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ON WEAK SYMMETRIES OF KÄHLER-NORDEN MANIFOLDS

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Abstract. The aim of the present paper is to study weakly symmetric and weakly Ricci symmetric Kähler-Norden manifolds.

Keywords: Manifold, symmetric spaces, Riemannian manifold, curvature.

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [4], who, in particular, obtained a classification of those spaces.

Let (M, g) be a Riemannian manifold of dimension n and ∇ be the Levi-Civita connection of (M, g). A Riemannian manifold is called locally symmetric [4] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). This condition of locally symmetry is equivalent to the fact that at every point $p \in M$, the local geodesic symmetry $F(p)$ is an isometry [14]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

During five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as recurrent manifolds by Walker [29], Ricci recurrent manifolds by Patterson [17], conformally symmetric manifolds by Chaki and Gupta [7], conformally recurrent manifolds by Adati and Miyazawa [1], pseudo symmetric manifolds by Chaki [5], weakly symmetric manifolds by Tamassy and Binh [26], projective symmetric manifolds by Soós [24] etc.

In 1989, the notions of weakly symmetric manifolds was introduced by Tamássy and Binh [26]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly symmetric if the curvature tensor R of type $(0, 4)$ satisfies the condition

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$$
\begin{aligned}
(\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V)+\beta(Y)R(X,Z,U,V)\\
&+ \gamma(Z)R(Y,X,U,V)+\delta(U)R(Y,Z,X,V)\\
&+ \rho(V)R(Y,Z,U,X),
\end{aligned}
\tag{1.1}
$$

for all $X, Y, Z, U, V \in \chi(M)$ and $\alpha, \beta, \gamma, \delta, \rho$ are 1-forms called the associated 1-forms which are not zero simultaneously and ∇ denotes covariant differentiation. Such a manifold is denoted by $(WS)_n$.

In a subsequent paper, the notion of weakly Ricci symmetric manifolds introduced by Tamássy and Binh [27]. A non-flat Riemannian manifold (M^n, g) is called weakly Ricci symmetric if the Ricci tensor S of type $(0,2)$ satisfies the condition

$$
(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z)+\beta(Y)S(X,Z)+\delta(Z)S(Y,X),
\tag{1.2}
$$

where α, β and δ are again 1-forms, not zero simultaneously. Such a manifold is denoted by $(WRS)_n$.

Weakly symmetric manifolds have been studied by Prvanović ([18], [19]), Binh [2], Özen and Altay ([15], [16]), De and Bandyopadhyay [10], De [9] and many others. If in (1.1) the 1-form α is replaced by 2α and ρ is equal to α, then the manifold is called a generalized pseudo symmetric manifold introduced and investigated by Chaki [6], and if in (1.2) the 1-form α is replaced by 2α, then the manifold is called a generalized pseudo Ricci symmetric manifold introduced by Chaki and Koley [8]. So the defining conditions of weakly symmetric and weakly Ricci symmetric manifolds are a little weaker than the generalized pseudo symmetric and generalized pseudo Ricci symmetric manifolds.

In [10], De and Bandyopadhyay gave an example of $(WS)_n$ and showed that in (1.1) necessarily $\gamma = \beta$ and $\rho = \delta$. So (1.1) takes the form:

$$
\begin{aligned}
(\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V)+\beta(Y)R(X,Z,U,V)\\
&+ \beta(Z)R(Y,X,U,V)+\delta(U)R(Y,Z,X,V)\\
&+ \delta(V)R(Y,Z,U,X).
\end{aligned}
\tag{1.3}
$$

Let A, B and P be the vector fields associated with the 1-forms α, β and δ respectively, that is, $g(X,A) = \alpha(X)$, $g(X,B) = \beta(X)$ and $g(X,P) = \delta(X)$ for all X. A, B and P are called the associated vector fields corresponding to the 1-forms α, β and δ respectively. In [28], Tamássy, De and Binh studied weakly symmetric and weakly Ricci symmetric Kähler manifolds and in [21] locally conformally Kähler manifolds. Also Velimirović et al. ([13], [25]) studied generalized Kählerian spaces. The Kähler manifolds with Norden metric have been studied in ([12], [20], [22], [23]) and many others. It may be mentioned that in a recent paper [12] Kim et al. proved the following:

Theorem 1.1. [12] *Every 4-dimensional Kähler-norden manifold is locally symmetric*

Hence in our paper we study weakly symmetric and weakly Ricci symmetric Kähler-Norden manifolds of dimension ≥ 6. The present paper is organized as follows: After preliminaries in section 3, we study weakly symmetric Kähler-Norden manifolds of dimension ≥ 6 and prove that a weakly symmetric Kähler-Norden manifold reduces to the recurrent one. Finally, we consider weakly Ricci symmetric Kähler-Norden manifolds of dimension ≥ 6 and prove that a weakly Ricci symmetric Kähler-Norden manifold is Ricci recurrent.

