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ON UNIQUE RANGE SET OF MEROMORPHIC FUNCTIONS WITH DEFICIENT POLES

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Abstract. With the aid of the notion of weighted sharing of sets we deal with the problem of Unique Range Sets for meromorphic functions and obtain a result which improves and extends some previous results. We exhibit two examples to show that a condition in one of our results is the best possible.

Keywords: Meromorphic function, deficient pole, root, complex number.

1. Introduction Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \qquad (r \longrightarrow \infty, r \notin E).$$

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, $r \notin E$.

We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [5]. For $a \in \mathbb{C} \cup \{\infty\}$, we define

$$\Theta(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have

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the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z : f(z)=a\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity, the set $\bigcup_{a\in S}\{z : f(z)=a\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$, we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values. First of all, we recall the following definitions.

A set $S\subset\mathbb{C}$ is called a unique range set for meromorphic functions (URSM), if for any two non-constant meromorphic functions f and g the condition $E_f(S)=E_g(S)$ implies $f\equiv g$. If this holds merely for entire functions, the set S will be termed as a unique range set for entire functions (URSE).

In 1926, R. Nevanlinna showed that a meromorphic function on the complex plane $\mathbb{C}$ is uniquely determined by the images, ignoring multiplicities, of 5 distinct values. A few years later, he showed that when multiplicities are considered, 4 points are sufficient (with one exceptional situation). A related problem, asked by Gross [4], is to find a finite set S so that an entire function is determined by the single pre-image, counting multiplicities of S.

The first example of URS for entire functions was found by F. Gross and C. C. Yang in 1982, that is

$$S=\{z\in\mathbb{C} : e^z+z=0\}.$$

Note that as S is an infinite set, the above result does not answer the question of Gross. Since then, there have been many efforts to study the problem of constructing unique range sets. The problem of determining a meromorphic (or entire) function on $\mathbb{C}$ by its single pre-images, counting with multiplicities, of finite sets cause an increasing interest among the researchers and naturally it has been investigated by many mathematicians. There are two main problems related to the study of unique range sets. The first problem is determining the minimum cardinality of a unique range set for entire and also for meromorphic functions. The second problem is characterizing unique range sets.

H. Fujimoto [3] first made a major contribution by highlighting a special property of monic polynomial of degree n with simple zeros which generates a U.R.S. So to study the behaviour of those monic polynomials which does not satisfy Fujimoto's condition will be of increasing interest. For these polynomials the zero sets will also form a U.R.S. provided we suppose some extra supposition on deficiency condition. In fact, examples of unique range sets given by most authors are sets of the form $\{z\in\mathbb{C} : z^n+az^r+b=0\}$ under suitable conditions on the constants a and b and on the positive integers n and r.

Answer to the question of Gross [4] and the analogous question for meromorphic functions on $\mathbb{C}$ were given by Yi [15] and Li and Yang [13, 14] who investigated the zero sets of polynomials P of the form

$$P(z) = z^n + az^{n-m} + b,$$

where $n > m \geq 1$ and a and b are so chosen so that P has n distinct roots. Addressing the question of Gross, in 1995 Yi [15] and independently Li-Yang [13] proved the following result for entire functions.

Theorem A. Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If f and g are two non-constant entire functions satisfying $E_f(S) = E_g(S)$ then $f \equiv g$.

Naturally one will be inquisitive about the case of meromorphic functions in the above theorem. In this direction in 1996 Yi proved the following theorem which also deals with the question of Gross.

Theorem B. [16] Let $S = \{z : z^n + az^{n-m} + b = 0\}$, where m, n are two positive integers such that m and n have no common factor, $n > 2m + 8$ $(m \geq 2)$ and a, b are nonzero constants such that the algebraic equation $z^n + az^{n-m} + b = 0$ has no multiple root. Then $E_f(S) = E_g(S)$ implies $f \equiv g$.

It is clear from the above theorem that a URS of meromorphic function of the form as given in *Theorem B* consists of atleast 13 elements. So far, the smallest unique range set for meromorphic functions has 11 elements and was given by Frank and Reinders in [2]. We note that in [16] and [2] both polynomials whose zeros generate the U.R.S. satisfies Fujimoto's condition.

