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CURVATURE BASED FUNCTIONS VARIATIONS

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Abstract. One of the methods of building the fundamentals of Differential Geometry is the shape operator approach and everything you could want to know about a surface's curvature is locked up in the shape operator. In this work we consider the variation of the shape operator and the normal, the mean and the principals curvatures and, at last, the Willmore energy, under infinitesimal bending of surface given in an explicit form.

Keywords: Shape operator, Willmore energy, infinitesimal bending, surface.

1. Introduction

The calculus of variations studies the extreme and critical points of functions. It has its roots in many areas, from geometry to optimization to mechanics.

The importance of checking the curvature of the surface was underlined at [1]. It is known that the magnitudes depending on the first fundamental form are stationary under infinitesimal bending [3]. The variation of the Willmore energy under infinitesimal bending of the surface is studied at [11]. Infinitesimal F-planar transformations were discussed in work [5].

2. Preliminary of Curvature Based Functions

In this section we give basic definitions and properties of the curvatures based functions, such as the shape operator, the normal curvature, the mean and the Gaussian curvatures, the principal curvatures and the Willmore energy at a surface point.

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2.1. Shape Operator

Two rudimentary ways to characterize the shape of a surface S are to consider how the unit normal ν behaves as we move around and to compare S to a sphere. The former of these methods is accomplished using the shape operator. It is a linear operator that calculates the bending of surface S.

The linear operator, the shape operator, applied to a tangent vector v_p is the negative of the derivative of ν in the direction v_p.

Definition 2.1. *[4] Let $S \subset \mathcal{R}^3$ be a regular surface, and let ν be a surface normal to S defined in the neighborhood of a point $p \in S$. For a tangent vector v_p to S at p we put*

$$\mathbf{S}(v_p) = -D_v \nu. \tag{2.1}$$

Then $\mathbf{S}$ *is called the* ***shape operator****.*

The shape operator of a plane is identically zero at all points of the plane. For a nonplanar surface, the surface normal ν will twist and turn from point to point, and $\mathbf{S}$ will be nonzero.

Now, we will show how to express the shape operator in terms of the coefficients of the first E, F, G and the second L, M, N fundamental form, that is proved in [4].

Theorem 2.1. *(The Weingarten equations) Let $\mathbf{r} : D \to \mathcal{R}^3, D \subset \mathcal{R}^2$ be a regular surface. Then the shape operator $\mathbf{S}$ of $\mathbf{r}$ is given in terms of the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ by*

$$\begin{cases} -\mathbf{S}(\mathbf{r}_u) = \nu_u = & \frac{MF-LG}{EG-F^2}\mathbf{r}_u + \frac{LF-ME}{EG-F^2}\mathbf{r}_v, \\ -\mathbf{S}(\mathbf{r}_v) = \nu_v = & \frac{NF-MG}{EG-F^2}\mathbf{r}u + \frac{MF-NE}{EG-F^2}\mathbf{r}_v. \end{cases} \tag{2.2} \qquad \blacksquare$$

From the last equation, we can express the shape operator in matrix form:

$$\mathbf{S} = \frac{1}{EG-F^2}\begin{pmatrix} LG-MF & ME-LF \\ MG-NF & NE-MF \end{pmatrix}. \tag{2.3}$$

If we collect the coefficients of the first E, F, G and the second L, M, N fundamental form into a 2×2 symmetric matrix called the first and the second fundamental form matrix,

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \mathcal{F}_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}, \tag{2.4}$$

from the Theorem (2.1) there is the direct Corollary:

Corollary 2.1. *Suppose $EG \neq F^2$, that is $\mathcal{F}_1$ is non-singular. The representative matrix of the shape operator* $\mathbf{S}$ *is*

$$\mathbf{S} = \mathcal{F}_1^{-1}\mathcal{F}_2. \ \blacksquare \tag{2.5}$$

2.2. Normal Curvature

While the shape operator is vector function that measures the bending of a surface, normal curvature is a real-valued function that does the same thing.

