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GENERALISED FRACTIONAL HERMITE-HADAMARD INEQUALITIES INVOLVING m-CONVEXITY AND (s,m)-CONVEXITY

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Abstract. Here we present generalised fractional Hermite-Hadamard type inequalities involving m-convexity and (s,m)-convexity. These inequalities are with respect to generalised Riemann-Liouville fractional integrals. Our work is motivated by and expands [7] to the greatest generality and all possible directions.

Keywords: fractional Hermite-Hadamard inequality, generalized fractional Riemann-Liouville integral, m-convexity, (s,m)-convexity.

1. Background

We use here the following generalised fractional integrals.

Definition 1.1. (see also [3, p. 99]) The left and right fractional integrals, respectively, of a function f with respect to a given function g are defined as follows:

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a,b])$ (absolutely continuous functions) and is strictly increasing, $f \in L_\infty([a,b])$. We set

(1) $$\left(I^{\alpha}_{a+;g} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t)\, dt, \quad x \geq a,$$

clearly

$$\left(I^{\alpha}_{a+;g} f\right)(a) = 0,$$

and

(2) $$\left(I^{\alpha}_{b-;g} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t)\, dt, \quad x \leq b,$$

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clearly

$$\left(I_{b-;g}^{\alpha} f\right)(b) = 0.$$

When g is the identity function id, we get that $I_{a+;id}^{\alpha} = I_{a+}^{\alpha}$ and $I_{b-;id}^{\alpha} = I_{b-}^{\alpha}$ the ordinary left and right Riemann-Liouville fractional integrals, where

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad x \geq a, \tag{3}$$

$(I_{a+}^{\alpha} f)(a) = 0$, and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x \leq b, \tag{4}$$

$\left(I_{b-}^{\alpha} f\right)(b) = 0$.

Remark 1.1. (see also [1]) We observe that

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} \left(f \circ g^{-1}\right)(g(t))\, g'(t)\, dt =$$

(by change of the variable for Lebesgue integrals)

$$\frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(f \circ g^{-1}\right)(z)\, dz = \left(I_{g(a)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \quad x \geq a, \tag{5}$$

equivalently $g(x) \geq g(a)$.

That is, in the terms and assumptions of Definition 1.1 we get

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \left(I_{g(a)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \quad \text{for } x \geq a. \tag{6}$$

Similarly, we observe that

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} \left(f \circ g^{-1}\right)(g(t))\, g'(t)\, dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(f \circ g^{-1}\right)(z)\, dz = \left(I_{g(b)-}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \tag{7}$$

for $x \leq b$.

That is,

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \left(I_{g(b)-}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \quad \text{for } x \leq b. \tag{8}$$

So by (6) and (8) we have reduced the general fractional integrals to the ordinary left and right Riemann-Liouville fractional integrals.

When $g(x) = e^x$, $x \in [a, b]$ we have the application

Definition 1.2. The left and right fractional exponential integrals are defined as follows: Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a,b])$. We set

$$\left(I^{\alpha}_{a+;e^x} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(e^x - e^t\right)^{\alpha-1} e^t f(t)\, dt, \quad x \geq a, \tag{9}$$

and

$$\left(I^{\alpha}_{b-;e^x} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(e^t - e^x\right)^{\alpha-1} e^t f(t)\, dt, \quad x \leq b. \tag{10}$$

Note 1.1. *We see that*

$$\left(I^{\alpha}_{a+;e^x} f\right)(x) = \left(I^{\alpha}_{e^a+} (f \circ \ln)\right)(e^x), \quad x \geq a, \tag{11}$$

and

$$\left(I^{\alpha}_{b-;e^x} f\right)(x) = \left(I^{\alpha}_{e^b-} (f \circ \ln)\right)(e^x), \quad x \leq b. \tag{12}$$

Another example follows:

Definition 1.3. Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a,b])$, $A > 1$. We introduce the fractional integrals:

$$\left(I^{\alpha}_{a+;A^x} f\right)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_a^x \left(A^x - A^t\right)^{\alpha-1} A^t f(t)\, dt, \quad x \geq a, \tag{13}$$

and

$$\left(I^{\alpha}_{b-;A^x} f\right)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_x^b \left(A^t - A^x\right)^{\alpha-1} A^t f(t)\, dt, \quad x \leq b. \tag{14}$$

We are motivated by the following theorem:

Theorem 1.1. (1881, Hermite-Hadamard inequality, [4]) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers, and $a, b \in I$, with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\, dt \leq \frac{f(a) + f(b)}{2}. \tag{15}$$

In addition to the classical convex functions, Toader [6], Hudzik and Maligranda [2] and Pinheiro [5] generalized the concepts of classical convex functions to the concepts of m-convex function and (s,m)-convex function.

