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GENERALIZATION OF TITCHMARCH'S THEOREM FOR THE JACOBI TRANSFORM

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Abstract. In this paper, we prove the generalization of Titchmarsh's theorem for the Jacobi transform for functions satisfying the k-Jacobi Lipschitz condition in the space $L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt)$.

Keywords: Titchmarsh's theorem, Jacobi Lipschitz condition, Jacobi Transform, Lipschitz condition.

1. Introduction and preliminaries

The Jacobi transform can be regarded as a generalization of the Helgason-Fourier transform of the radial function on the Riemannian symmetric spaces of the noncompact type and rank one. For more details about this transform see [1, 6, 7].

In this paper, we prove the generalization of Titchmarsh's theorem for the Jacobi transform for functions satisfying the k-Jacobi Lipschitz condition in the space $L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt)$. For this purpose, we use the generalized translation operator. This operator is one of the most important generalized translations on the half-line $\mathbb{R}^+ = [0, \infty)$ (see [5]).

In [3] we proved an analog of Titchmarsh's theorem for the Jacobi transform in the space $L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt)$ for functions satisfying the Jacobi-Lipschitz condition. In this paper, we prove the generalization of this result for functions satisfying the k-Jacobi Lipschitz class ($k \in \{1, 2, ..\}$).

In this section, we collect some basic facts of the Jacobi transform. More about the Jacobi transform can be found in [7].

Let

$$(a)_0 = 1,\ (a)_m = a(a+1)...(a+m-1).$$

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The Gaussian hypergeometric function is defined by

$$F(a,b,c,z)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}z^m,\ |z|<1,$$

where $a,b,z\in\mathbb{C}$ and $c\notin -\mathbb{N}$.

The function $z\longrightarrow F(a,b,c,z)$ is a unique solution to the differential equation

$$z(1-z)u''(z)+(c-(a+b+1)z)u'(z)-abu(z)=0,$$

which is regular in 0 and equals 1 there.

The Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ is defined by

$$\varphi_\lambda(t)=\varphi_\lambda^{(\alpha,\beta)}(t)=\mathrm{F}(\frac{1}{2}(\rho-i\lambda),\frac{1}{2}(\rho+i\lambda),\alpha+1,-\sinh^2 t),$$

where $\alpha\geq-\frac{1}{2}$, $\alpha>\beta\geq-\frac{1}{2}$ and $\rho=\alpha+\beta+1$.

The Jacobi operator

$$\mathrm{D}=\mathrm{D}_{\alpha,\beta}=\frac{d^2}{dt^2}+((2\alpha+1)\coth t+(2\beta+1)\tanh t)\frac{d}{dt}.$$

By means of which the Jacobi function φ_λ may alternatively be characterized as a unique solution to

$$\mathrm{D}\varphi+(\lambda^2+\rho^2)\varphi=0,$$

on $\mathbb{R}^+$ satisfying

$$\varphi_\lambda(0)=1,\ \ \varphi_\lambda'(0)=0$$

and $\lambda\longrightarrow\varphi_\lambda(t)$ is analytic for all $t\geq 0$.

Lemma 1.1. *The following inequalities are valid for a Jacobi function* $\varphi_\lambda(t),(\lambda,t\in\mathbb{R}^+)$:

1. $|\varphi_\lambda(t)|\leq 1$
2. $1-\varphi_\lambda(t)\leq t^2(\lambda^2+\rho^2)$
3. *there is a constant* $c>0$ *such that*

$$1-\varphi_\lambda(t)\geq c$$

for $\lambda t\geq 1$

Proof. Analog of Lemmas 3.1-3.2 in [8] □

Consider the Hilbert space

$$\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+) = \mathrm{L}^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt)$$

with the norm

$$\|f\| = \|f\|_{2,(\alpha,\beta)} = \left(\int_0^\infty |f(x)|^2 \Delta_{\alpha,\beta}(x)dx\right)^{1/2},$$

where

$$\Delta_{\alpha,\beta}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}.$$

For a function $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$, the Jacobi transform is defined by the formula

$$\widehat{f}(\lambda) = \int_0^\infty f(t)\varphi_\lambda(t)\Delta_{\alpha,\beta}(t)dt.$$

The inversion formula (cf. [7])

$$f(t) = \frac{1}{2\pi}\int_0^\infty \widehat{f}(\lambda)\varphi_\lambda(t)d\mu(\lambda),$$

where $d\mu(\lambda) = |C(\lambda)|^{-2}d\lambda$ and the C-function $C(\lambda)$ is defined by

$$C(\lambda) = \frac{2^\rho\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)},$$

where $\alpha \geq -\frac{1}{2}$ and $\alpha > \beta \geq -\frac{1}{2}$.