Definition 2.2. *[4] Let u_p is the tangent vector of regular surface $S \subset \mathcal{R}^3$ that is $||u_p|| = 1$. Then the **normal curvature** of S in direction u_p will be equal:*

$$k_n(u_p) = \mathbf{S}(u_p) \cdot u_p. \tag{2.6}$$

In general case, if v_p is an arbitrary non-zero tangent vector of S in $p \in S$, then

$$k_n(v_p) = \frac{\mathbf{S}(v_p) \cdot v_p}{||v_p||^2}. \tag{2.7}$$

If we use that for two arbitrary unit tangent vectors v and w in base $\{x_u, x_v\}$:

$$v = v_1 x_u + v_2 x_v, \qquad w = w_1 x_u + w_2 x_v,$$

the scalar product is equal:

$$\begin{aligned} v \cdot w &= v_1 w_1 (x_u \cdot x_u) + v_1 w_2 (x_u \cdot x_v) + v_2 w_1 (x_v \cdot x_u) + v_2 w_2 (x_v \cdot x_v) = \\ &= v_1 w_1 E + v_1 w_2 F + v_2 w_1 F + v_2 w_2 G = \\ &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v^T \mathcal{F}_1 w, \end{aligned}$$

for normal curvatures we obtain:

$$k_n(u_p) = \mathbf{S}(u_p) \cdot u_p = \mathcal{F}_1^{-1} \mathcal{F}_2(u_p) \cdot u_p = (\mathcal{F}_1^{-1} \mathcal{F}_2(u_p))^T \mathcal{F}_1(u_p).$$

Since $(AB)^T = B^T A^T$ for arbitrary matrix A and B, and $\mathcal{F}_1$ and $\mathcal{F}_2$ are symmetric, it is true:

$$\begin{aligned} k_n(u_p) = \mathbf{S}(u_p) \cdot u_p &= (u_p)^T \cdot \mathcal{F}_2^T (\mathcal{F}_1^{-1})^T \mathcal{F}_1(u_p) = (u_p)^T \cdot \mathcal{F}_2 \mathcal{F}_1^{-1} \mathcal{F}_1(u_p) = \\ &= (u_p)^T \cdot \mathcal{F}_2(u_p) = \mathcal{F}_2(u_p) \cdot u_p \,. \end{aligned}$$

At a given point $p \in S$ there is a normal curvature associated with every direction from point p. We can make a clever choice of vector $(u)_s$ to represent all directions:

$$(u(t))_s = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s, \tag{2.8}$$

where $t \in [0, 2\pi)$ and index s presents a vector in standard base $\{x_u, x_v\}$:

$$(x_u)_s = (1, 0), \qquad (x_v)_s = (0, 1).$$

Then, for normal curvature it is true:

$$\begin{aligned} k_n(t) &= \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s = \\ &= \begin{pmatrix} L\cos t + M\sin t \\ M\cos t + N\sin t \end{pmatrix}_s \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s = \\ &= L\cos^2 t + 2M\sin t\cos t + N\sin^2 t. \end{aligned}$$

2.3. Mean and Gaussian Curvature

Gaussian and mean curvatures are the most important curvatures in surface theory.

Definition 2.3. *[4] Gaussian and mean curvatures are the functions $K, H : S \to R$ defined as:*

$$K(p) = det(\mathbf{S}(p)) \tag{2.9}$$

and

$$H(p) = \frac{1}{2} tr(\mathbf{S}(p)). \tag{2.10}$$

In the following Theorem, proved in [4], the Gaussian and mean curvatures are expressed due to coefficients of the first and second fundamental form:

Theorem 2.2. *Let $\mathbf{r} : S \to \mathcal{R}^3$ be a regular surface. Gaussian and mean curvatures of surface $\mathbf{r}$ are equal:*

$$K = \frac{LN - M^2}{EG - F^2}, \tag{2.11}$$

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}, \tag{2.12}$$

where L, M, N are the coefficients of the first, and E, F, G are the coefficients of the second fundamental form of $\mathbf{r}$.