Definition 1.4. The function $f : [0, b^*] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$ and $b^* > 0$ if for every $x, y \in [0, b^*]$ and $t \in [0,1]$, we have

$$f(tx + m(1-t)y) \leq t f(x) + m(1-t) f(y). \tag{16}$$

Definition 1.5. The function $f : [0, b^*] \to \mathbb{R}$ is said to be (s, m)-convex, where $(s, m) \in [0, 1]^2$ and $b^* > 0$, if for every $x, y \in [0, b^*]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s) f(y). \tag{17}$$

We need the following list of lemmas and theorems from [7].

Lemma 1.1. *Let $\alpha > 0$, $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) with $a < b$. If $f'' \in L_1([a, b])$, then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right) =$$

$$\frac{(b-a)^2}{2}\int_0^1 m(t) f''(ta + (1-t)b)\,dt, \tag{18}$$

where

$$m(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[0, \frac{1}{2}\right), \\ 1 - t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[\frac{1}{2}, 1\right). \end{cases} \tag{19}$$

Lemma 1.2. *Let $\alpha > 0$, $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) with $a < b$. If $f'' \in L_1([a, b])$, $r > 0$, then*

$$\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right] =$$

$$(b-a)^2 \int_0^1 k(t) f''(ta + (1-t)b)\,dt, \tag{20}$$

where

$$k(t) = \begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{t}{r+1}, & t \in \left[0, \frac{1}{2}\right), \\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1}, & t \in \left[\frac{1}{2}, 1\right). \end{cases} \tag{21}$$

Lemma 1.3. *Let $\alpha > 0$, $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) with $a < mb \leq b$. If $f'' \in L_1([a, b])$, $r > 0$, then*

$$\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb) + I_{mb-}^\alpha f(a)\right] =$$

$$(mb-a)^2 \int_0^1 k(t) f''(ta + m(1-t)b)\,dt, \tag{22}$$

where $k(t)$ is defined in (21).

The following fractional m-convex Hermite-Hadamard type inequalities also come from [7].

Theorem 1.2. *Let* $f:[0,b^*]\to\mathbb{R}$ *be a twice differentiable mapping with* $b^*>0$, $\alpha>0$. *If* $\left|f''\right|^q$ *is measurable and m-convex on* $\left[a,\frac{b}{m}\right]$ *for some fixed* $q\geq 1$, $0\leq a<b$ *and* $m\in(0,1]$ *with* $\frac{b}{m}\leq b^*$, $r>0$, *then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I^\alpha_{a+}f(b)+I^\alpha_{b-}f(a)\right]\right|$$

$$\leq(b-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right).$$

$$\left(\frac{\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R^m_1(f). \tag{23}$$

Theorem 1.3. *Let* $f:[0,b^*]\to\mathbb{R}$ *be a twice differentiable mapping with* $b^*>0$, $\alpha>0$. *If* $\left|f''\right|^q$ *is measurable and m-convex on* $\left[a,\frac{b}{m}\right]$ *for some fixed* $q>1$, $0\leq a<b$ *and* $m\in(0,1]$ *with* $\frac{b}{m}\leq b^*$, $r>0$, *then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I^\alpha_{a+}f(b)+I^\alpha_{b-}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R^m_2(f), \tag{24}$$

where

$$\frac{1}{p}+\frac{1}{q}=1.$$

Theorem 1.4. *Let* $f:[0,b^*]\to\mathbb{R}$ *be a twice differentiable mapping with* $b^*>0$, $\alpha>0$. *If* $\left|f''\right|^q$ *is measurable and m-convex on* $\left[a,\frac{b}{m}\right]$ *for some fixed* $q>1$, $0\leq a<b$ *and* $m\in(0,1]$ *with* $\frac{b}{m}\leq b^*$, $r>0$, *then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I^\alpha_{a+}f(b)+I^\alpha_{b-}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R^m_3(f). \tag{25}$$

Theorem 1.5. *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $\left|f''\right|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I_{a+}^\alpha f(b)+I_{b-}^\alpha f(a)\right]\right|$$

$$\le\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}.$$

(26) $$\left(\frac{\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R_4^m(f),$$

where

$$\frac{1}{p}+\frac{1}{q}=1.$$

Theorem 1.6. *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $\left|f''\right|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I_{a+}^\alpha f(b)+I_{b-}^\alpha f(a)\right]\right|$$

$$\le\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}.$$

(27) $$\left(\frac{\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R_5^m(f).$$

The following fractional (s,m)-convex Hermite-Hadamard type inequalities also come from [7].