2. Priliminaries

By a Kählerian manifold with Norden metric (Kähler-Norden or, Anti-Kähler in short) [11] we mean a triple (M, J, g), where M is a connected differentiable manifold of dimension $n = 2m$, J is a $(1, 1)$-tensor field and g is a pseudo-Riemannian metric on M satisfying the conditions

$$J^2 = -I, \qquad g(JX, JY) = -g(X, Y), \qquad \nabla J = 0$$

for every $X, Y \in \chi(M)$ is the Lie algebra of vector fields on M and ∇ is the Levi-Civita connection of g.

Let (M, J, g) be a Kähler-Norden manifold. Since in dimension two such a manifold is flat, we assume in the sequel that $dimM \geq 4$. Let $R(X, Y)$ be the curvature operator

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

and let R be the Riemann-Christoffel curvature tensor,

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

The Ricci tensor S is defined as

$$S(X, Y) = trace\{Z \longrightarrow R(Z, X)Y\}.$$

These tensors have the following properties [3]

$$R(JX, JY) = -R(X, Y), \qquad R(JX, Y) = R(X, JY), \qquad \nabla_X JY = J\nabla_X Y \tag{2.1}$$

$$S(JY, Z) = trace\{X \longrightarrow R(JX, Y)Z\}, \qquad S(JX, Y) = S(JY, X),$$

$$S(JX, JY) = -S(X, Y).$$

Let Q be the Ricci operator. Then we have $S(X, Y) = g(QX, Y)$ and

$$QY = -\sum_i \epsilon_i R(e_i, Y)e_i.$$

In the above and in the sequel, $\{e_1, e_2, ..., e_n\}$ is an orthonormal frame and ϵ_i are the indicators of e_i,

$$\epsilon_i = g(e_i, e_i) = \pm 1.$$

The scalar curvature r and the $*$-scalar curvature r^*, which are defined as the trace of Q and JQ respectively.

3. Weakly Symmetric Kähler-Norden Manifolds

In this section we suppose that (M^n, g) is a $(WS)_n$ Kähler-Norden manifold. Now from (2.1) we find

$$\begin{aligned}(\nabla_X R)(JY, JZ, U, V) &= \nabla_X R(JY, JZ, U, V) - R(\nabla_X JY, JZ, U, V)\\ &\quad -R(JY, \nabla_X JZ, U, V) - R(JY, JX, \nabla_X U, V)\\ &\quad -R(JY, JZ, U, \nabla_X V).\\ &= \nabla_X R(JY, JZ, U, V) - R(J\nabla_X Y, JZ, U, V)\\ &\quad -R(JY, J\nabla_X Z, U, V) - R(JY, JX, \nabla_X U, V)\\ &\quad -R(JY, JZ, U, \nabla_X V). \qquad using (2.1)\\ &= -\nabla_X R(Y, Z, U, V) + R(\nabla_X Y, Z, U, V)\\ &\quad +R(Y, \nabla_X Z, U, V) + R(Y, X, \nabla_X U, V)\\ &\quad +R(Y, Z, U, \nabla_X V). \qquad using (2.1)\\ &= -(\nabla_X R)(Y, Z, U, V).\end{aligned} \tag{3.1}$$

Similarly,

$$(\nabla_X R)(Y, Z, JU, JV) = -(\nabla_X R)(Y, Z, U, V). \tag{3.2}$$

From (1.3) and (3.1) we have

$$\begin{aligned}\beta(Y)R(X, Z, U, V) + \beta(Z)R(Y, X, U, V) &= -\beta(JY)R(X, JZ, U, V)\\ &\quad -\beta(JZ)R(JY, X, U, V).\end{aligned} \tag{3.3}$$