In [16] Yi asked the following question:

What can be said if $m = 1$ in *Theorem B*?

In connection to his question Yi [16] proved the following theorem.

Theorem C. [16] Let

$$S = \{z : z^n + az^{n-1} + b = 0\},$$

where n (≥ 11) is an integer, a and b are two nonzero constants such that the algebraic equation $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g are non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ then either $f \equiv g$ or

$$f = -\frac{ah(h^{n-1} - 1)}{h^n - 1},\ g = -\frac{a(h^{n-1} - 1)}{h^n - 1},$$

where $h = \frac{f}{g}$.

In the meantime Fang and Hua [1] extended *Theorem A* to meromorphic functions with the help of some additional conditions imposing on the ramification indexes of f and g. Fang and Hua [1] proved the following theorem.

Theorem D. [1] Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If two meromorphic functions f and g are such that $\Theta(\infty; f) > \frac{11}{12}$, $\Theta(\infty; g) > \frac{11}{12}$ and $E_f(S) = E_g(S)$ then $f \equiv g$.

To proceed further we require the following definition known as weighted sharing of sets and values which renders a useful tool for the purpose of relaxation of the nature of sharing the sets.

Definition 1.1. [7, 8] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k. We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [7] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Improving *Theorem D*, Lahiri [9] proved the following theorem.

Theorem E. [9] Let S be defined as in *Theorem D*. If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > \frac{3}{2}$ and $E_f(S, 2) = E_g(S, 2)$ then $f \equiv g$.

In 2004 Lahiri-Banerjee [10] further improved *Theorem C* in a more compact and convenient way and obtained the following result.

Theorem F. [10] Let

$$S = \{z : z^n + az^{n-1} + b = 0\},$$

where n (≥ 9) be an integer and a, b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If $E_f(S, 2) = E_g(S, 2)$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$ then $f \equiv g$.

We now give the following example which establishes the fact that the set S in *Theorems E-F* cannot be replaced by any arbitrary set containing six distinct elements.

Example 1.1. Let

$$f(z) = \sqrt{\alpha\beta\gamma}\, e^{z}, \; g(z) = \sqrt{\alpha\beta\gamma}\, e^{-z}$$

and

$$S = \{\alpha\sqrt{\beta}, \alpha\sqrt{\gamma}, \beta\sqrt{\alpha}, \beta\sqrt{\gamma}, \gamma\sqrt{\alpha}, \gamma\sqrt{\beta}\},$$

where α, β and γ are three nonzero distinct complex numbers. Clearly $E_f(S, \infty) = E_g(S, \infty)$ but $f \not\equiv g$.

So it remains an open problem for investigations whether the degree of the equation defining S in Theorem F can be reduced to six and at the same time the conditions over ramification indexes can be further weakened. In the paper we are taking up this problem and provide a solution in this regard. Actually the purpose of the paper is to continue the investigations of further improvement and extensions of theorems E- F. The following theorem is the main result of the paper.

Theorem 1.1. Let

$$S = \{z : z^n + az^{n-1} + b = 0\},$$

where n (≥ 6) be an integer and a, b be two nonzero constants such that

$$z^n + az^{n-1} + b = 0$$

has no multiple root. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S, m) = E_g(S, m)$. If

(i) $m \geq 2$ and $\Theta_f + \Theta_g > \max\{\frac{10-n}{2}, \frac{4}{n-1}\}$

(ii) or if $m = 1$ and $\Theta_f + \Theta_g > \max\{\frac{11-n}{2}, \frac{4}{n-1}\}$

(iii) or if $m = 0$ and $\Theta_f + \Theta_g > \max\{\frac{16-n}{3}, \frac{4}{n-1}\}$

then $f \equiv g$, where $\Theta_f = \Theta(0; f) + \Theta(\infty; f)$ and Θ_g can be similarly defined.