Theorem 1.7. *Let $f:[0,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\le a<mb\le b$, $\alpha>0$. If $\left|f''\right|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q\ge1$ and $(s,m)\in(0,1]^2$, $r>0$, then*

$$H_s^m(f):=$$

$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb)+I_{mb-}^\alpha f(a)\right]\right|$$

$$\leq (mb-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}} \tag{28}$$

$$\times\left[\left|f''(a)\right|^q I+m\left|f''(b)\right|^q\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right]^{\frac{1}{q}}=:R_{1s}^m(f),$$

where

$$I=\frac{1}{r(s+1)(s+\alpha+2)}-\frac{1}{r(\alpha+1)}B(s+1,\alpha+2)$$
$$+\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right).$$

Theorem 1.8. *Let $f:[0,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\leq a<mb\leq b$, $\alpha>0$. If $\left|f''\right|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$ and $(s,m)\in(0,1]^2$ $r>0$, then*

$$H_s^m(f):=$$
$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[I_{a+}^{\alpha}f(mb)+I_{mb-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(mb-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\left|f''(a)\right|^q+\frac{ms}{s+1}\left|f''(b)\right|^q\right)^{\frac{1}{q}} \tag{29}$$

$$=:R_{2s}^m(f),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.9. *Let $f:[0,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\leq a<mb\leq b$, $\alpha>0$. If $\left|f''\right|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$ and $(s,m)\in(0,1]^2$, $r>0$, then*

$$H_s^m(f):=$$
$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[I_{a+}^{\alpha}f(mb)+I_{mb-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(mb-a)^2}{r(\alpha+1)}\left[\left|f''(a)\right|^q\left(\frac{1}{s+1}-\frac{1}{q(s+1)+s+1}-B(s+1,q(\alpha+1)+1)\right)\right. \tag{30}$$
$$+m\left|f''(b)\right|^q\left(\frac{s}{s+1}-\frac{2}{q(\alpha+1)+1}+\frac{1}{q(\alpha+1)+s+1}\right.$$
$$\left.\left.+B(s+1,q(\alpha+1)+1)\right)\right]=:R_{3s}^m(f).$$

Theorem 1.10. *Let $f : [0, b] \to \mathbb{R}$ be a twice differentiable mapping with $0 \leq a < mb \leq b$, $\alpha > 0$. If $\left|f''\right|^q$ is measurable and (s, m)-convex on $[a, b]$ for some fixed $q > 1$ and $(s, m) \in (0, 1]^2$, $r > 0$, then*

$$H_s^m(f) :=$$

$$\left| \frac{f(a) + f(mb)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb) + I_{mb-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(mb-a)^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{p+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}} \cdot$$

$$\left(\frac{1}{s+1}\left|f''(a)\right|^q + \frac{ms}{s+1}\left|f''(b)\right|^q\right)^{\frac{1}{q}} =: R_{4s}^m(f), \tag{31}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.11. *Let $f : [0, b] \to \mathbb{R}$ be a twice differentiable mapping with $0 \leq a < mb \leq b$, $\alpha > 0$. If $\left|f''\right|^q$ is measurable and (s, m)-convex on $[a, b]$ for some fixed $q > 1$ and $(s, m) \in (0, 1]^2$, $r > 0$, then*

$$H_s^m(f) :=$$

$$\left| \frac{f(a) + f(mb)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb) + I_{mb-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(mb-a)^2}{r+1}\left[\left|f''(a)\right|^q H + m\left|f''(b)\right|^q\left(\frac{2}{q+1}\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+1}\right.\right.$$

$$\left.\left. - \frac{2}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1} - H\right)\right] =: R_{5s}^m(f), \tag{32}$$

where

$$H = \int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)} + t\right)^q t^s dt + \int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)} + 1 - t\right)^q t^s dt. \tag{33}$$

The aim of this article is to extend the results of [7] to generalized fractional integrals (1) and (2), in particular to fractional exponential integrals (9), (10) and to fractional trigonometric integrals (60), (61), that is, to produce very general fractional m-convex and (s, m)-convex Hermite-Hadamard type inequalities.

