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A NOTE ON MULTIPLIERS OF WEAK TYPE ON THE DYADIC GROUP

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Abstract. In this paper we give an example that shows the sharpness of Quek′s result on weak type multipliers for Lipschitz functions on the dyadic group.

Keywords: Weak multipliers, Lipschitz functions, dyadic group.

1. Introduction

Let $\mathbb{Z}_2$ denote the discrete cyclic group $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is addition modulo 2. If $|E|$ denotes the measure of the subset $E \subset \mathbb{Z}_2$, then $|\{0\}| = |\{1\}| = \frac{1}{2}$.

The dyadic group G is obtained from $G = \prod_{i=0}^{\infty} \mathbb{Z}_2$, where topology and measure are obtained by the product.

Let $x = (x_n)_{n\geq 0} \in G$. The sets

$$I_n(x) := \{y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\},\ n \geq 1$$

and $I_0(x) := G$ are dyadic intervals of G. Set $e_n := (\delta_{in})_i$.

We define the right shift by

$$e_i.x := \sum_{j=i}^{\infty} x_{j-i} e_j.$$

The Walsh-Paley system Γ is defined as the set of Walsh-Paley functions

$$\psi_i(x) = \prod_{k=0}^{\infty} (r_k(x))^{i_k},\ i \in \mathbb{N},\ x \in G,$$

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where $i = \sum_{k=0}^{\infty} i_k 2^k$ and $r_k(x) = (-1)^{x_k}$.

Let Γ_n denote the finite subset of characters taking the value 1 on the subgroup $I_n(0)$. It is easily seen that

$$\Gamma_n = \{\psi_0, \psi_1, \dots, \psi_{2^n-1}\}.$$

Also, $(\Gamma_n)_n$ is an increasing sequence and $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$.

Let μ, λ be the Haar measures on G and Γ respectively. These are chosen so that

$$\mu(G) = \lambda(\Gamma_0) = 1$$

and

$$\mu(I_n) = (\lambda(\Gamma_n))^{-1} = 2^{-n},$$

for every $n \geq 1$.

On the set Γ we define the metric $|\xi - \eta| = 2^n$, if

$$\xi \cdot \eta^{-1} \in \Gamma_n \backslash \Gamma_{n-1}.$$

Let $f \in L^\infty(\Gamma)$. f^n will denote the function

$$f \cdot 1_{\Gamma_{n+1}} - f \cdot 1_{\Gamma_n},$$

where 1_A is the characteristic function of the set A. In other words, f^n is the restriction of f on the set $\Gamma_{n+1} \setminus \Gamma_n$, and f^n vanishes outside that set.

For any $\beta > 0$, the set Λ_β is the Lipschitz space of functions $f \in L^\infty(\Gamma)$ so that

$$\|f\|_{\Lambda_\beta} = \sup_{\psi_i \neq \psi_j} \frac{|f(\psi_i) - f(\psi_j)|}{|\psi_i - \psi_j|^\beta} < \infty.$$

The Fourier transform and the inverse Fourier transform are denoted by $\wedge$ and $\vee$ respectively.

Namely, if φ is integrable on G, then,

$$\varphi^\wedge(\psi_i) = \int \varphi(x)\overline{\psi}_i(x)dx,$$

where $\psi_i \in \Gamma, i \in \mathbb{N}$.

Now, if $f \in L^\infty(\Gamma)$, then

$$f^\vee(x) = \sum_{i=0}^{\infty} f(\psi_i)\psi_i(x), \qquad x \in G.$$

It is easily seen that $(2^n 1_{G_n})^\wedge = 1_{\Gamma_n}$.

On the set of real numbers, the question on how little regularity is needed for the restriction of a function on bounded intervals in order to construct an L^p-multiplier emerged in [5].

Sufficient conditions have been studied in [1] and [2].