The following examples show that the condition $\Theta_f + \Theta_g > \frac{4}{n-1}$ is sharp in *Theorem 1.1.* when $n \geq 9$ and $m \geq 2$.

Example 1.2. {[10], *Example 2*} Let

$$f = -a\frac{1 - h^{n-1}}{1 - h^n}, \ \ g = -ah\frac{1 - h^{n-1}}{1 - h^n},$$

where

$$h = \frac{\alpha^2(e^z - 1)}{e^z - \alpha}, \ \alpha = exp(\frac{2\pi i}{n})$$

and $n(\geq 3)$ is an integer.

Then $T(r, f) = (n-1)T(r, h) + O(1)$ and $T(r, g) = (n-1)T(r, h) + O(1)$ and $T(r, h) = T(r, e^z) + O(1)$. Further we see that $h \neq \alpha, \alpha^2$ and so for any complex number $\gamma \neq \alpha, \alpha^2$, $\overline{N}(r, \gamma; h) \sim T(r, h)$. We also note that a root of $h = 1$ is not a pole and zero of f and g. Hence $\Theta(\infty; f) = \Theta(\infty; g) = \frac{2}{n-1}$. On the other hand

$$\Theta(0, f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\sum_{k=1}^{n-2} \overline{N}(r, \beta^k; h) + \overline{N}(r, \infty; h)}{(n-1)T(r, h) + O(1)} = 0$$

and

$$\Theta(0, g) = 1 - \limsup_{r\longrightarrow\infty} \frac{\sum_{k=1}^{n-2} \overline{N}(r, \beta^k; h) + \overline{N}(r, 0; h)}{(n-1)T(r, h) + O(1)} = 0,$$

where $\beta = exp\ (\frac{2\pi i}{n-1})$. Clearly $E_f(S, \infty) = E_g(S, \infty)$ because $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$ but $f \not\equiv g$.

Example 1.3. Let f and g be given as in *Example 1.2*, where

$$h = \frac{\alpha(\alpha e^z - 1)}{e^z - 1}, \ \ \alpha = exp(\frac{2\pi i}{n})$$

and $n(\geq 3)$ is an integer.

We now explain some definitions and notations which are used in the paper.

Definition 1.3. [6] For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by

$$N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$$

the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also

$$N(r, a; f \mid< m), N(r, a; f \mid> m), \overline{N}(r, a; f \mid< m) \text{ and } \overline{N}(r, a; f \mid> m).$$

are defined analogously.

Definition 1.4. [17] Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the reduced counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$, by $\overline{N}_E^{(2}(r, a; f)$ the reduced counting function of those a-points of f and g where $p = q \geq 2$. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$. In a similar manner we can define $\overline{N}_L(r, a; f)$ and $\overline{N}_L(r, a; g)$ for $a \in \mathbb{C} \cup \{\infty\}$.

When f and g share (a, m), $m \geq 1$ then $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

Definition 1.5. We denote by $\overline{N}(r, a; f \mid= k)$ the reduced counting function of those a-points of f whose multiplicities are exactly k, where $k \geq 2$ is an integer.

Definition 1.6. [7, 8] Let f, g share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows:

$$F = \frac{f^{n-1}(f + a)}{-b}, G = \frac{g^{n-1}(g + a)}{-b}. \tag{2.1}$$

Henceforth we shall denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.2}$$

Lemma 2.1. [12] Let f be a non-constant meromorphic function and let

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.2. [17] If F, G are two non-constant meromorphic functions such that they share (1,0) and $H \not\equiv 0$ then

$$N_E^{1)}(r, 1; F \mid= 1) = N_E^{1)}(r, 1; G \mid= 1) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Lemma 2.3. Let

$$S = \{z : z^n + az^{n-1} + b = 0\},$$

where a, b are nonzero constants such that

$$z^n + az^{n-1} + b = 0$$

has no repeated root, n (≥ 3) is an integer and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S, 0) = E_g(S, 0)$ and $H \not\equiv 0$ then

$$\begin{aligned} & N(r, \infty; H) \\ \leq\ & \overline{N}(r, 0, f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; nf + a(n-1)) \\ & + \overline{N}(r, 0; ng + a(n-1)) + \overline{N}_*(r, 1; F, G) + \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g'), \end{aligned}$$

where $\overline{N}_0(r, 0; f')$ is the reduced counting function of those zeros of f' which are not the zeros of f and $(F-1)$ and $\overline{N}_0(r, 0; g')$ is similarly defined.