2.4. Principals Curvatures

The principal curvatures measure the maximum and minimum bending of regular surface S at each point $p \in S$.

Definition 2.4. *[4] Let $S \subset \mathcal{R}^3$ be a regular surface and $p \in S$. The maximal and the minimal value of normal curvature k_n of S in point p are called* ***the principal curvatures*** *of S in point p and denote k_1 and k_2.*

The next Theorem, proved in [4], is often used as a definition of Gaussian and mean curvatures:

Theorem 2.3. *Let k_1 and k_2 are the principal curvatures of regular surface $S \subset \mathcal{R}^3$. Gaussian curvature of surface S is in form:*

$$K = k_1 k_2. \tag{2.13}$$

Mean curvature of surface S is in form:

$$H = \frac{1}{2}(k_1 + k_2). \blacksquare \tag{2.14}$$

From the Theorem (2.3) there is the Corollary:

Corollary 2.2. *Principal curvatures k_1 and k_2 are the solutions to the equation:*

$$k^2 - 2Hk + K = 0. \tag{2.15}$$

k_1 and k_2 can be expressed as:

$$k_1 = H + \sqrt{H^2 - K} \text{ and } k_2 = H - \sqrt{H^2 - K}. \blacksquare \tag{2.16}$$

2.5. Willmore Energy at a Surface Point

Another way to characterize the shape of a surface S is to compare S to a sphere. This is what the Willmore energy does. The Willmore energy is a quantitative measure of how much a given surface deviates from a round sphere.

Definition 2.5. *Let H and K be the mean and the Gaussian curvature, respectively, of the surface S. **Willmore energy at a surface point** $p \in S$ is given by:*

$$W(p) = H(p)^2 - K(p). \tag{2.17}$$

The next example presents the visualization of the mean, the Gaussian curvature and the Willmore energy comparable, applying the Mathematica computer program [4], [12].

Example 2.1. *For the Gaudi surface*

$$S : \mathbf{r}(u, v) = (u, v, \frac{1}{2}u \sin \frac{v}{2})$$

(Fig. 2.1) the mean, the Gaussian curvature and the Willmore energy are respectively shown:

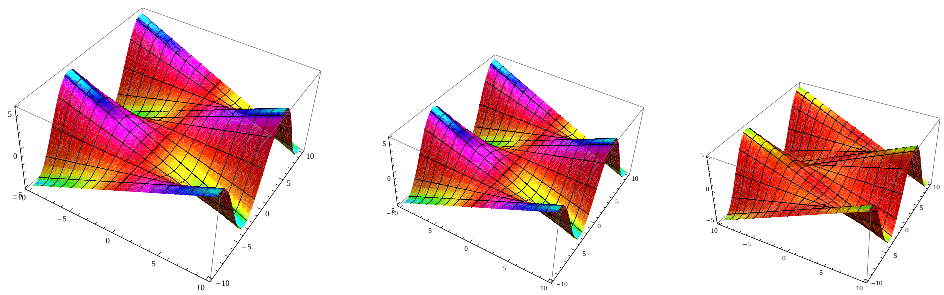

FIG. 2.1: The mean, the Gaussian curvature and the Willmore energy of a Gaudi surface

3. Infinitesimal Bending of a Surface

In the sequel, we will give some basic facts about infinitesimal bending of surfaces according to [9] and [10].

Let a regular surface S of class C^3 be given in the vector form

$$S : \mathbf{r} = \mathbf{r}(u, v),\ (u, v) \in D,\ D \subset \mathcal{R}^2, \tag{3.1}$$

included in the family of surfaces

$$S_\epsilon : \mathbf{r}_\epsilon(u, v, \epsilon) = \mathbf{r}(u, v) + \epsilon \mathbf{z}(u, v), \tag{3.2}$$

where $\epsilon \in (-1, 1)$, $\mathbf{r}_0(u, v, 0) = \mathbf{r}(u, v)$ and $\mathbf{z} \in C^3$ is given field.