From [7], the Plancherel formula for the Jacobi transform is written as

$$\|f\| = \|f\|_{\mathrm{L}^2(\mathbb{R}^+,\Delta_{\alpha,\beta}(t)dt)} = \|\widehat{f}\|_{\mathrm{L}^2(\mathbb{R}^+,\frac{1}{2\pi}d\mu(\lambda))}.$$

We have

$$\widehat{(\mathrm{D}f)}(\lambda) = -(\lambda^2+\rho^2)\widehat{f}(\lambda). \tag{1.1}$$

The generalized translation operator was defined by Flensted-Jensen and Koornwinder ([4], Formula (5.1)) given by

$$\mathrm{T}_h f(x) = \int_0^\infty f(z)K(x,h,z)\Delta_{\alpha,\beta}(z)dz,$$

where K is an explicitly known kernel function such that

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)K(x,y,z)\Delta_{\alpha,\beta}(z)dz,$$

with the kernel

$$\begin{aligned} K(x,y,z) &= \frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x\cosh y\cosh z)^{\alpha-\beta-1}}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})(\sinh x\sinh y\sinh z)^{2\alpha}}(1-B^2)^{\alpha-\frac{1}{2}} \\ &\times \quad F((\alpha+\beta),\alpha-\beta,\alpha+\frac{1}{2},\frac{1}{2}(1-B)) \end{aligned}$$

for $|x-y|<z<x+y$ and $K(x,y,z)=0$ elsewhere and

$$B = \frac{\cosh^2 x+\cosh^2 y+\cosh^2 z-1}{2\cosh x\cosh y\cosh z}$$

In [2], we have

$$\widehat{(\mathrm{T}_h f)}(\lambda) = \varphi_\lambda(h)\widehat{f}(\lambda). \tag{1.2}$$

The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = \mathrm{T}_h f(x) - f(x) = (\mathrm{T}_h - \mathrm{E})f(x),$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1}f(x)) = (\mathrm{T}_h-\mathrm{E})^k f(x) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\mathrm{T}_h^i f(x), \tag{1.3}$$

where

$$\mathrm{T}_h^0 f(x) = f(x),\ \mathrm{T}_h^i f(x) = \mathrm{T}_h(\mathrm{T}_h^{i-1}f(x)),\ (i=1,2,...,k;\ k=1,2,....)$$

and E is a unit operator in $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$.

Let $\mathrm{W}^k_{2,\alpha,\beta}$ be the Sobolev space constructed by the Jacobi operator D, i.e.,

$$\mathrm{W}^k_{2,\alpha,\beta} = \{f\in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+);\ \mathrm{D}^j f\in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+),\ j=1,2,...,k\}.$$

In [3], we have the following result:

Theorem 1.1. *Let $\delta\in(0,1)$ and assume that $f\in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$. Then the following are equivalents*

1. $\|\mathrm{T}_h f(x) - f(x)\|_{\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)} = O(h^\delta)$ *as* $h\longrightarrow 0$,
2. $\int_r^\infty |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(r^{-2\delta})$ *as* $r\longrightarrow +\infty$.

The main aim of this paper is to establish an analog of Theorem 1.1.

2. Main Results

In this section we present the main result of this paper. We first need to define the k-Jacobi Lipschitz class.

Definition 2.1. Let $\delta \in (0,1)$. A function $f \in W^k_{2,\alpha,\beta}$ is said to be in the k-Jacobi Lipschitz class, denoted by $Lip(\delta, 2, k, r)$, if

$$\|\Delta_h^k D^r f(x)\| = O(h^\delta) \text{ as } h \longrightarrow 0,$$

where $r = 0, 1, ..., k$.

Lemma 2.1. *Let $f \in W^k_{2,\alpha,\beta}$. Then*

$$\|\Delta_h^k D^r f(x)\|^2 = \int_0^\infty (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda(h)|^{2k} |\widehat{f}(\lambda)|^2 d\mu(\lambda),$$

where $r = 0, 1, ..., k$.

Proof. From formula (1.1), we obtain

$$(\widehat{D^r f})(\lambda) = (-1)^r (\lambda^2+\rho^2)^r \widehat{f}(\lambda). \tag{2.1}$$

From formulas (1.2) and (2.1), we conclude that

$$(\widehat{T_h^i D^r f})(\lambda) = (-1)^r (\lambda^2+\rho^2)^r \varphi_\lambda^i(h) \widehat{f}(\lambda) \tag{2.2}$$

From formulas (1.3) and (2.2) follows that the Jacobi transform of

$$\Delta_h^k D^r f(x)$$

is $(-1)^r(\lambda^2+\rho^2)^r(1-\varphi_\lambda(h))^k \widehat{f}(\lambda)$.

By Parseval's identity we have the result □

Theorem 2.1. *Let $f \in W^k_{2,\alpha,\beta}$. Then the following are equivalents*

1. $f \in Lip(\delta, 2, k, r)$,
2. $\int_s^\infty (\lambda^2+\rho^2)^{2r} |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(s^{-2\delta})$ *as* $s \longrightarrow +\infty$.