2. Main Results

Combining Theorems 1.2-1.6 we get the following m-convex Hermite-Hadamard type inequality.

Theorem 2.1. *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $\left|f''\right|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$H^m(f)\le\min\left\{R_1^m(f),R_2^m(f),R_3^m(f),R_4^m(f),R_5^m(f)\right\}. \tag{34}$$

Combining Theorems 1.7-1.11 we obtain the following (s,m)-convex Hermite-Hadamard type inequality.

Theorem 2.2. *Let $f:[0,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\le a<mb\le b$, $\alpha>0$. If $\left|f''\right|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$ and $(s,m)\in(0,1]^2$, $r>0$, then*

$$H_s^m(f)\le\min\left\{R_{1s}^m(f),R_{2s}^m(f),R_{3s}^m(f),R_{4s}^m(f),R_{5s}^m(f)\right\}. \tag{35}$$

Next we generalize Lemmas 1.1-1.3.

Lemma 2.1. *Let $\alpha>0$, $a<b$, $f\in C([a,b])$, $g\in C^1([a,b])$, g strictly increasing on $[a,b]$, $\left(f\circ g^{-1}\right)$ is twice differentiable function on $(g(a),g(b))$ with $\left(f\circ g^{-1}\right)''\in L_1([g(a),g(b)])$. Then*

$$\frac{\Gamma(\alpha+1)}{2(g(b)-g(a))^\alpha}\left[I_{a+;g}^\alpha f(b)+I_{b-;g}^\alpha f(a)\right]-\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right)=$$

$$\frac{(g(b)-g(a))^2}{2}\int_0^1 m(t)\left(f\circ g^{-1}\right)''(tg(a)+(1-t)g(b))\,dt, \tag{36}$$

where $m(t)$ as in (19).

Lemma 2.2. *Let all as in Lemma 2.1, $r>0$. Then*

$$\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right)$$

$$-\frac{\Gamma(\alpha+1)}{r(g(b)-g(a))^\alpha}\left[I_{a+;g}^\alpha f(b)+I_{b-;g}^\alpha f(a)\right]$$

$$=(g(b)-g(a))^2\int_0^1 k(t)\left(f\circ g^{-1}\right)''(tg(a)+(1-t)g(b))\,dt, \tag{37}$$

where $k(t)$ as in (21).

Lemma 2.3. *Let all as Lemma 2.2, with $g(a) < mg(b) \leq g(b)$. Then*

$$\frac{f(a) + \left(f \circ g^{-1}\right)(mg(b))}{r(r+1)} + \frac{2}{r+1}\left(f \circ g^{-1}\right)\left(\frac{g(a) + mg(b)}{2}\right)$$

$$-\frac{\Gamma(\alpha+1)}{r(mg(b) - g(a))^{\alpha}}\left[I^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)(mg(b)) + I^{\alpha}_{mg(b)-}\left(f \circ g^{-1}\right)(g(a))\right]$$

$$= (mg(b) - g(a))^2 \int_0^1 k(t)\left(f \circ g^{-1}\right)''(tg(a) + m(1-t)g(b))\,dt, \tag{38}$$

where $k(t)$ as in (21).

We apply Lemmas 2.1-2.3 to $g(x) = e^x$.

Lemma 2.4. *Let $\alpha > 0$, $a < b$, $f \in C([a,b])$, $(f \circ \ln)$ is twice differentiable function on $\left(e^a, e^b\right)$ with $(f \circ \ln)'' \in L_1\left(\left[e^a, e^b\right]\right)$. Then*

$$\frac{\Gamma(\alpha+1)}{2\left(e^b - e^a\right)^{\alpha}}\left[I^{\alpha}_{a+;e^x} f(b) + I^{\alpha}_{b-;e^x} f(a)\right] - (f \circ \ln)\left(\frac{e^a + e^b}{2}\right) =$$

$$\frac{\left(e^b - e^a\right)^2}{2}\int_0^1 m(t)(f \circ \ln)''\left(te^a + (1-t)e^b\right)dt, \tag{39}$$

where $m(t)$ as in (19).