Definition 3.1. *The surfaces (3.2) are* ***infinitesimal bending*** *of the surface (3.1) if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon), \tag{3.3}$$

i.e. if the difference of the squares of the line elements of these surfaces is of the order higher then the first.

The field $\mathbf{z}(u, v)$ is ***infinitesimal bending field*** *of the infinitesimal bending.*

Definition 3.2. *Bending field is* ***trivial****, i.e. it is a field of the rigid motion of the surface, if it can be given in the form $\mathbf{z} = \mathbf{a} \times \mathbf{r} + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.*

Definition 3.3. *A surface is* ***rigid*** *if it doesn't allow any bending fields other than the trivial one; otherwise, the surface is* ***flexible****.*

Theorem 3.1. *[3], [10] Necessary and sufficient condition for the surface S_ϵ (3.2) to be infinitesimal bending is*

$$d\mathbf{r} \cdot d\mathbf{z} = 0, \tag{3.4}$$

where $\cdot$ stands for the scalar product in R^3.

The equation (3.4) is equivalent to the next three partial differential equations:

$$\mathbf{r}_u \mathbf{z}_u = 0,\ \mathbf{r}_u \mathbf{z}_v + \mathbf{r}_v \mathbf{z}_u = 0,\ \mathbf{r}_v \mathbf{z}_v = 0. \tag{3.5}$$

4. Variation of Curvatures and Functions of Curvatures Under Infinitesimal Bending of Surface

In case of bendable surfaces it is useful to discuss the variation of magnitudes such as the curvatures of a surface. Variation of geometric magnitudes will be defined according to [8].

Definition 4.1. *Let $\mathcal{A} = \mathcal{A}(u, v)$ be the magnitude that characterizes a geometric property on the surface S and $\tilde{\mathcal{A}}(u, v, \epsilon)$ the corresponding magnitude on the surface S_ϵ being infinitesimal bending of the surface S. Then*

$$\delta\mathcal{A} = \frac{d}{d\epsilon}\tilde{\mathcal{A}}(u, v, \epsilon)\Big|_{\epsilon=0} \tag{4.1}$$

is called **variation** *of the geometric magnitude $\mathcal{A}$ under infinitesimal bending S_ϵ of the surface S.*

The equation (4.1) can be written in equivalent form

$$\delta A = \lim_{\epsilon\to 0}\frac{\Delta A}{\epsilon} = \lim_{\epsilon\to 0}\frac{\tilde{A}(u^1, u^2, \epsilon) - A(u^1, u^2)}{\epsilon}. \tag{4.2}$$

It is easy to prove (see [9])

$$\textbf{a)}\quad \delta(AB) = A\delta B + B\delta A, \qquad \textbf{b)}\quad \delta\Big(\frac{\partial A}{\partial u^i}\Big) = \frac{\partial(\delta A)}{\partial u^i}. \tag{4.3}$$

In [3], [6], [8], [10] it is shown that variations of some geometric magnitudes that depend on coefficients of the first fundamental form of the surface are zero under infinitesimal bending of the surface at $\mathcal{R}^3$. Cristoffel's symbols, the first fundamental form, determinant of the first and the second fundamental form, area of a region on the surface, Gaussian and geodesic curvature are stationary under infinitesimal bending of surface.

