CHARACTERIZATIONS OF A RIEMANNIAN MANIFOLD ADMITTING RICCI SOLITIONS

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Abstract. The object of the present paper is to characterize a Riemannian manifold admitting Ricci solitons $(g, \xi, \lambda)$.

1. Introduction

In 1982 Hamilton [8] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$  

A Ricci soliton [9] is a generalization of the Einstein metric and is defined on a Riemannian manifold $(M, g)$ by

$$\mathcal{L}_{\xi} g_{ij} + 2R_{ij} + 2\lambda g_{ij} = 0$$

for some constant $\lambda$, a vector field $\xi$ on $M$ where $R_{ij}$ is the Ricci tensor. Clearly, a Ricci soliton with $\xi$ zero or a Killing vector field [18] reduces to an Einstein manifold. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively. Ricci solitons in contact metric manifolds were studied in ([2], [5], [11], [14]) and many others. Recently, Bejan and Crasmareanu [1] studied Ricci solitons in a manifold of quasi-constant curvature. There are two aspects to study Ricci solitons in a Riemannian manifold $(M, g)$.

(i) Given $\xi$, to find the nature of $M$.

(ii) Given the properties of $R_{ij}$, to find the nature of $\xi$.

The object of the present paper is to characterize a Riemannian manifold in respect of $R_{ij}$ when the nature of the vector field $\xi$ is given.

On the other hand, a Riemannian manifold is said to be a quasi-Einstein manifold [3] if its Ricci tensor is non-zero and satisfies

$$R_{ij} = a g_{ij} + b A_i A_j,$$
where $a$ and $b$ are scalars and $A_i$ is a unit covariant vector. Such a manifold is denoted by $(QE)_n$. In 2008, De and Gazi [4] introduced the notion of nearly quasi-Einstein manifolds.

A Riemannian manifold is said to be a nearly quasi-Einstein manifold [4] if its Ricci tensor is non-zero and satisfies

$$R_{ij} = ag_{ij} + bE_{ij},$$

where $E_{ij}$ is a symmetric $(0,2)$ tensor. From the definition it follows that every quasi-Einstein manifold is a nearly quasi-Einstein manifold, but the converse is not necessarily true.

N. S. Sinyukov [13] and E. N. Sinyukova [12] investigated manifolds whose Ricci tensor satisfies

$$\nabla_i R_{jk} = \sigma_i g_{jk} + \nu_j g_{ik} + \nu_k g_{ij}, \quad (1.2)$$

where $\sigma_i$ and $\nu_j$ are some covariant vectors. Such manifolds are known under different names (see [7], [12], [13]). In what follows a Riemannian (or pseudo-Riemannian) manifold satisfying (1.2) with non-constant scalar curvature will be called a Sinyukov manifold. Such manifolds always admit non-trivial geodesic mappings and every Sinyukov manifold is nearly conformally symmetric.

The paper is organized in the following way.

In Section 2, we prove that if the vector field $\xi$ is a torseforming vector field, then the manifold reduces to a nearly quasi-Einstein manifold and the Ricci soliton is steady under certain condition. As a particular case of a torseforming vector field we find the nature of the Riemannian manifold and the Ricci solitons. Finally, we obtain the nature of the Riemannian manifold when $\xi$ induces infinitesimal transformations.

### 2. $\xi$ as a torseforming vector field

In this section we determine the nature of the Riemannian manifold if the vector field $\xi$ is torseforming.

The vector field $\xi$ is called a torseforming vector field [17] if

$$\nabla_j \xi_i = \rho g_{ij} + \xi_i \omega_j, \quad (2.1)$$

where $V$ denotes covariant differentiation, $\rho$ is a scalar and $\omega_j$ is a covariant vector. Using (2.1) we get

$$\mathcal{L}_\xi g_{ij} = \nabla_j \xi_i + \nabla_i \xi_j$$

$$= 2\rho g_{ij} + 2\mu_{ij}, \quad (2.2)$$

where

$$\mu_{ij} = \frac{1}{2}(\xi_i \omega_j + \xi_j \omega_i).$$
Therefore
\[ \mathcal{L}_\xi g_{ij} + 2R_{ij} + 2\lambda g_{ij} = 0 \]
implies
\begin{equation}
R_{ij} = ag_{ij} + b\mu_{ij}, \tag{2.3}
\end{equation}
where \( a = -(\rho + \lambda) \), \( b = -1 \neq 0 \), which implies that the manifold is a nearly quasi-Einstein manifold \([4]\). From (2.3) it follows that
\begin{equation}
r = -(\rho + \lambda)n - \mu = -(\rho n + \lambda n + \mu), \tag{2.4}
\end{equation}
where \( r \) is the scalar curvature and \( \mu = \mu_{ij}g^{ij} \). From (2.4) we obtain
\[ \lambda = -\frac{r}{n} - \rho - \frac{\mu}{n}. \]

Hence we have the following theorem:

**Theorem 2.1.** If in a Riemannian manifold admitting Ricci soliton \( (g, \xi, \lambda) \) the vector field \( \xi \) is torseforming, then the manifold is a nearly quasi-Einstein manifold and the Ricci soliton is steady provided the value of the scalar curvature is \( -(\rho n + \mu) \).

Now we consider the following cases:

Case i) Suppose \( \xi \) is a unit torseforming vector field.