Proof. 1) $\Longrightarrow$ 2) Assume that $f \in Lip(\delta, 2, k, r)$. Then we have

$$\|\Delta_h^k \mathrm{D}^r f(x)\| = O(h^\delta) \ as\ h \longrightarrow 0.$$

From Lemma 2.1, we have

$$\|\Delta_h^k \mathrm{D}^r f(x)\|^2 = \int_0^\infty (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda(h)|^{2k} |\widehat{f}(\lambda)|^2 d\mu(\lambda).$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $\lambda h \geq 1$ and (3) of Lemma 1.1 implies that

$$1 \leq \frac{1}{c^{2k}} |1-\varphi_\lambda(h)|^{2k}.$$

Then

$$\begin{aligned}
\int_{1/h}^{2/h} (\lambda^2+\rho^2)^{2r} |\widehat{f}(\lambda)|^2 d\mu(\lambda) &\leq \frac{1}{c^{2k}} \int_{1/h}^{2/h} (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda(h)|^{2k} |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq \frac{1}{c^{2k}} \int_0^\infty (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda(h)|^{2k} |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&= O(h^{2\delta}).
\end{aligned}$$

Therefore

$$\int_s^{2s} (\lambda^2+\rho^2)^{2r} |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(s^{-2\delta}) \ as\ s \longrightarrow +\infty \tag{2.3}$$

It follows from (2.3) that

$$\int_s^{2s} (\lambda^2+\rho^2)^{2r} |\widehat{f}(\lambda)|^2 d\mu(\lambda) \leq c_1 s^{-2\delta},$$

where $c_1 > 0$ is some constant. From this inequality we obtain

$$\begin{aligned}
\int_s^\infty (\lambda^2+\rho^2)^{2r} |\widehat{f}(\lambda)|^2 d\mu(\lambda) &= \sum_{j=0}^\infty \int_{2^j s}^{2^{j+1}s} (\lambda^2+\rho^2)^{2r} |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq c_1 \sum_{j=0}^\infty (2^j s)^{-2\delta} \\
&\leq c_2 s^{-2\delta}.
\end{aligned}$$

This proves that

$$\int_s^\infty (\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(s^{-2\delta}) \text{ as } s \longrightarrow +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_s^\infty (\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(s^{-2\delta}) \text{ as } s \longrightarrow +\infty.$$

We have to show that

$$\int_0^\infty (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(h^{2\delta}) \text{ as } h \longrightarrow 0.$$

We write

$$\int_0^\infty (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2 d\mu(\lambda) = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{1/h} (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2 d\mu(\lambda)$$

and

$$\mathrm{I}_2 = \int_{1/h}^\infty (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2 d\mu(\lambda)$$

Estimate the summands I_1 and I_2 from above. It follows from (1) of Lemma 1.1 that

$$\mathrm{I}_2 \leq 4^k \int_{1/h}^\infty (\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(h^{2\delta}).$$

To estimate I_1, we use the inequalities (1.1) and (2) of Lemma 1.1

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/h} (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq 2^{2k} \int_0^{1/h} (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)||\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq 4^k h^2 \int_0^{1/h} (\lambda^2+\rho^2)^{2r}(\lambda^2+\rho^2)|\widehat{f}(\lambda)|^2 d\mu(\lambda) = \mathrm{I}_3 + \mathrm{I}_4,
\end{aligned}$$

where

$$\mathrm{I}_3 = 4^k\rho^2h^2\int_0^{1/h}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2d\mu(\lambda)$$

and

$$\mathrm{I}_4 = 4^kh^2\int_0^{1/h}\lambda^2(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2d\mu(\lambda)$$

Note that

$$\begin{aligned}\mathrm{I}_3 &\leq 4^k\rho^2h^2\int_0^{\infty}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2d\mu(\lambda)\\ &= 4^k\rho^2h^2\|\mathrm{D}^r f(x)\|^2 = O(h^{2\delta}),\end{aligned}$$

since $2\delta < 2$.

For a while, we put

$$\psi(s) = \int_s^{\infty}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2d\mu(\lambda).$$

Using integration by parts, we find that

$$\begin{aligned}\mathrm{I}_4 &= 4^kh^2\int_0^{1/h}(-s^2\psi'(s))ds\\ &= 4^kh^2\left(-\frac{1}{h^2}\psi(\frac{1}{h})+2\int_0^{1/h}s\psi(s)ds\right)\\ &= -4^k\psi(\frac{1}{h})+2\times 4^kh^2\int_0^{1/h}s\psi(s)ds.\end{aligned}$$

Since $\psi(s) = O(s^{-2\delta})$, we have $s\psi(s) = O(s^{1-2\delta})$ and

$$\int_0^{1/h}s\psi(s)ds = O\left(\int_0^{1/h}s^{1-2\delta}ds\right) = O(h^{2\delta-2}).$$

Then

$$\mathrm{I}_4 \leq 2\times 4^kC_3h^2h^{2\delta-2}.$$

Finally

$$\mathrm{I}_4 = O(h^{2\delta}),$$

which completes the proof of the theorem $\square$

Corollary 2.1. *Let* $f \in W^k_{2,\alpha,\beta}$, *and let*

$$f \in Lip(\delta, 2, k, r).$$

Then

$$\int_s^\infty |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(s^{-4r-2\delta}) \text{ as } s \longrightarrow +\infty.$$

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