By means of an analogue of Calderón-Zygmund decomposition [3], Quek [6] obtained the following result for L^p-multipliers on totally disconnected groups which are nothing but generalizations of the dyadic group:

Theorem 1.1. *([6]) Let $1 < p < 2$ and let $f \in L^\infty(\Gamma)$. Suppose there exists $\beta > \frac{2-p}{2p}$ such that*

$$\|f^j\|_{\Lambda_\beta} \leq C \cdot 2^{-j\beta}, j \in \mathbb{N}, \tag{1.1}$$

where C is a constant independent of j. Then f is an $L^p(G)$-multiplier.

Gaudry and Inglis [4] proved that for $\beta < \frac{2-p}{2p}$, there exists a function f on the dyadic group satisfying (1.1), but f is not an L^p-multiplier.

The following open question appeared in [6]:

If $\beta = \frac{2-p}{2p}$, and $f \in L^\infty(\Gamma)$ satisfies (1.1). Is f a multiplier of weak type (p,p) on G?

However, we prove:

Theorem 1.1. *If G is the dyadic group, $1 < p < 2$, then there exists a function $f \in L^\infty(\Gamma)$ having the property (1.1), but f is not a multiplier of weak type (p,p) on G.*

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following lemma.

Lemma 2.1. *Let G be the dyadic group. There exists a function $f \in L^\infty(\Gamma)$ so that*

$$|(f^k)^\vee(t)| = 2^{k(\frac{1}{2}-\beta)},$$

for every even positive integer k and for every $t \in G$.

Proof. Let $f \in L^\infty(\Gamma)$ be a real function such that $f^0 = f^k = 0$, for every odd number k and f^2 takes the values $f^2(\psi_4) = f^2(\psi_5) = f^2(\psi_6) = -f^2(\psi_7) = 2^{-2\beta}$.

We construct recursively f^{2k+2} from f^{2k} assuming that $(f^{2k})^\vee$ takes only the values $\pm 2^{2k(\frac{1}{2}-\beta)}$, which is easily verified for $(f^2)^\vee$. In fact, since $(f^2)^\vee$ is constant on I_3-cosets, it suffices to calculate its values on

$$0,\ e_0,\ e_1,\ e_0+e_1,\ e_2,\ e_2+e_0,\ e_2+e_1,\ e_2+e_0+e_1.$$

Simple calculations give that $|(f^2)^\vee| = 2^{2(\frac{1}{2}-\beta)}$.

Using the notations above we have for $k \geq 1$,

$$\Gamma_{2k+1} \setminus \Gamma_{2k} = \{\psi_{2^{2k}}, \ldots, \psi_{2^{2k+1}-1}\}.$$

It is easily seen that

$$\begin{aligned} \Gamma_{2k+3} \setminus \Gamma_{2k+2} &= \{\psi_{2^{2k+2}}, \ldots, \psi_{2^{2k+3}-1}\} \\ &= \{\psi_{4i}, \psi_{4i+1}, \psi_{4i+2}, \psi_{4i+3}; \psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}\}. \end{aligned}$$

We put

$$f^{2k+2}(\psi_{4i}) = f^{2k+2}(\psi_{4i+1}) = f^{2k+2}(\psi_{4i+2}) = -f^{2k+2}(\psi_{4i+3}) = 2^{-2\beta} f^{2k}(\psi_i),$$

for $\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}$.

Assume that for some $k \geq 1$, $(f^{2k})^\vee$ takes only the values $\pm 2^{2k(\frac{1}{2}-\beta)}$. We prove that $(f^{2k+2})^\vee$ only takes the values $\pm 2^{(2k+2)(\frac{1}{2}-\beta)}$. It suffices to verify this fact only on the set of representatives of all classes from G/I_{2k+3}.