Then (2.1) can be written as
\begin{equation}
\nabla_j \xi_i = \rho (g_{ij} - \xi_i \xi_j). \tag{2.5}
\end{equation}
Then
\[ \mathcal{L}_\xi g_{ij} = \nabla_j \xi_i + \nabla_i \xi_j = 2\rho (g_{ij} - \xi_i \xi_j). \]
Thus \( \mathcal{L}_\xi g_{ij} + 2R_{ij} + 2\lambda g_{ij} = 0 \) implies
\begin{equation}
R_{ij} = -(\rho + \lambda)g_{ij} + \rho \xi_i \xi_j. \tag{2.6}
\end{equation}
Therefore the manifold becomes a quasi-Einstein manifold.

From (2.6) we get
\[ -R_{ij} \xi^i \xi^j = \lambda, \]
since \( \xi_i \) is a unit vector. Thus we can state the following:

**Theorem 2.2.** If \( \xi \) is a unit torseforming vector field in a Riemannian manifold admitting Ricci soliton \( (g, \xi, \lambda) \) then the manifold becomes a quasi-Einstein manifold and the Ricci soliton \( (g, \xi, -R_{ij} \xi^i \xi^j) \) is shrinking if \( R_{ij} \xi^i \xi^j \) is positive or expanding if \( R_{ij} \xi^i \xi^j < 0 \).
Case ii) If $\omega_i = 0$ in (2.1), that is, if $\xi$ is a concircular vector field [18], then (2.3) reduces to

$$R_{ij} = a g_{ij},$$

where $a = - (\lambda + \rho)$, which implies that the manifold is an Einstein manifold. Hence we obtain the following:

**Theorem 2.3.** If $\xi$ is a concircular vector field in a Riemannian manifold admitting Ricci soliton, then the manifold is an Einstein manifold and the Ricci soliton $(g, \xi, - (\xi_n + \rho))$ is steady provided $r = - \rho n$.

Case iii) If $\xi$ is a parallel vector field, then it can be easily seen that the manifold becomes an Einstein manifold.

3. **Infinitesimal transformation induced by $\xi$**

In this section we determine the nature of the Riemannian manifold $M$ if the vector field $\xi$ induces various types of infinitesimal transformations. Recently, Velimirovic et al. [15] studied infinitesimal rigidity and flexibility of a non-symmetric affine connection space.

I. First suppose that $\xi$ is a Killing vector [18]. That is, $\mathcal{L}_\xi g_{ij} = 0$. Then from (1.1) it can be easily seen that

$$R_{ij} = - \lambda g_{ij},$$

which implies that $M$ is an Einstein manifold and $\lambda = - \frac{\rho}{n}$. Thus we obtain the following:

**Theorem 3.1.** If $\xi$ is a Killing vector field, then the manifold is an Einstein manifold and the Ricci soliton $(g, \xi, - \frac{\rho}{n})$ is expanding or shrinking according as $r < 0$ or $r > 0$.

II. Suppose $\xi$ induces an affine connection. Then $\nabla_k \mathcal{L}_\xi g_{ij} = 0$. [10] which implies that $M$ is Ricci parallel.

In 1923, Eisenhart [6] proved that if a positive definite Riemannian manifold $(M, g)$ admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible.

Hence we can state the following:

**Theorem 3.2.** If $\xi$ induces an affine connection in $(M, g)$ admitting a Ricci soliton $(g, \xi, \lambda)$, then the manifold $(M, g)$ is Ricci parallel and reducible.

III. If $\xi$ is a conformal motion, then we have

$$(3.1) \quad \mathcal{L}_\xi g_{ij} = 2 \sigma g_{ij},$$

where $\sigma$ is a scalar. If $\sigma = \text{constant}$, then the conformal motion is called homothetic.

Suppose $\xi$ is a homothetic motion. Then from (3.1) it follows that $\nabla_k \mathcal{L}_\xi g_{ij} = 0$. 
Hence from (1.1) we get
\[ \nabla_k R_{ij} = 0. \]
Thus we get the same conclusion as in II if \( \xi \) induces a homothetic motion.

IV. Suppose \( \xi \) is projective collineation. In this case we have [10]
\[ E_\xi \{ h_{ij} \} = \delta_i^h \phi_j + \delta_j^h \phi_i \]
where \( \phi = \phi(x) \) is a scalar function and \( \phi_j = \nabla_j \phi \). The commutation formula gives
\[
\nabla_k E_\xi g_{ij} - E_\xi \nabla_k g_{ij} = g_{hi} E_\xi \{ h_{jk} \} + g_{hk} E_\xi \{ h_{ij} \} = 2 g_{ij} \phi_k + g_{ij} \phi_i + g_{ik} \phi_j.
\]
Hence
\[ (3.2) \quad \nabla_k E_\xi g_{ij} = 2 g_{ij} \phi_k + g_{ij} \phi_i + g_{ik} \phi_j. \]
Now from (1.1) we get
\[ \nabla_k E_\xi g_{ij} + 2 \nabla_k R_{ij} = 0 \]
Using (3.2) in the above equation we get
\[ (3.3) \quad \nabla_k R_{ij} = - (\phi_k g_{ij} + \frac{1}{2} \phi_i g_{kj} + \frac{1}{2} \phi_j g_{ik}), \]
which can be written as
\[ \nabla_k R_{ij} = \sigma_k g_{ij} + \nu_l g_{kj} + \nu_i g_{ki}, \]
where \( \sigma_k = -\phi_k, \nu_i = -\frac{1}{2} \phi_i. \)
Equation (3.3) implies that the manifold is a Sinyukov manifold.
Thus we can state the following:

Theorem 3.3. If \( \xi \) induces a projective collineation in \((M, g)\) admitting a Ricci soliton then the manifold is a Sinyukov manifold.

REFERENCES


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