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FIELD EXTENSION AND DESIGNING A NEW 2×2 FULL-DIVERSITY, FULL-RATE AND INFORMATION LOSSLESS SPACE-TIME BLOCK CODE *

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Abstract. Based on field extensions we propose a novel aspect for designing linear dispersion space-time block codes (STBCs). The proposed code achieves the maximum attainable diversity. This code is also full-rate and proved to be information lossless. The simulation results also show that the proposed scheme outperforms the Golden code when BPSK constellation is utilized.

Keywords: Field extension, block code, diversity, Golden code.

1. Introduction

Digital wireless communication is an inseparable part of nowadays life. Achieving a high rate reliable data transmission is the main goal and many works have been done on different parts of a digital wireless communication system. As the bit rate increases, the characteristics of the channel change. As the time period of each symbol becomes shorter, the wideband and the narrowband fading grows. In a fading channel, transmitter has to use decades of dBs more than the case of simple additive white Gaussian noise (AWGN) channel in order to achieve the same performance. Obviously, the bit error rate in a digital communication system with AWGN depends on the mean distance of symbols. But, when we have fading in the channel, as an extreme upper bound, we could assume that symbols of the constellation are multiplied by a random number. Therefore, the mean distances of the symbols change. For instance, one can experimentally show that for a binary phase shift keying (BPSK) modulation, the mean distance is 2, but in a Rayleigh fading channel the mean distance would be approximately 1.2, as the mean of the coefficients is 0.62. In other words, when there is a deep fade, power of the noise would be comparable to power of the signal, and therefore detector would easily make mistakes.

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The solution to the fading effect of the channel is suggested to be diversity. The simplest definition by means of diversity is to have different replicas of signal at the receiver, and then as a result one can choose the best copy of the transmitted signal with highest SNR. To be more accurate the aim of this technique is to provide a signal with a fading coefficient, which has a desirable distribution. For instance, using maximal combining ratio would provide a Rayleigh coefficient for the transmitted signal with a larger mean, than the fading coefficients of the channel [3].
Multi antenna is the implementation of space diversity. That is, using multiple antennas at the receiver provides diversity gain and also could achieve the array gain which means higher signal to noise ratio compared to single antenna.

Using multiple antennas at the transmitter individually is not useful unlike receiver diversity [2]. In order to take the advantage of multi antenna at the transmitter one should also take time slots into account. For example in a multi-input single-output (MISO) system with two transmit antennas and two time slots, one can simply transmit one symbol with first antenna at the first time slot, and can use the second time slot and antenna to transmit the same symbol. This technique could outperform the performance of a fading channel at high SNRs, which is the technical definition of the diversity gain [15]. Obviously, sending no data with any of antennas in each time slot is not reasonable and loses the rate. Therefore, many works have been done in this area so far. The code presented in [12], also known as its designer Alamouti, is a space-time block code (STBC) whose rate is one symbol per channel use with linear complexity optimum decoder. The general forms of structure used in the Alamouti code are discussed in [14, 10], known as Orthogonal STBCs (OSTBCs). OSTBCs have linear decoders, but they have limited and low rates. Quasi-Orthogonal STBCs solve the problem of the low-rate property of the OSTBCs in the expense of Maximum-likelihood (ML) decoding [7]. References [9, 16] also introduce linear codes but their optimum performance is attained by using a large number of antennas.

Linear dispersion (LD) STBCs is the next generation of STBCs [6]. LD STBCs are defined as the matrix

$$X = \sum_{k=1}^{K} A_k s_k + \sum_{k=1}^{K} B_k s_k^*,$$

where K is the number of symbols; $\mathbf{s} = [s_1, s_2, \cdots, s_k]^T$ denotes a $K\times 1$ symbol vector; and A_k, B_k are $M \times T$ matrices [6]. These codes provide superior performance and useful properties, some of them will be discussed in next sections. There are many LD codes proposed in the literature [11, 13]. In this paper, we use a new method to design an information lossless full-rate STBCs with high coding gain and maximum achievable diversity.

The rest of this paper is organized as follows: In the next section, system model used in STBCs is described. Mathematical backgrounds for code construction and its properties are provided in section 3. In section 4, we propose a novel STBC and prove its advantages. Section 5 contains the simulation results, and finally in the last section the conclusions of the paper is presented.

