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CHARACTERIZATIONS OF A RIEMANNIAN MANIFOLD ADMITTING RICCI SOLITONS

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Abstract. The object of the present paper is to characterize a Riemannian manifold admitting Ricci solitons (g, ξ, λ).

1. Introduction

In 1982 Hamilton [8] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton [9] is a generalization of the Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\pounds_\xi g_{ij} + 2R_{ij} + 2\lambda g_{ij} = 0 \tag{1.1}$$

for some constant λ, a vector field ξ on M where R_{ij} is the Ricci tensor. Clearly, a Ricci soliton with ξ zero or a Killing vector field [18] reduces to an Einstein manifold. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Ricci solitons in contact metric manifolds were studied in ([2], [5], [11], [14]) and many others. Recently, Bejan and Crasmareanu [1] studied Ricci solitons in a manifold of quasi-constant curvature. There are two aspects to study Ricci solitons in a Riemannian manifold (M, g).

(i) Given ξ, to find the nature of M.

(ii) Given the properties of R_{ij}, to find the nature of ξ.

The object of the present paper is to characterize a Riemannian manifold in respect of R_{ij} when the nature of the vector field ξ is given.

On the other hand, a Riemannian manifold is said to be a quasi-Einstein manifold [3] if its Ricci tensor is non-zero and satisfies

$$R_{ij} = a g_{ij} + b A_i A_j,$$

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where a and b are scalars and A_i is a unit covariant vector. Such a manifold is denoted by $(QE)_n$. In 2008, De and Gazi [4] introduced the notion of nearly quasi-Einstein manifolds.

A Riemannian manifold is said to be a nearly quasi-Einstein manifold [4] if its Ricci tensor is non-zero and satisfies

$$R_{ij} = ag_{ij} + bE_{ij},$$

where E_{ij} is a symmetric (0,2) tensor. From the definition it follows that every quasi-Einstein manifold is a nearly quasi-Einstein manifold, but the converse is not necessarily true.

N. S. Sinyukov [13] and E. N. Sinyukova [12] investigated manifolds whose Ricci tensor satisfies

$$\nabla_i R_{jk} = \sigma_i g_{jk} + \nu_j g_{ik} + \nu_k g_{ij}, \tag{1.2}$$

where σ_i and ν_i are some covariant vectors. Such manifolds are known under different names (see [7], [12], [13]). In what follows a Riemannian (or pseudo-Riemannian) manifold satisfying (1.2) with non-constant scalar curvature will be called a Sinyukov manifold. Such manifolds always admit non-trivial geodesic mappings and every Sinyukov manifold is nearly conformally symmetric.

The paper is organized in the following way.

In Section 2, we prove that if the vector field ξ is a torseforming vector field, then the manifold reduces to a nearly quasi-Einstein manifold and the Ricci soliton is steady under certain condition. As a particular case of a torseforming vector field we find the nature of the Riemannian manifold and the Ricci solitons. Finally, we obtain the nature of the Riemannian manifold when ξ induces infinitesimal transformations.

2. ξ as a torseforming vector field

In this section we determine the nature of the Riemannian manifold if the vector field ξ is torseforming.

The vector field ξ is called a torseforming vector field [17] if

$$\nabla_j \xi_i = \rho g_{ij} + \xi_i \omega_j, \tag{2.1}$$

where ∇ denotes covariant differentiation, ρ is a scalar and ω_j is a covariant vector. Using (2.1) we get

$$\begin{aligned} \pounds_\xi g_{ij} &= \nabla_j \xi_i + \nabla_i \xi_j \\ &= 2\rho g_{ij} + 2\mu_{ij}, \end{aligned} \tag{2.2}$$

where

$$\mu_{ij} = \frac{1}{2}(\xi_i \omega_j + \xi_j \omega_i).$$

Therefore

$$\pounds_\xi g_{ij} + 2R_{ij} + 2\lambda g_{ij} = 0$$

implies

$$R_{ij} = a g_{ij} + b\mu_{ij}, \tag{2.3}$$

where $a = -(\rho + \lambda)$, $b = -1 \neq 0$, which implies that the manifold is a nearly quasi-Einstein manifold [4]. From (2.3) it follows that

$$r = -(\rho + \lambda)n - \mu = -(\rho n + \lambda n + \mu), \tag{2.4}$$

where r is the scalar curvature and $\mu = \mu_{ij} g^{ij}$. From (2.4) we obtain

$$\lambda = -\frac{r}{n} - \rho - \frac{\mu}{n}.$$

Hence we have the following theorem:

Theorem 2.1. *If in a Riemannian manifold admitting Ricci soliton (g, ξ, λ) the vector field ξ is torseforming, then the manifold is a nearly quasi-Einstein manifold and the Ricci soliton is steady provided the value of the scalar curvature is $-(\rho n + \mu)$.*

Now we consider the following cases:

Case i) Suppose ξ is a unit torseforming vector field.