Lemma 2.5. *Let all as in Lemma 2.4, $r > 0$. Then*

$$\frac{f(a) + f(b)}{r(r+1)} + \frac{2}{r+1}(f \circ \ln)\left(\frac{e^a + e^b}{2}\right) - \frac{\Gamma(\alpha+1)}{r\left(e^b - e^a\right)^{\alpha}}\left[I^{\alpha}_{a+;e^x} f(b) + I^{\alpha}_{b-;e^x} f(a)\right]$$

$$= \left(e^b - e^a\right)^2 \int_0^1 k(t)(f \circ \ln)''\left(te^a + (1-t)e^b\right)dt, \tag{40}$$

where $k(t)$ as in (21).

Lemma 2.6. *Let all as in Lemma 2.5, with $e^a < me^b \leq e^b$. Then*

$$\frac{f(a) + (f \circ \ln)\left(me^b\right)}{r(r+1)} + \frac{2}{r+1}(f \circ \ln)\left(\frac{e^a + me^b}{2}\right)$$

$$-\frac{\Gamma(\alpha+1)}{r\left(me^b - e^a\right)^{\alpha}}\left[I^{\alpha}_{e^a+}(f \circ \ln)\left(me^b\right) + I^{\alpha}_{me^b-}(f \circ \ln)\left(e^a\right)\right]$$

$$= \left(me^b - e^a\right)^2 \int_0^1 k(t)(f \circ \ln)''\left(te^a + (1-t)e^b\right)dt, \tag{41}$$

where $k(t)$ as in (21).

We need the following notations.

Notation 2.1. *We denote by*

$$H^m(f,g) := \left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right) - \right.$$

$$\left. \frac{\Gamma(\alpha+1)}{r(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{a+;g}f(b) + I^{\alpha}_{b-;g}f(a)\right]\right|, \tag{42}$$

$$R_1^m(f,g) := (g(b)-g(a))^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right)\cdot$$

$$\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q + m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{43}$$

$$R_2^m(f,g) := \frac{(g(b)-g(a))^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\cdot$$

$$\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q + m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{44}$$

where

$$\frac{1}{p}+\frac{1}{q}=1,$$

$$R_3^m(f,g) := \frac{(g(b)-g(a))^2}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\cdot$$

$$\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q + m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{45}$$

$$R_4^m(f,g) := \left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(g(b)-g(a))^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1} - \right.$$

$$\left. -\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q + m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{46}$$

where

$$\frac{1}{p}+\frac{1}{q}=1,$$

and

$$R_5^m(f,g):=\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(g(b)-g(a))^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\right.$$

(47) $$\left.\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q+m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}.$$

We present the following fractional generalised m-convex Hermite-Hadamard type inequality.

Theorem 2.3. *Let all as in Notation 2.1. Here* $\alpha>0$, $b^*>0$, $f\in C([0,b^*])$, $g\in C^1([0,b^*])$, *g is strictly increasing on* $[0,b^*]$ *with* $g(0)=0$. *Assume that*

$$f\circ g^{-1}:[0,g(b^*)]\to\mathbb{R}$$

is twice differentiable mapping. If $\left|\left(f\circ g^{-1}\right)''\right|^q$ *is measurable and m-convex on* $\left[g(a),\frac{g(b)}{m}\right]$ *for some fixed* $q>1$, $0\le a<b\le b^*$ *and* $m\in(0,1]$ *with* $\frac{g(b)}{m}\le g(b^*)$, $r>0$, *then*

(48) $$H^m(f,g)\le\min\left\{R_1^m(f,g),\ R_2^m(f,g),\ R_3^m(f,g),\ R_4^m(f,g),\ R_5^m(f,g)\right\}.$$

Proof. By Theorem 2.1. □

We need additional notations.

Notation 2.2. *We denote by*

$$H_s^m(f,g):=\left|\frac{f(a)+\left(f\circ g^{-1}\right)(mg(b))}{r(r+1)}+\frac{2}{r+1}\left(f\circ g^{-1}\right)\left(\frac{g(a)+mg(b)}{2}\right)-\right.$$

(49) $$\left.\frac{\Gamma(\alpha+1)}{r(mg(b)-g(a))^\alpha}\left[I^\alpha_{g(a)+}\left(f\circ g^{-1}\right)(mg(b))+I^\alpha_{mg(b)-}\left(f\circ g^{-1}\right)(g(a))\right]\right|,$$

$$R_{1s}^m(f,g):=(mg(b)-g(a))^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}}\cdot$$