Let $(t_j)_{j\in J}$ be a fixed set of representatives of all I_{2k+1} cosets. We can generate a complete set of representatives of I_{2k+3} cosets if we consider elements of the form

$t_j^1 = e_2.t_j$, $t_j^2 = e_2.t_j + e_0$, $t_j^3 = e_2.t_j + e_1$ and $t_j^4 = e_2.t_j + e_0 + e_1$, $j \in J$.

Now,

$$\begin{aligned}
(f^{2k+2})^{\vee}(t_j^1) &= \sum_{\psi_i \in \Gamma_{2k+3} \setminus \Gamma_{2k+2}} f^{2k+2}(\psi_i)\psi_i(t_j^1) \\
&= \sum_{\psi_i \in \Gamma_{2k+3} \setminus \Gamma_{2k+2}} f(\psi_i)\psi_i(t_j^1) \\
&= \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i})\psi_{4i}(t_j^1) + \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i+1})\psi_{4i+1}(t_j^1) \\
&\quad + \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i+2})\psi_{4i+2}(t_j^1) + \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i+3})\psi_{4i+3}(t_j^1) \\
&= 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) + 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) \\
&\quad + 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) - 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) \\
&= 2^{-2\beta+1} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f^{2k}(\psi_i)\psi_i(t_j) \\
&= 2^{-2\beta+1}(f^{2k})^{\vee}(t_j),
\end{aligned}$$

because

$$\psi_{4i}(t_j^1) = \psi_{4i+1}(t_j^1) = \psi_{4i+2}(t_j^1) = \psi_{4i+3}(t_j^1) = \psi_i(t_j).$$

In a similar way we have

$$\begin{aligned}
(f^{2k+2})^{\vee}(t_j^2) &= \sum_{\psi_i \in \Gamma_{2k+3} \setminus \Gamma_{2k+2}} f^{2k+2}(\psi_i)\psi_i(t_j^2) \\
&= \sum_{\psi_i \in \Gamma_{2k+3} \setminus \Gamma_{2k+2}} f(\psi_i)\psi_i(t_j^2) \\
&= \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i})\psi_{4i}(t_j^2) + \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i+1})\psi_{4i+1}(t_j^2) \\
&\quad + \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i+2})\psi_{4i+2}(t_j^2) + \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_{4i+3})\psi_{4i+3}(t_j^2) \\
&= 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) - 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) \\
&\quad + 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) + 2^{-2\beta} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f(\psi_i)\psi_i(t_j) \\
&= 2^{-2\beta+1} \sum_{\psi_i \in \Gamma_{2k+1} \setminus \Gamma_{2k}} f^{2k}(\psi_i)\psi_i(t_j) = 2^{-2\beta+1}(f^{2k})^{\vee}(t_j),
\end{aligned}$$

because
$$\psi_{4i}(t_j^2) = \psi_{4i+2}(t_j^2) = -\psi_{4i+1}(t_j^2) = -\psi_{4i+3}(t_j^2) = \psi_i(t_j).$$

Since
$$\psi_{4i}(t_j^3) = \psi_{4i+1}(t_j^3) = -\psi_{4i+2}(t_j^3) = -\psi_{4i+3}(t_j^3) = \psi_i(t_j),$$

and

$$\psi_{4i}(t_j^4) = -\psi_{4i+1}(t_j^4) = -\psi_{4i+2}(t_j^4) = \psi_{4i+3}(t_j^4) = \psi_i(t_j),$$

we similarly get

$$(f^{2k+2})^\vee(t_j^3) = -(f^{2k+2})^\vee(t_j^4) = 2^{-2\beta+1}(f^{2k})^\vee(t_j),$$

which ends the proof of the lemma. □

The following is the proof of Theorem 1.1.

Proof. We use the function f from Lemma 2.1. It is easily seen that the property (1.1) is satisfied. We prove the existence of locally constant functions $(\varphi_n)_n$ and positive numbers $(\sigma_n)_n$, so that

$$\mu(\{t \in G : |f^\vee * \varphi_n| > \sigma_n\}) = 1$$

for every n, and
$$\frac{\|\varphi_n\|_p^p}{\sigma_n^p} \to 0$$
as $n \to \infty$.