Let the surface S be a regular surface, parameterized by

$$\mathbf{r}(u, v) = (u, v, f(u, v)), \tag{4.4}$$

and infinitesimal bending field by

$$\mathbf{z}(u, v) = (\xi(u, v), \eta(u, v), \zeta(u, v)). \tag{4.5}$$

Then the equations (3.5) take the form:

$$\xi_u = -f_u\zeta_u,\ \xi_v + \eta_u = -f_v\zeta_u - f_u\zeta_v,\ \eta_v = -f_v\zeta_v,$$

and we get the partial differential equation of the second order

$$f_{uu}\zeta_{vv} - 2f_{uv}\zeta_{uv} + f_{vv}\zeta_{uu} = 0. \tag{4.6}$$

Using the standard machinery of differential geometry [7] it can get that the coefficients of the first and the second fundamental form (see [2]) of surface S_ϵ.

$$\begin{aligned} S_\epsilon : \tilde{\mathbf{r}}(u,v,\epsilon) &= \mathbf{r}(u,v) + \epsilon \mathbf{z}(u,v) = \\ &= (u + \epsilon\xi(u,v), v + \epsilon\eta(u,v), f(u,v) + \epsilon\zeta(u,v)). \end{aligned}$$

$$\tilde{E} = \tilde{\mathbf{r}}_u \cdot \tilde{\mathbf{r}}_u = 1 + f_u^2 + \epsilon^2(\xi_u^2 + \eta_u^2 + \zeta_u^2), \tag{4.7}$$

$$\tilde{F} = \tilde{\mathbf{r}}_u \cdot \tilde{\mathbf{r}}_v = f_u f_v + \epsilon^2(\xi_u\xi_v + \eta_u\eta_v + \zeta_u\zeta_v), \tag{4.8}$$

$$\tilde{G} = \tilde{\mathbf{r}}_v \cdot \tilde{\mathbf{r}}_v = 1 + f_v^2 + \epsilon^2(\xi_v^2 + \eta_v^2 + \zeta_v^2, \tag{4.9}$$

$$\tilde{L} = \frac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{uu}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \frac{1}{\sqrt{\tilde{g}}}[f_{uu} + \epsilon\zeta_{uu}(1 + f_u^2 + f_v^2) + \epsilon^2 A_1 + \epsilon^3 A_2], \tag{4.10}$$

$$\tilde{M} = \frac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{uv}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \frac{1}{\sqrt{\tilde{g}}}[f_{uv} + \epsilon\zeta_{uv}(1 + f_u^2 + f_v^2) + \epsilon^2 B_1 + \epsilon^3 B_2], \tag{4.11}$$

$$\tilde{N} = \frac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{vv}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \frac{1}{\sqrt{\tilde{g}}}[f_{vv} + \epsilon\zeta_{vv}(1 + f_u^2 + f_v^2) + \epsilon^2 C_1 + \epsilon^3 C_2]. \tag{4.12}$$

Functions A_i, B_i, C_i, $i = 1,2$, are obtained in development of corresponding determinants and

$$\tilde{g} = \tilde{E}\tilde{G} - \tilde{F}^2 = 1 + f_u^2 + f_v^2 + \epsilon^2 \ldots + \epsilon^4 \ldots.$$

From (4.7) it is implied that

$$\tilde{\mathbf{r}}_u = (1 + \epsilon\xi_u, \epsilon\eta_u, f_u + \epsilon\zeta_u), \tag{4.13}$$

$$\tilde{\mathbf{r}}_v = (\epsilon\xi_v, 1 + \epsilon\eta_v, f_v + \epsilon\zeta_v). \tag{4.14}$$

Using (4.1) or (4.2), variation of these vectors will be in the form:

$$\delta\mathbf{r}_u = (\xi_u, \eta_u, \zeta_u) = \mathbf{z}_u, \tag{4.15}$$

$$\delta\mathbf{r}_v = (\xi_v, \eta_v, \zeta_v) = \mathbf{z}_v. \tag{4.16}$$

4.1. Variation of the Shape Operator

Since the shape operator is the function of coefficients of the first and the second fundamental form, we can calculate its variation under infinitesimal bending of the surface. Let us first express the coefficients $\tilde{a}_{ij}$ of the surface S_ϵ.