(50) $$\left[\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q I+m\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q\cdot\right.$$

$$\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\Bigg]^{\frac{1}{q}},$$

where

$$I:=\frac{1}{r(s+1)(s+\alpha+2)}-\frac{1}{r(\alpha+1)}B(s+1,\alpha+2)$$

$$+\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right), \tag{51}$$

$$R_{2s}^{m}(f,g):=\frac{(mg(b)-g(a))^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\cdot$$

$$\left(\frac{1}{s+1}\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q+\frac{ms}{s+1}\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q\right)^{\frac{1}{q}}, \tag{52}$$

where $\frac{1}{p}+\frac{1}{q}=1$,

$$R_{3s}^{m}(f,g):=\frac{(mg(b)-g(a))^2}{r(\alpha+1)}\Bigg[\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q\left(\frac{1}{s+1}-\frac{1}{q(\alpha+1)+s+1}\right.$$

$$\left.-B(s+1,q(\alpha+1)+1)\right)+m\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q\left(\frac{s}{s+1}-\frac{2}{q(\alpha+1)+1}\right. \tag{53}$$

$$\left.+\frac{1}{q(\alpha+1)+s+1}+B(s+1,q(\alpha+1)+1)\right)\Bigg],$$

$$R_{4s}^{m}(f,g):=\frac{(mg(b)-g(a))^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\Bigg[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-$$

$$\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\Bigg]^{\frac{1}{p}}\left(\frac{1}{s+1}\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q+\frac{ms}{s+1}\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q\right)^{\frac{1}{q}}, \tag{54}$$

where

$$\frac{1}{p}+\frac{1}{q}=1,$$

and

$$R_{5s}^{m}(f,g):=\frac{(mg(b)-g(a))^2}{r+1}\Bigg[\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q H+m\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q\cdot$$

$$\left(\frac{2}{q+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\frac{2}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}-H\right)\Bigg], \tag{55}$$

where

$$H=\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^q t^s dt+\int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^q t^s dt. \tag{56}$$

Next we present a fractional generalised (s,m)-convex Hermite-Hadamard type inequality.

Theorem 2.4. *Here all as in Notation 2.2. Let $\alpha>0$, $b>0$, $f\in C([0,b])$, $g\in C^1([0,b])$, g is strictly increasing on $[0,b]$ with $g(0)=0$. Assume that $f\circ g^{-1}:[0,g(b)]\to\mathbb{R}$ is twice differentiable mapping, with $0\le g(a)<mg(b)\le g(b)$, $a\in[0,b]$. If $\left|\left(f\circ g^{-1}\right)''\right|^q$ is measurable and (s,m)-convex on $[g(a),g(b)]$ for some fixed $q>1$ and $(s,m)\in(0,1]^2$, $r>0$, then*

$$H_s^m(f,g)\le\min\left\{R_{1s}^m(f,g),\ R_{2s}^m(f,g),\ R_{3s}^m(f,g),\ R_{4s}^m(f,g),\ R_{5s}^m(f,g)\right\}. \tag{57}$$

Proof. By Theorem 2.2. □

The case $q=1$ is met separately.

Proposition 2.3. *Here $H^m(f,g)$ as in (42) of Notation 2.1. The rest of the assumptions as in Theorem 2.3 with $q=1$. Then*

$$H^m(f,g)\le(g(b)-g(a))^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)$$

$$\times\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|+m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|}{2}\right). \tag{58}$$

Proof. By Theorem 1.2. □

Proposition 2.4. *Here $H_s^m(f,g)$ as in (49) of Notation 2.2. The rest of the assumptions as in Theorem 2.4 with $q=1$. Then*

$$H_s^m(f,g)\le(mg(b)-g(a))^2\left[\left|\left(f\circ g^{-1}\right)''(g(a))\right|I+m\left|\left(f\circ g^{-1}\right)''(g(b))\right|\cdot\right.$$

$$\left.\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right], \tag{59}$$

where I as in (51).