Contracting (3.3) with respect to the pair of arguments Z, U (that is, taking $Z = U = e_i$ into (3.3), multiplying by ϵ_i and summing up over i), we have

$$\begin{aligned}\beta(Y)S(X, V) + g(R(X, Y)V, B) &= \sum_i \epsilon_i \beta(JY) g(R(V, e_i)X, Je_i)\\ &\quad -\sum_i \epsilon_i g(B, Je_i) g(R(JY, X)e_i, V).\end{aligned} \tag{3.4}$$

Again contracting (3.4) with respect to the pair of arguments X, V (that is, taking $X = V = e_j$ into (3.4), multiplying by ϵ_j and summing up over j) and using (2.1) we have

$$\beta(Y)r - S(Y,B) = -\sum_i \epsilon_i \beta(JY)S(e_i, Je_i) + S(JY, JB), \tag{3.5}$$

where r is the scalar curvature. Since in a Kähler-Norden manifold

$$S(JY, JB) = -S(Y,B),$$

then from (3.5) we get

$$\beta(Y)r = -\beta(JY)\sum_i \epsilon_i S(e_i, Je_i). \tag{3.6}$$

This implies

$$\beta(Y)r = -\beta(JY)r^*, \tag{3.7}$$

where r^* is the trace of JQ.

Similarly the formulas (1.3) and (3.2) yields

$$\delta(Y)r = -\delta(JY)r^*. \tag{3.8}$$

Putting $Y = JY$ in (3.7) we have

$$\beta(JY)r = \beta(Y)r^*, \tag{3.9}$$

Thus in view of (3.7) and (3.9) we have

$$\beta(Y)r^2 = -\beta(JY)rr^*$$

and

$$\beta(JY)r^* r = \beta(Y)r^{*2},$$

that is,

$$\beta(Y)(r^2 + r^{*2}) = 0. \tag{3.10}$$

Thus if $r \neq 0$ and $r^* \neq 0$, then from (3.10) we get $\beta(Y) = 0$.

In the similar way, it follows from (3.8) that if $r \neq 0$ and $r^* \neq 0$, then $\delta(Y) = 0$.

Using $\beta(Y) = 0$ and $\delta(Y) = 0$ in (1.3) we have

$$(\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V), \tag{3.11}$$

that is, the weakly symmetric Kähler-Norden manifold reduces to recurrent one. Therefore we can state the following:

Theorem 3.1. *The weakly symmetric Kähler-Norden manifold of dimension* ≥ 6, *with non-zero scalar curvature and non-zero* $*$*-scalar curvature reduces to recurrent one.*

Suppose $r^* = 0$. Therefore $r = 0$. Again from (1.3) we have

$$\begin{aligned}(\nabla_X S)(Y, V) = \alpha(X)S(Y, V) &+ \beta(Y)S(X, V) + \beta(R(X, Y)V) \\ &+ \delta(R(X, V)Y) + \delta(V)S(X, Y).\end{aligned} \tag{3.12}$$

Contracting (3.12) with respect to the pair of arguments Y, V (that is, taking $Y = V = e_i$ into (3.12), multiplying by ϵ_i and summing up over i), we have

$$X(r) = \alpha(X)r + 2S(X, B) + 2S(X, P). \tag{3.13}$$

Since $r = 0$, therefore from (3.13) we have

$$2S(X, B) + 2S(X, P) = 0, \tag{3.14}$$

that is,

$$S(X, B + P) = 0, \tag{3.15}$$

which shows that $B + P$ is the eigenvector of the Ricci tensor S corresponding to the eigenvalue zero. Therefore we can state the following:

Theorem 3.2. *In a weakly symmetric Kähler-Norden manifold of dimension* ≥ 6, $B + P$ *is the eigenvector of the Ricci tensor* S *corresponding to the eigenvalue zero provided* $r^* = 0$.