$$\tilde{a}_{11} = \frac{\tilde{M}\tilde{F} - \tilde{L}\tilde{G}}{\tilde{g}}, \qquad \tilde{a}_{12} = \frac{\tilde{L}\tilde{F} - \tilde{M}\tilde{E}}{\tilde{g}},$$

$$\tilde{a}_{21} = \frac{\tilde{N}\tilde{F} - \tilde{M}\tilde{G}}{\tilde{g}}, \qquad \tilde{a}_{22} = \frac{\tilde{M}\tilde{F} - \tilde{N}\tilde{E}}{\tilde{g}}.$$

Variation of the coefficients a_{ij}, using that

$$\delta a_{ij} = \lim_{\epsilon\to 0}\frac{\tilde{a}_{ij} - a_{ij}}{\epsilon}, \tag{4.17}$$

will be in the form:

$$\delta a_{11} = \frac{\zeta_{uv} f_u f_v - \zeta_{uu}(1+f_v^2)}{\sqrt{1+f_u^2+f_v^2}}, \qquad \delta a_{12} = \frac{\zeta_{uu} f_u f_v - \zeta_{uv}(1+f_u^2)}{\sqrt{1+f_u^2+f_v^2}},$$

$$\delta a_{21} = \frac{\zeta_{vv} f_u f_v - \zeta_{uv}(1+f_v^2)}{\sqrt{1+f_u^2+f_v^2}}, \qquad \delta a_{22} = \frac{\zeta_{uv} f_u f_v - \zeta_{vv}(1+f_u^2)}{\sqrt{1+f_u^2+f_v^2}}.$$

Applying (4.3) under **a)**, we obtain the variation of the shape operator:

Theorem 4.1. *Variation of the shape operator of surface (4.4) under infinitesimal bending (3.2) is given in the form:*

$$\begin{aligned}\delta\mathbf{S}(\mathbf{r}_u) &= \tfrac{1}{\sqrt{g}}\Big((\zeta_{uu}(1+f_v^2) - \zeta_{uv} f_u f_v)\mathbf{r}_u + (\zeta_{uv}(1+f_u^2) - \zeta_{uu} f_u f_v)\mathbf{r}_v\Big) + \\ &\quad + \tfrac{1}{g^{\frac{3}{2}}}\Big((f_{uu}(1+f_v^2) - f_{uv} f_u f_v)\mathbf{z}_u + (f_{uv}(1+f_u^2) - f_{uu} f_u f_v)\mathbf{z}_v\Big),\\ \delta\mathbf{S}(\mathbf{r}_v) &= \tfrac{1}{\sqrt{g}}\Big((\zeta_{uv}(1+f_v^2) - \zeta_{vv} f_u f_v)\mathbf{r}_u + (\zeta_{vv}(1+f_u^2) - \zeta_{uv} f_u f_v)\mathbf{r}_v\Big) + \\ &\quad + \tfrac{1}{g^{\frac{3}{2}}}\Big((f_{uv}(1+f_v^2) - f_{vv} f_u f_v)\mathbf{z}_u + (f_{vv}(1+f_u^2) - f_{uv} f_u f_v)\mathbf{z}_v\Big),\end{aligned}$$

where $g = 1 + f_u^2 + f_v^2$.

From the equation (4.18) follows the next Corollary:

Corollary 4.1. *Variation of the shape operator of surface (4.4) under infinitesimal bending (3.2) will be equal to zero if the second order partial derivatives of functions f and its bending field z are zero, i.e. these functions are linear.*

4.2. Variation of the Normal Curvature

Using the expression (2.9) of normal curvature at a surface point $p \in S$ in direction of vector

$$(u(t))_s = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}_s, \qquad t \in [0, 2\pi),$$

its variation can be calculated via the coefficients of the second fundamental form:

$$k_n(t) = L\cos^2 t + 2M\sin t\cos t + N\sin^2 t.$$

For that purpose, we should first calculate the variation of the coefficients of the second fundamental form.