Proof. By Theorem 1.7. □

We need

Definition 2.1. Let $a, b \in \left[0, \frac{\pi}{2}\right]$, $a < b$, $\alpha > 0$, $f \in L_\infty([a,b])$. We consider the left and right fractional trigonometric integrals of f with respect to sine function denoted by sin :

$$\left(I^{\alpha}_{a+;\sin} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\sin x - \sin t)^{\alpha-1} \cos t f(t)\, dt, \quad x \geq a, \tag{60}$$

and

$$\left(I^{\alpha}_{b-;\sin} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\sin t - \sin x)^{\alpha-1} \cos t f(t)\, dt, \quad x \leq b. \tag{61}$$

We need

Notation 2.5. *We denote by*

$$H^m_*(f, \sin) := \left| \frac{f(a) + f(b)}{r(r+1)} + \frac{2}{r+1}\left(f \circ \sin^{-1}\right)\left(\frac{\sin(a) + \sin(b)}{2}\right) - \right.$$

$$\left. \frac{\Gamma(\alpha+1)}{r(\sin(b) - \sin(a))^{\alpha}} \left[I^{\alpha}_{a+;\sin} f(b) + I^{\alpha}_{b-;\sin} f(a)\right] \right|, \tag{62}$$

$$R^m_{1*}(f, \sin) := (\sin(b) - \sin(a))^2 \left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right) \cdot$$

$$\left(\frac{\left|\left(f \circ \sin^{-1}\right)''(\sin(a))\right|^q + m\left|\left(f \circ \sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^q}{2} \right)^{\frac{1}{q}}, \tag{63}$$

$$R^m_{2*}(f, \sin) := \frac{(\sin(b) - \sin(a))^2}{r(\alpha+1)} \left(1 - \frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \cdot$$

$$\left(\frac{\left|\left(f \circ \sin^{-1}\right)''(\sin(a))\right|^q + m\left|\left(f \circ \sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^q}{2} \right)^{\frac{1}{q}}, \tag{64}$$

where $\frac{1}{p} + \frac{1}{q} = 1$,

$$R^m_{3*}(f, \sin) := \frac{(\sin(b) - \sin(a))^2}{r(\alpha+1)} \left(\frac{q(\alpha+1) - 1}{q(\alpha+1)+1}\right)^{\frac{1}{q}} \cdot$$

$$\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{65}$$

$$R_{4*}^{m}(f,\sin):=\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(\sin(b)-\sin(a))^{2}}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\right.$$

$$\left.\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{66}$$

where

$$\frac{1}{p}+\frac{1}{q}=1,$$

and

$$R_{5*}^{m}(f,\sin):=\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(\sin(b)-\sin(a))^{2}}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\right.$$

$$\left.\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{67}$$

We present the following fractional generalised m-convex Hermite-Hadamard type inequality for sin function. So here $g(x)=\sin(x)$, $x\in\left[0,\frac{\pi}{2}\right]$.

Theorem 2.5. *Let all as in Notation 2.5. Here $\alpha>0$, $f\in C\left(\left[0,\frac{\pi}{2}\right]\right)$. Assume that $f\circ\sin^{-1}:[0,1]\to\mathbb{R}$ is twice differentiable mapping. If $\left|\left(f\circ\sin^{-1}\right)''\right|^{q}$ is measurable and m-convex on $\left[\sin(a),\frac{\sin(b)}{m}\right]$ for some fixed $q>1$, $0\le a<b\le\frac{\pi}{2}$ and $m\in(0,1]$ with $\sin(b)\le m$, $r>0$, then*

$$H_{*}^{m}(f,\sin)\le$$

$$\min\left\{R_{1*}^{m}(f,\sin),\ R_{2*}^{m}(f,\sin),\ R_{3*}^{m}(f,\sin),\ R_{4*}^{m}(f,\sin),\ R_{5*}^{m}(f,\sin)\right\}. \tag{68}$$

Proof. By Theorem 2.3. □

We need

Notation 2.6. *We denote by*

$$H_{s*}^{m}(f,\sin) := \left|\frac{f(a)+\left(f\circ\sin^{-1}\right)(m\sin(b))}{r(r+1)}+\right.$$

$$\frac{2}{r+1}\left(f\circ\sin^{-1}\right)\left(\frac{\sin(a)+m\sin(b)}{2}\right)-\frac{\Gamma(\alpha+1)}{r(m\sin(b)-\sin(a))^{\alpha}}\cdot$$

$$\left.\left[I_{\sin(a)+}^{\alpha}\left(f\circ\sin^{-1}\right)(m\sin(b))+I_{m\sin(b)-}^{\alpha}\left(f\circ\sin^{-1}\right)(\sin(a))\right]\right|, \tag{69}$$

$$R_{1s*}^{m}(f,\sin) := (m\sin(b)-\sin(a))^{2}\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}}\cdot$$