4. Weakly Ricci Symmetric Kähler-Norden Manifolds

In this section we suppose that the Kähler-Norden manifold is a $(WRS)_n$, that is,

$$(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \delta(Z)S(Y, X). \tag{4.1}$$

We know that

$$(\nabla_X S)(JY, JZ) = -(\nabla_X S)(Y, Z). \tag{4.2}$$

From (4.1) and (4.2) we have

$$-\beta(Y)S(X, Z) - \delta(Z)S(Y, X) = \beta(JY)S(X, JY) + \delta(JZ)S(JY, X). \tag{4.3}$$

Contracting (4.3) with respect to the pair of arguments X, Z (that is, taking $X = Z = e_i$ into (4.3), multiplying by ϵ_i and summing up over i), we have

$$\beta(Y)r = -\beta(JY)r^*. \tag{4.4}$$

Similarly, contracting (4.3) with respect to the pair of arguments X, Y we get

$$\delta(Z)r = -\delta(JZ)r^*. \tag{4.5}$$

Putting $Y = JY$ in (4.4) we have

$$\beta(JY)r = \beta(Y)r^*, \tag{4.6}$$

Thus, in view of (4.4) and (4.6), we have

$$\beta(Y)r^2 = -\beta(JY)rr^*$$

and

$$\beta(JY)r^*r = \beta(Y)r^{*2},$$

that is,

$$\beta(Y)(r^2 + r^{*2}) = 0. \tag{4.7}$$

Thus if $r \neq 0$ and $r^* \neq 0$, then from (4.7) we get $\beta(Y) = 0$.

In the similar way, it follows from (4.5) that if $r \neq 0$ and $r^* \neq 0$, then $\delta(Y) = 0$. Using $\beta(Y) = 0$ and $\delta(Y) = 0$ in (1.2) we have

$$(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z), \tag{4.8}$$

that is, the weakly Ricci symmetric Kähler-Norden manifold reduces to Ricci recurrent one. Therefore we can state the following:

Theorem 4.1. *The weakly Ricci symmetric Kähler-Norden manifold of dimension* ≥ 6, *with non-zero scalar curvature and non-zero* $*$*-scalar curvature reduces to Ricci recurrent one.*

Suppose $r^* = 0$. Therefore from (4.4) we obtain $r = 0$.

Contracting (4.1) with respect to the pair of arguments Y, Z (that is, taking $Y = Z = e_i$ into (4.1), multiplying by ϵ_i and summing up over i), we have

$$X(r) = \alpha(X)r + S(X, B) + S(X, P). \tag{4.9}$$

Again contracting (4.1) with respect to the pair of arguments X, Z (that is, taking $X = Z = e_i$ into (4.1), multiplying by ϵ_i and summing up over i), we get

$$\frac{1}{2}Y(r) = S(Y, A) + \beta(Y)r + S(Y, P). \tag{4.10}$$

Similarly by contracting X,Y we obtain

$$\frac{1}{2}Z(r) = S(Z, A) + S(Z, B) + \delta(Z)r. \tag{4.11}$$

Since $r = 0$, we have from (4.9), (4.10), (4.11) we have

$$S(X, A + B + P) = 0. \tag{4.12}$$

Thus $A + B + P$ is the eigenvector of the Ricci tensor S corresponding to the eigenvalue zero. Therefore we can state the following:

Theorem 4.2. *In a weakly Ricci symmetric Kähler-Norden manifold of dimension* ≥ 6, $A + B + P$ *is the eigenvector of the Ricci tensor* S *corresponding to the eigenvalue zero provided* $r^* = 0$.

Now we consider 4-dimensional weakly Ricci symmetric Kähler-Norden manifolds. Since every 4-dimensional Kähler-Norden manifold is Einstein [12], therefore

$$S(X, Y) = \lambda g(X, Y), \tag{4.13}$$

where λ is a non-zero constant and $\lambda = \frac{r}{4}$. From (4.9), (4.10) and (4.11) we have

$$2X(r) = r\{\alpha(X) + \beta(X) + \delta(X)\} + 2\{S(X, A) + S(X, B) + S(X, P)\}. \tag{4.14}$$

Using (4.13) and then $\lambda = \frac{r}{4}$ in (4.14) we have

$$X(r) = \frac{3r}{4}(\alpha(X) + \beta(X) + \delta(X)). \tag{4.15}$$

Since every 4-dimensional Kähler-Norden manifold is Einstein [12], therefore the scalar curvature is non-zero constant. Hence (4.15) becomes

$$\alpha(X) + \beta(X) + \delta(X) = 0. \tag{4.16}$$

Thus we can state the following:

Corollary 4.1. *In a* 4*-dimensional weakly Ricci symmetric Kähler-Norden manifold the sum of the associated* 1*-forms is zero.*

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