Then (2.1) can be written as

$$\nabla_j \xi_i = \rho(g_{ij} - \xi_i \xi_j). \tag{2.5}$$

Then

$$\begin{aligned} \pounds_\xi g_{ij} &= \nabla_j \xi_i + \nabla_i \xi_j \\ &= 2\rho(g_{ij} - \xi_i \xi_j). \end{aligned}$$

Thus $\pounds_\xi g_{ij} + 2R_{ij} + 2\lambda g_{ij} = 0$ implies

$$R_{ij} = -(\rho + \lambda) g_{ij} + \rho \xi_i \xi_j. \tag{2.6}$$

Therefore the manifold becomes a quasi-Einstein manifold.

From (2.6) we get

$$-R_{ij}\xi^i \xi^j = \lambda,$$

since ξ_i is a unit vector. Thus we can state the following:

Theorem 2.2. *If ξ is a unit torseforming vector field in a Riemannian manifold admitting Ricci soliton (g, ξ, λ) then the manifold becomes a quasi-Einstein manifold and the Ricci soliton $(g, \xi, -R_{ij}\xi^i\xi^j)$ is shrinking if $R_{ij}\xi^i\xi^j$ is positive or expanding if $R_{ij}\xi^i\xi^j < 0$.*

Case ii) If $\omega_i = 0$ in (2.1), that is, if ξ_i is a concircular vector field [18], then (2.3) reduces to

$$R_{ij} = ag_{ij},$$

where $a = -(\lambda + \rho)$, which implies that the manifold is an Einstein manifold. Hence we obtain the following:

Theorem 2.3. *If ξ is a concircular vector field in a Riemannian manifold admitting Ricci soliton, then the manifold is an Einstein manifold and the Ricci soliton $(g, \xi, -(\frac{r}{n} + \rho))$ is steady provided $r = -\rho n$.*

Case iii) If ξ is a parallel vector field, then it can be easily seen that the manifold becomes an Einstein manifold.

3. Infinitesimal transformation induced by ξ

In this section we determine the nature of the Riemannian manifold M if the vector field ξ induces various types of infinitesimal transformations. Recently, Velimirovic et al. [15] studied infinitesimal rigidity and flexibility of a non-symmetric affine connection space.

I. First suppose that ξ is a Killing vector [18]. That is, $£_\xi g_{ij} = 0$. Then from (1.1) it can be easily seen that

$$R_{ij} = -\lambda g_{ij},$$

which implies that M is an Einstein manifold and $\lambda = -\frac{r}{n}$. Thus we obtain the following:

Theorem 3.1. *If ξ is a Killing vector field, then the manifold is an Einstein manifold and the Ricci soliton $(g, \xi, -\frac{r}{n})$ is expanding or shrinking according as $r < 0$ or $r > 0$.*

II. Suppose ξ induces an affine connection. Then $\nabla_k £_\xi g_{ij} = 0$. [10] which implies that M is Ricci parallel.

In 1923, Eisenhart [6] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible.

Hence we can state the following:

Theorem 3.2. *If ξ induces an affine connection in (M, g) admitting a Ricci soliton (g, ξ, λ), then the manifold (M, g) is Ricci parallel and reducible.*

III. If ξ is a conformal motion, then we have

$$£_\xi g_{ij} = 2\sigma g_{ij}, \tag{3.1}$$

where σ is a scalar. If $\sigma = constant$, then the conformal motion is called homothetic.

Suppose ξ is a homothetic motion. Then from (3.1) it follows that $\nabla_k £_\xi g_{ij} = 0$.

Hence from (1.1) we get

$$\nabla_k R_{ij} = 0.$$

Thus we get the same conclusion as in II if ξ induces a homothetic motion.

IV. Suppose ξ is projective collineation. In this case we have [10]

$$£_\xi \left\{ {h \atop ij} \right\} = \delta_i^h \phi_j + \delta_j^h \phi_i$$

where $\phi = \phi(x)$ is a scalar function and $\phi_j = \nabla_j \phi$. The commutation formula gives

$$\begin{aligned} \nabla_k £_\xi g_{ij} - £_\xi \nabla_k g_{ij} &= g_{hj} £_\xi \left\{ {h \atop ik} \right\} + g_{ih} £_\xi \left\{ {h \atop jk} \right\} \\ &= 2 g_{ij}\phi_k + g_{kj}\phi_i + g_{ik}\phi_j. \end{aligned}$$

Hence

$$\nabla_k £_\xi g_{ij} = 2 g_{ij}\phi_k + g_{kj}\phi_i + g_{ik}\phi_j. \tag{3.2}$$

Now from (1.1) we get

$$\nabla_k £_\xi g_{ij} + 2\nabla_k R_{ij} = 0$$

Using (3.2) in the above equation we get

$$\nabla_k R_{ij} = -(\phi_k g_{ij} + \frac{1}{2}\phi_i g_{kj} + \frac{1}{2}\phi_j g_{ik}), \tag{3.3}$$

which can be written as $\nabla_k R_{ij} = \sigma_k g_{ij} + \nu_i g_{kj} + \nu_j g_{ik}$,

where $\sigma_k = -\phi_k$, $\nu_i = -\frac{1}{2}\phi_i$.

Equation (3.3) implies that the manifold is a Sinyukov manifold.

Thus we can state the following:

Theorem 3.3. *If ξ induces a projective collineation in (M, g) admitting a Ricci soliton then the manifold is a Sinyukov manifold.*

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