$$\left[\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}I+m\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^{q}\cdot\right. \tag{70}$$

$$\left.\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right]^{\frac{1}{q}},$$

where

$$I := \frac{1}{r(s+1)(s+\alpha+2)}-\frac{1}{r(\alpha+1)}B(s+1,\alpha+2)$$

$$+\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right), \tag{71}$$

$$R_{2s*}^{m}(f,\sin) := \frac{(m\sin(b)-\sin(a))^{2}}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\cdot$$

$$\left(\frac{1}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+\frac{ms}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^{q}\right)^{\frac{1}{q}}, \tag{72}$$

where $\frac{1}{p}+\frac{1}{q}=1$,

$$R_{3s*}^{m}(f,\sin) :=$$

$$\frac{(m\sin(b)-\sin(a))^{2}}{r(\alpha+1)}\left[\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}\left(\frac{1}{s+1}-\frac{1}{q(\alpha+1)+s+1}\right.\right.$$

$$\left.-B(s+1,q(\alpha+1)+1)\right)+m\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^{q}\left(\frac{s}{s+1}-\frac{2}{q(\alpha+1)+1}\right. \tag{73}$$

$$\left.\left.+\frac{1}{q(\alpha+1)+s+1}+B(s+1,q(\alpha+1)+1)\right)\right],$$

$$R_{4s*}^{m}(f,\sin):=\frac{(m\sin(b)-\sin(a))^{2}}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\right.$$

$$\left.-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\left(\frac{1}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+\right.$$

$$\left.\frac{ms}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^{q}\right)^{\frac{1}{q}},\tag{74}$$

where

$$\frac{1}{p}+\frac{1}{q}=1,$$

and

$$R_{5s*}^{m}(f,\sin):=$$

$$\frac{(m\sin(b)-\sin(a))^{2}}{r+1}\left[\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}H+m\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^{q}\cdot\right.$$

$$\left.\left(\frac{2}{q+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\frac{2}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}-H\right)\right],\tag{75}$$

where

$$H=\int_{0}^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^{q}t^{s}dt+\int_{\frac{1}{2}}^{1}\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^{q}t^{s}dt.\tag{76}$$

Next we present a fractional generalised (s,m)-convex Hermite-Hadamard type inequality involving $g(x)=\sin x$, $x\in\left[0,\frac{\pi}{2}\right]$.

Theorem 2.6. *Here all as in Notation 2.6. Let $\alpha>0$, $a,b\in\left[0,\frac{\pi}{2}\right]$, $a<b$, $f\in C([0,b])$. Assume that $f\circ\sin^{-1}:[0,\sin(b)]\to\mathbb{R}$ is twice differentiable mapping, with $0\leq\sin(a)<m\sin(b)\leq\sin(b)$. If $\left|\left(f\circ\sin^{-1}\right)''\right|^{q}$ is measurable and (s,m)-convex on $[\sin(a),\sin(b)]$ for some fixed $q>1$ and $(s,m)\in(0,1]^{2}$, $r>0$, then*

$$H_{s*}^{m}(f,\sin)\leq$$

$$\min\left\{R_{1s*}^{m}(f,\sin),\ R_{2s*}^{m}(f,\sin),\ R_{3s*}^{m}(f,\sin),\ R_{4s*}^{m}(f,\sin),\ R_{5s*}^{m}(f,\sin)\right\}.\tag{77}$$

Proof. By Theorem 2.4. □

Finally we treat the case of $q=1$ when $g(x)=\sin x$, $x\in\left[0,\frac{\pi}{2}\right]$.

Proposition 2.7. *Here $H_*^m(f,\sin)$ as in (62) of Notation 2.5. The rest of the assumptions as in Theorem 2.5 with $q=1$. Then*

$$H_*^m(f,\sin) \le (\sin(b)-\sin(a))^2 \left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right)\cdot$$

(78) $$\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right| + m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|}{2}\right).$$

Proof. By Proposition 2.3. □

Proposition 2.8. *Here $H_{s*}^m(f,\sin)$ as in (69) of Notation 2.6. The rest of the assumptions as in Theorem 2.6 with $q=1$. Then*

$$H_{s*}^m(f,\sin) \le (m\sin(b)-\sin(a))^2 \left[\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right| I + \right.$$

(79) $$\left. m\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right| \left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)} - I\right)\right],$$

where I as in (51).

Proof. By Proposition 2.4. □

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