begin{array}{c}
D^{\frac{7}{2}} x(t) + \frac{1}{8(t+2)^2}\left(\frac{|y(t)|}{1+|y(t)|} + \frac{t\left|D^{\frac{1}{5}} y(t)\right|}{2\pi\left(1+\left|D^{\frac{1}{5}} y(t)\right|\right)}\right) + \cos\left(1 + t + t^2\right) = 0,\, t \in [0,1], \\
D^{\frac{7}{2}} y(t) + \frac{1}{32\left(t^2+1\right)}\left(\sin x(t) + \frac{t+t^2}{8\pi}\sin\left(2\pi D^{\frac{1}{4}} x(t)\right)\right) + \cosh\left(2 + t^2\right) = 0,\, t \in [0,1], \\
x(0) = 5,\, y(0) = \sqrt{10}, \\
|x'(0)| + |x''(0)| + |y'(0)| + |y''(0)| = 0, \\
x'''(0) = J^{\frac{2}{5}} x\left(\frac{2}{13}\right),\, y'''(0) = J^{\frac{3}{5}} y\left(\frac{3}{11}\right),
\end{array}\right. \tag{4.4}$$

where

$$\begin{aligned}
f_1(t, x, y) &= \frac{1}{8(t+2)^2}\left(\frac{|x|}{1+|x|} + \frac{t|y|}{2\pi\left(1+|y|\right)}\right) + \cos\left(1 + t + t^2\right),\, t \in [0,1],\, x, y \in \mathbb{R}, \\
f_2(t, x, y) &= \frac{1}{32\left(t^2+1\right)}\left(\sin x + \frac{t+t^2}{8\pi}\sin(2\pi y)\right) + \cosh\left(2 + t^2\right),\, t \in [0,1],\, x, y \in \mathbb{R}.
\end{aligned}$$

For $t \in [0, 1]$ and $x, y, x_1, y_1 \in \mathbb{R}$, we have:

$$\begin{aligned}
\left|f_1(t, x, y) - f_1(t, x_1, y_1)\right| &\le \frac{1}{8(t+2)^2}|x - x_1| + \frac{t}{16\pi(t+2)^2}\left|y - y_1\right|, \\
\left|f_2(t, x, y) - f_2(t, x_1, y_1)\right| &\le \frac{1}{32\left(t^2+1\right)}|x - x_1| + \frac{1}{256\pi\left(t^2+1\right)}\left|y - y_1\right|,
\end{aligned}$$

with

$$a_1(t) = \frac{1}{8(t+2)^2},\, b_1(t) = \frac{t}{16\pi(t+2)^2}, \tag{4.5}$$

and

$$a_2(t) = \frac{1}{32\left(t^2+1\right)},\, b_2(t) = \frac{1}{256\pi\left(t^2+1\right)}. \tag{4.6}$$

It follows then that

$$\begin{aligned}\omega_1 &= \frac{1}{32}, \omega_2 = \frac{1}{400\pi},\\ \varpi_1 &= \frac{1}{32}, \varpi_2 = \frac{1}{256},\\ N_1 &= 0,0859751, N_2 = 0,2413984, N_3 = 0,0860209, N_4 = 0,1129376,\end{aligned}$$

and

$$(N_1 + N_2)(\omega_1 + \omega_2) + (N_3 + N_4)(\varpi_1 + \varpi_2) = 0,0176382 < 1. \tag{4.7}$$

Hence by Theorem 3.1, we conclude that the system (4.4) has a unique solution on $[0,1]$.

Example 4.3. Taking

$$\begin{cases} D^{\frac{11}{3}} x(t) + \frac{\cos\left(y(t)+D^{\frac{1}{2}} y\right)}{16+t+t^2} = 0, t \in [0,1],\\ D^{\frac{10}{3}} y(t) + \frac{\left(\sin x(t)+D^{\frac{1}{4}} x(t)\right)}{18+t+t^2} = 0, t \in [0,1],\\ x(0) = \sqrt{2}, y(0) = \sqrt{3},\\ |x'(0)| + |x''(0)| + \left|y'(0)\right| + \left|y''(0)\right| = 0,\\ x'''(0) = J^{\frac{2}{5}} x\left(\frac{3}{7}\right), y'''(0) = J^{\frac{3}{4}} y\left(\frac{1}{3}\right). \end{cases} \tag{4.8}$$

Then, we have

$$\begin{aligned} f_1(t,x,y) &= \frac{\cos(x+y)}{16+t+t^2}, t \in [0,1], x, y \in \mathbb{R},\\ f_2(t,x,y) &= \frac{\sin(x+y)}{18+t+t^2}, t \in [0,1], x, y \in \mathbb{R}. \end{aligned}$$

Let $x, y \in \mathbb{R}$ and $t \in [0,1]$. Then

$$\left|f_1(t,x,y)\right| \leq \frac{1}{16+t+t^2}, \left|f_2(t,x,y)\right| \leq \frac{1}{18+t+t^2}. \tag{4.9}$$

So we take

$$l_1(t) = \frac{1}{16+t+t^2}, l_2(t) = \frac{1}{18+t+t^2}. \tag{4.10}$$

Then,

$$\theta_1 = \frac{1}{16}, \theta_2 = \frac{1}{18}. \tag{4.11}$$

Thanks to Theorem 3.2, the system (4.8) has at least one solution on $[0,1]$.

5. Conclusion

We have presented some existence and uniqueness results for a nonlinear coupled system of fractional differential equations involving Caputo derivative. The proof of the existence results is based on Schaefer fixed point theorem, and the uniqueness result is proved by applying Banach contraction principle. This work can be extended to coupled systems of nonlinear fractional differential equations involving Riemann-Liouville fractional derivative.

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