2. System Formulation

Consider a multi-input multi-output (MIMO) system with M transmit antennas and N receive antennas which sends data simultaneously within each of T time slots. The received signal at the j^{th} receive antenna ($j = 1, 2, \cdots, N$), during the t^{th} time slot ($t = 1, 2, \cdots, T$) is given by:

$$y_j^t = \sum_{i=1}^{M} c_i^t h_{i,j}^t + \mathcal{N}_j^t \tag{2.1}$$

Where c_i^t is the symbol transmitted from i^{th} the transmit antenna during t^{th} time period, and $h_{i,j}^t$ is the channel fading coefficient of the path between transmit antenna i and receive antenna j within time slot t. Where $h_{i,j}^t$s are independent of t, in other words when

$$h_{i,j}^t = h_{i,j}, \qquad \forall t = 1, 2, \cdots, T$$

the channel is called quasi-static. It means that the fading coefficient of each path stays constant for at least one block of STBC.

The design criteria of STBCs for quasi-static channels are provided in [15]. In this case, formula 2.1 can be expressed in matrix form as :

$$Y = HX + W \tag{2.2}$$

Where Y is the $N \times T$ received matrix, H is the $N \times M$ channel matrix, W is the $N \times T$ additive noise matrix and X is the $M \times T$ matrix of a STBC associated with each information symbol vector $\mathbf{s}$. In family of LD codes, X could be presented by $M \times T$ matrices A_k and B_k for $k = 1, 2, \cdots K$, where K is the number of symbols coded in each codeword of STBC. The relation of vector $\mathbf{s} = (s_1, s_2, \cdots, s_K)$, where s_i's are taken from a given constellation and the matrices A_k, B_k and X is:

$$X = \sum_{k=1}^{K} A_k s_k + \sum_{k=1}^{K} B_k s_k^* \tag{2.3}$$

Where $(.)^*$ denotes the complex conjugate.

3. Mathematical Bases

Before introducing the code structure, some mathematical backgrounds are provided below.

Definition 3.1. A field F is said to be an extension of field K provided that K is a subfield of F. Let K be an arbitrary field. If there exists a least positive integer n such that $na = 0$ for all $a \in K$, then K is said to have characteristic n. If no such n exists K is said to have characteristic zero (The characteristic of is denoted by $char(K)$). Let F be an extension field of K. An element u of F is said to be algebraic over K provided that u is a root of some nonzero polynomial $f \in K[x]$.

Let K be a field and $f \in K[x]$ a polynomial of degree n. An extension field f of K is said to be a splitting field over K of the polynomial f if f splits in $F[x]$ and $F = K(u_1, u_2, \cdots, u_n)$ where $u_1, u_2, \cdots, u_n$ are the roots of f in F.

A splitting field F over a field K of $x^n - 1_K$ is called a cyclomatic extension of order n. If F is an extension field of K then F is a vector space over K. Throughout this paper the dimension of the K-vector space F will be denoted by $[F : K]$.

Example 3.1. $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ and $\{1, i\}$ is the basis of vector space $\mathbb{Q}(i)$ over $\mathbb{Q}$.

Theorem 3.1. *Let F be an extension field of E and E an extension field of K. Then* $[F : K] = [F : E][E : K]$ [5].

Theorem 3.2. *If F is an extension field of K and $u \in F$ is algebraic over K and*

$$[K(u) : K] = n$$

then $\{1_K, u, u^2, \cdots, u^{n-1}\}$ is a basis of the vector space $K(u)$ over K and so u is not the root of any nonzero polynomial in $K[x]$ of degree less than n. [5]

Theorem 3.3. *Let n be a positive integer, K a field such $char(K)$ that does not divide n and F a cyclomatic extension of K of order n then $F = K(\zeta)$, where $\zeta \in F$ is a primitive nth root of unity [5].*

Definition 3.2. The nth cyclotomic polynomial, for any positive integer n is the monic polynomial

$$\Phi_n(x) = \prod_{w} (x - w),$$

where the product is over all nth primitive roots of unity w in an algebraically closed field.