Indeed, let

$$\sigma_k = 2^{k(\frac{1}{2}-\beta)}, k \geq 0, \quad \varphi_n = 3\sum_{k=0}^{n-1} \epsilon_k \frac{\sigma_n}{n\sigma_k}(\triangle_{k+1} - \triangle_k),$$

where $\triangle_k = 2^k 1_{G_k}$, and $\epsilon_k(t) = \mathrm{sgn}((f^k)^\vee(t))$. Since,

$$\begin{aligned}(f^k)^\vee(t) &= (f \cdot 1_{\Gamma_{k+1}} - f \cdot 1_{\Gamma_k})^\vee(t) \\ &= (f^\vee * (1_{\Gamma_{k+1}} - 1_{\Gamma_k})^\vee)(t) \\ &= f^\vee * (\triangle_{k+1} - \triangle_k)(t),\end{aligned}$$

then,

$$\epsilon_k(t) = \mathrm{sgn}(f^\vee * (\triangle_{k+1} - \triangle_k)(t)).$$

We have

$$\begin{aligned} f^\vee * \varphi_n &= 3\sum_k \epsilon_k \frac{\sigma_n}{n\sigma_k} f^\vee * (\triangle_{k+1} - \triangle_k) \\ &= 3\sum_{k=0}^{n-1} \frac{\sigma_n}{n\sigma_k} |f^\vee * (\triangle_{k+1} - \triangle_k)(t)| \\ &= 3\sum_{k=0}^{n-1} \frac{\sigma_n}{n\sigma_k} |(f^k)^\vee(t)| > \sigma_n, \end{aligned}$$

for every $t \in G$.

Besides, by Lemma 2.2 in [6], we obtain

$$\begin{aligned} \|\varphi_n\|_p^p &= \|3\sum_{k=0}^{n-1} \epsilon_k \frac{\sigma_n}{n\sigma_k}(\triangle_{k+1} - \triangle_k)\|_p^p \\ &= \|3\sum_{k=0}^{n-1}([\epsilon_k \frac{\sigma_n}{n\sigma_k}(\triangle_{k+1} - \triangle_k)]^\wedge)^\vee\|_p^p \\ &= \|3\sum_{k=0}^{n-1}([\epsilon_k \frac{\sigma_n}{n\sigma_k}(\triangle_{k+1} - \triangle_k)]^\wedge \cdot 1_{\Gamma_{k+1}\backslash Gamma_k})^\vee\|_p^p \\ &\leq C\sum_{k=0}^{n-1} \frac{\sigma_n^p}{n^p\sigma_k^p}\|\triangle_{k+1} - \triangle_k\|_p^p, \end{aligned}$$

for some constant C independent on the choice of n.

The assumption $\beta = \frac{2-p}{2p}$ implies that $\|\triangle_{k+1} - \triangle_k\|_p^p = \sigma_k^p$.

Then, $\frac{\|\varphi_n\|_p^p}{\sigma_n^p} \leq C\frac{n}{n^p} \to 0$, as $n \to \infty$. This completes the proof of Theorem 1.1. □

3. Conclusion

Analyzing the works of [5] and [6], we see that a certain form of regularity of the restrictions of a given function to bounded intervals is sufficient to obtain either multipliers or weak type multipliers. The regularity needed for weak type multipliers was proved to be sharp in Theorem 3 of [6]. However, the regularity obtained for multipliers is only valid for $\beta > \frac{2-p}{2p}$. The question about the nature of functions constructed under similar conditions but for $\beta = \frac{2-p}{2p}$ remained unanswered.

Our Theorem 1.1 provides a counterexample which shows that for $\beta = \frac{2-p}{2p}$, the obtained function need not be even a weak type multiplier. From which we conclude that Theorem 2 in [6] is very sharp.

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