Lemma 4.1. *Variation of the coefficients of the second fundamental form is given in the expression:*

$$\delta L = \zeta_{uu}\sqrt{g}, \qquad \delta M = \zeta_{uv}\sqrt{g}, \qquad \delta N = \zeta_{vv}\sqrt{g}. \tag{4.18}$$

Proof. Providing that the coefficients of the second fundamental form of a surface (4.7) is given in (4.10), and that variation of magnitudes is determined by (4.2), one can get (4.18).

This implies that the variation of the normal curvature of a surface will be in the form:

$$\delta k_n(t) = \sqrt{g}(\zeta_{uu}\cos^2 t + 2\zeta_{uv}\sin t\cos t + \zeta_{vv}\sin^2 t), \tag{4.19}$$

where $g = 1 + f_u^2 + f_v^2$.

Theorem 4.2. *Variation of the normal curvature of a surface (4.4) under infinitesimal bending (4.7) is given using the equation (4.19).*

4.3. Variation of the Mean and Gaussian Curvature

Since the Gaussian curvature is stationary under infinitesimal bending of a surface [8], [9], only the variation of the mean curvature of a surface will be considered. The mean curvature of a surface (4.7) is in the form:

$$\tilde{H} = \frac{\tilde{E}\tilde{N} - 2\tilde{F}\tilde{M} + \tilde{G}\tilde{L}}{2(\tilde{E}\tilde{G} - \tilde{F}^2)},$$

and the variation of it is given in ([2]):

Theorem 4.3. *The variation of the mean curvature of a surface (4.4) under infinitesimal bending of a surface (4.7) is given with the equation (4.20):*

$$\delta H = \frac{(1 + f_u^2)\zeta_{vv} - 2f_u f_v\zeta_{uv} + (1 + f_v^2)\zeta_{uu}}{2\sqrt{1 + f_u^2 + f_v^2}}. \tag{4.20}$$

4.4. Variation of the Principal Curvatures

Expressing the principal curvatures against the mean and the Gaussian curvatures of a surface, (2.16):

$$k_{1,2} = H \pm \sqrt{H^2 - K},$$

the variation of the principal curvatures will be determined applying the next Lemma:

Lemma 4.2. *The variation of the principal curvatures under infinitesimal bending of a surface, expressed against the mean and the Gaussian curvatures of a surface is equal to:*

$$\delta k_{1,2} = \delta H \frac{\sqrt{H^2 - K} \pm H}{\sqrt{H^2 - K}}. \tag{4.21}$$

Proof. Since the variation of the Gaussian curvature is equal to zero, using the derivative of a square root, we can get:

$$\delta k_{1,2} = \delta H \pm \frac{H \delta H}{\sqrt{H^2 - K}}.$$

The simplifying of the last expression implies (4.21).

If we express the mean and the Gaussian curvatures against the coefficients of the first and the second fundamental form, and apply the variation of the mean curvature, we can get the variation of the principal curvatures, and the next Theorem is valid:

Theorem 4.4. *The variation of the principal curvatures of a surface (4.4) under infinitesimal bending of a surface (4.7) is given with the equation:*

$$\delta k_{1,2} = \delta H \frac{\sqrt{\mathcal{A}} \pm \mathcal{B}}{\sqrt{\mathcal{A}}}, \tag{4.22}$$

where

$$\begin{aligned}
\mathcal{A} &= \Big(f_{uu}(1 + f_v^2) - f_{vv}(1 + f_u^2) \Big)^2 + \\
&+ 4\Big(f_{uv}(1 + f_v^2) - f_{vv} f_u f_v \Big)\Big(f_{uv}(1 + f_u^2) - f_{uu} f_u f_v \Big), \\
\mathcal{B} &= f_{uu}(1 + f_v^2) - 2 f_{uv} f_u f_v + f_{vv}(1 + f_u^2),
\end{aligned}$$

and δH is given with (4.20).