Theorem 3.4. *The nth cyclotomic polynomial Φ_n is a polynomial in $\mathbb{Q}[x]$ and is an irreducible polynomial over $\mathbb{Q}$* [4].

Theorem 3.5. *Suppose that p is a prime number. Then*

$$\Phi_p(x) = \sum_{i=1}^{p-1} x^i$$

and so $\deg(\Phi_p) = p - 1$ [4].

Now we introduce our main theorem that it's Proof is easy and direct by previous theorems and definitions.

Theorem 3.6. *For every positive integer n there exists a cyclomatic extension F of order $m > n$ of the field $\mathbb{Q}(i)$ such that $[F : \mathbb{Q}(i)] \geqslant n$ and $F = \mathbb{Q}(i, \zeta)$ where $\zeta = e^{i\theta}$.*

Corollary 3.1. *For every positive integer n there is a real number $\zeta = e^{i\theta}$ such that ζ is not the root of any nonzero polynomial in $\mathbb{Z}(i)[x]$ of degree less than n.*

4. STBCs Design Based on Field Extension

4.1. Construction of the Proposed STBC

In this section we propose an information lossless full-rate, full-diversity STBC as follows:

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 + \varphi^2 s_3 & \varphi s_2 + \varphi^3 s_4 \\ -\varphi s_2 + \varphi^3 s_4 & s_1 - \varphi^2 s_3 \end{pmatrix}, \tag{4.1}$$

where $\mathbf{s} = (s_1, s_2, s_3, s_4)$ are the symbol vector taken from a given constellation (QAM or PSK) and the scalar$\varphi = e^{i\theta}$ is not the root of any nonzero polynomial in $\mathbb{Z}(i)[x]$ of degree less than 7.

4.2. The Diversity Order and the Coding Gain of the Proposed STBC

4.2.1. Diversity Order

It is well-known that an STBC is full diversity if for any two different code words C and $\hat{C}$ in the code book, $\Delta = C - \hat{C}$ be a full-rank matrix [3]. For a full-diversity STBC, coding gain is defined as the minimum determinant of A for any two distinct codewords where

$$A = \Delta^H \Delta. \tag{4.2}$$

As a consequence, if the minimum determinant of matrix A for any two distinct codewords is nonzero, the STBC benefits from the full-diversity property.

Table 4.1: The optimum values of parameter φ.

	Step	0.1	0.01	0.001	0.0001
$4-QAM$	*CG*	1.751	1.751	1.751	1.7942
	θ	5.9210	4.0790	4.079	6.0795
BPSK	*CG*	8	8	8	8
	θ	2.8010	0.1710	0.1670	4.2100

In order to prove that the proposed code in equation 4.1 is full-diversity, we show that minimum determinant of matrix A for each pair of different codewords is not zero.

According to 4.1, we can write $\det(A)$ for our proposed code as

$$\begin{aligned}\det(\Delta^H\Delta) &= \mid \det(\Delta) \mid^2 \\ &= \frac{1}{4} \mid \omega_1^2 + \varphi^2\omega_2^2 - \varphi^4\omega_3^2 - \varphi^6\omega_4^2 \mid^2,\end{aligned}$$

where $\omega_i = s_i - \hat{s}_i$, $i = 1, 2, 3, 4$ and s_i, $\hat{s}_i$ are the i^{th} symbol of C and $\hat{C}$ respectively taken from a given constellation. Based on Corollary 3.1, we can choose ϕ such that it is not the root of any nonzero polynomial in $\mathbb{Z}(i)[x]$ of degree less than 7. For every nonzero vector $(\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2) \in \mathbb{Z}(i)^4$, $\det(\Delta^H\Delta)$ is the value of a nonzero polynomial in of degree 6 in and so it is not zero. Therefore, the newly proposed STBC is full-diversity over any constellation that is a subspace of $\mathbb{Z}(i)$ (like M-QAM).

4.2.2. Coding gain

In order to obtain the best performance of the code, we must find the value of which ϕ maximize the coding gain. By numerical search, we found the optimal values of ϕ presented in Tabel4.1.

4.3. Information losslessness

It is known that the STBC **X** is information lossless if the capacity of the new precoded channel, $\mathcal{H}$ obtained by considering the STBC as a part of the channel, has the same capacity as the original channel H. The difference between the capacities of the new and the original channels represents the information loss of the STBC.