Proof. If we express the mean and the Gaussian curvature against the coefficients of the first and the second fundamental form of a surface in equation (4.21), we can get:

$$\begin{aligned}
\sqrt{H^2 - K} &= \sqrt{\frac{(LG - NE)^2 + 4(MG - FN)(ME - FL)}{4(EG - F^2)^2}} = \\
&= \frac{1}{2(EG - F^2)} \sqrt{(LG - NE)^2 + 4(MG - FN)(ME - FL)}.
\end{aligned}$$

Using that

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)},$$

then, after simplification:

$$\frac{\sqrt{H^2-K}\pm H}{\sqrt{H^2-K}}=\frac{\sqrt{(LG-NE)^2+4(MG-FN)(ME-FL)}\pm(LG-2MF+NE)}{\sqrt{(LG-NE)^2+4(MG-FN)(ME-FL)}}.$$

Expression of the coefficients of the first and the second fundamental form of a surface (4.4) and the variation of the mean curvature (4.20) implies (4.22).

4.5. Variation of the Willmore Energy at a Surface Point

In paper [13] we considered a variation of the Willmore energy at a surface point under infinitesimal bending of the surface.

The Willmore energy at a surface point $p\in S$ is defined as:

$$W(p)=H(p)^2-K(p),$$

and its variation will be in the form:

$$\delta W(p)=2H(p)\delta H(p), \tag{4.23}$$

i.e.the next Lemma is valid:

Lemma 4.3. *The variation of the Willmore energy at a surface point (4.4) under infinitesimal bending of the surface (4.7) is given in the equation (4.23), where $H(p)$ is the mean curvature and $\delta H(p)$ is its variation.*

Expressing H and δH, we can obtain a variation of the Willmore energy at a surface point $p\in S$:

$$\delta W(u,v)= \tag{4.24}$$

$$=\frac{\big((1+f_u^2)f_{vv}-2f_uf_vf_{uv}+(1+f_v^2)f_{uu}\big)\big((1+f_u^2)\zeta_{vv}-2f_uf_v\zeta_{uv}+(1+f_v^2)\zeta_{uu}\big)}{2(1+f_u^2+f_v^2)}.$$

If we denote:

$$\mathbf{a}=\mathbf{a}(u,v)=(1+f_u^2,-\sqrt{2}f_uf_v,1+f_v^2), \tag{4.25}$$

$$\mathbf{b}=\mathbf{b}(u,v)=(f_{vv},\sqrt{2}f_{uv},f_{uu}), \tag{4.26}$$

$$\mathbf{c}=\mathbf{c}(u,v)=(\zeta_{vv},\sqrt{2}\zeta_{uv},\zeta_{uu}), \tag{4.27}$$

using

$$\mathbf{b} \cdot \mathbf{c} = (f_{uu} + f_{vv})(\zeta_{uu} + \zeta_{vv}), \tag{4.28}$$

and

$$\|\mathbf{a}\| = \sqrt{1 + g^2}, \tag{4.29}$$

where $g = 1 + f_u^2 + f_v^2$ is the determinant of the first fundamental form of surface (4.4), then follows the next Theorem:

Theorem 4.5. *The variation of the Willmore energy at a surface point (4.4) under infinitesimal bending of the surface (4.7) is given in the equation:*

$$\delta W(u, v) = \frac{(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})}{2(\|\mathbf{a}\|^2 - 1)}, \tag{4.30}$$

where **a**, **b** *and* **c** *are defined in (4.25), (4.26) and (4.27).*

5. Conclusion

Since the shape is an important feature of objects and can be immensely useful in characterizing objects, analytic gauges and visualization options can be applied to surfaces, providing immediate feedback by measuring deviation, curvature, rate of curvature changes, and surface smoothness during geometry manipulation. In shape analysis it is useful to discus the variation of magnitudes that characterize the shape itself.

In the above discussion, we found the variation of the shape operator, the normal curvature, the mean curvature, the principal curvatures and the Willmore energy at a surface point under infinitesimal bending of a surface given in an explicit form.

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