From [1], we have the following theorem for a full-rate STBC to be information lossless:

Theorem 4.1. *Let $K = N_tT$ (the STBC is full-rate). then, subject to the power constraint*

$$\sum_{k=1}^{K} tr(A_k A_k^H + B_k B_k^H) = N_t T \tag{4.3}$$

The LD-STBC is information lossless if and only if the matrix $\mathcal{F}$ is unitary, where

$$\mathcal{F} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^* & \mathcal{A}^* \end{pmatrix} \tag{4.4}$$

and

$$\begin{aligned} \mathcal{A} &= [vecA_1, vecA_2, \cdots, vecA_K], \\ \mathcal{B} &= [vecB_1, vecB_2, \cdots, vecB_K]. \end{aligned}$$

where $vec(.)$ is operator stacking all columns of a matrix on top of each other and $(.)^*$ is complex conjugate and $(.)^H$ is the conjugate and transpose of a matrix.

In our STBC, we have $\mathcal{B} = 0$ and so it is enough to show that matrix $\mathcal{A}$ is unitary. It is easy to see that

$$\mathcal{A}_C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \varphi^2 & 0 \\ 0 & -\varphi & 0 & \varphi^3 \\ 0 & \varphi & 0 & \varphi^3 \\ 1 & 0 & -\varphi^2 & 0 \end{pmatrix} \tag{4.5}$$

is a unitary matrix and, therefore, based on Theorem 4.1, our STBC is information lossless.

5. Simulation Results

Simulation results are presented in Figure5.1 for BPSK modulation. The optimum ML is used for detection. The number of receive antennas is two. The parameter φ is taken to be 4.21 as the optimum value of TABLE 4.1 for 1bit/sec/Hz bandwidth efficiency.

In what follows, we compare the performance of our proposed STBC with those of the Golden code and the code proposed in [11].

The Golden code is a 2×2 linear dispersion algebraic space-time code with non-vanishing determinants [8].

$$X = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha(a + b\theta) & \alpha(c + d\theta) \\ \gamma\bar{\alpha}(c + d\bar{\theta}) & \bar{\alpha}(a + b\bar{\theta}) \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}[i],$$

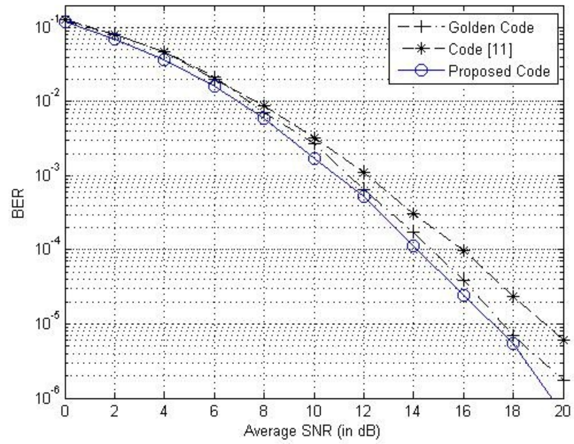

FIG. 5.1: BER versus SNR for the quasi-static channel, BPSK constellation (bit/sec/Hz)

where

$$\theta = \frac{1+\sqrt{5}}{2},\ \bar{\alpha} = 1 + i(1-\bar{\theta}),\ \bar{\theta} = 1-\theta = \frac{1-\sqrt{5}}{2},\ \gamma = e^{i\varphi} \in \mathbb{C}.$$

The code proposed in [11] has the following structure:

$$B_{2,\varphi} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 + \varphi s_2 & \theta(s_3 + \varphi s_4) \\ \theta(s_3 - \varphi s_4) & s_1 - \varphi s_2 \end{pmatrix} \tag{5.1}$$

where $\theta^2 = \varphi$ and $\varphi = e^{i\gamma}$ and $\gamma = \frac{1}{2}$ is optimum value of γ.

As Figure 5.1 shows, the proposed code outperforms the Golden code and has a superior performance than code in $B_{2,\varphi}$.

6. Conclusion

In this paper, we constructed a novel 2×2 STBC with a new algebraic point of view. This code is full rate. The proposed code is also full diversity and information lossless.

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