Proof. Since $E_f(S, 0) = E_g(S, 0)$ it follows that F and G share $(1, 0)$. We have from (2.1) that

$$F' = [nf + (n-1)a]\, f^{n-2} f' / (-b)$$

and

$$G' = [ng + (n-1)a]\, g^{n-2} g' / (-b).$$

We can easily verify that possible poles of H occur at

(i) zeros of f and g,

(ii) zeros of $nf + a(n-1)$ and $ng + a(n-1)$,

(iii) poles of f and g,

(iv) those 1-points of F and G with different multiplicities,

(v) zeros of $f^{'}$ which are not the zeros of $f(F-1)$, (v) zeros of $g^{'}$ which are not zeros of $g(G-1)$. Since H has only simple poles, the lemma follows from above. This proves the lemma.

Lemma 2.4. ([10], Lemma 1) Let f, g be two nonconstant meromorphic functions. Then $f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b$, where a, b are nonzero finite constants and $n\ (\geq 5)$ is an integer.

Lemma 2.5. Let f, g be two non-constant meromorphic functions such that

$$\Theta(0; f) + \Theta(\infty; f) + \Theta(0; g) + \Theta(\infty; g) > \frac{4}{n-1}$$

then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where $n\ (\geq 3)$ is an integer and a is a nonzero finite constant.

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a) \tag{2.3}$$

and suppose $f \not\equiv g$. We consider two cases:

Case I. Let $y = \frac{g}{f}$ be a constant. Then from (2.3) it follows that $y \neq 1$, $y^{n-1} \neq 1$, $y^{n} \neq 1$ and $f \equiv -a\frac{1-y^{n-1}}{1-y^{n}}$, a constant, which is impossible.

Case II. Let $y = \frac{g}{f}$ be non-constant. Then

$$f \equiv -a\frac{1-y^{n-1}}{1-y^{n}} \equiv a\left(\frac{y^{n-1}}{1+y+y^{2}+\ldots+y^{n-1}} - 1\right). \tag{2.4}$$

From (2.4) we see by *Lemma 2.1* that

$$\begin{aligned} T(r,f) &= T(r, \sum_{j=0}^{n-1}\frac{1}{y^{j}}) + O(1) \\ &= (n-1)\ T(r, \frac{1}{y}) + S(r,y) \\ &= (n-1)\ T(r,y) + S(r,y). \end{aligned}$$

We first note that the zeros of $1 + y + y^2 + \ldots + y^{n-2}$ contribute to the zeros of both f and g. In addition to this, the poles of y contribute to the zeros of f and since $g = fy$ the zeros of y contribute to the zeros of g. So from (2.4) we see that

$$\sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \overline{N}(r, \infty; y) \leq \overline{N}(r, 0; f), \sum_{k=1}^{n-1} \overline{N}(r, u_k; y) \leq \overline{N}(r, \infty; f)$$

where

$$u_k = \exp\left(\frac{2k\pi i}{n}\right),\ k = 1, 2, \ldots, n-1$$
$$v_j = \exp\left(\frac{2j\pi i}{n-1}\right)\ j = 1, 2, \ldots, n-2.$$

By the second fundamental theorem we get

$$\begin{aligned}
& (2n-4)\ T(r, y) \\
\leq\ & \overline{N}(r, \infty; y) + \sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \sum_{k=1}^{n-1} \overline{N}(r, u_k; y) + S(r, y) \\
\leq\ & \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, y) \\
\leq\ & (2 - \Theta(0; f) - \Theta(\infty; f) + \varepsilon)\ T(r, f) + S(r, y) \\
=\ & (n-1)\ (2 - \Theta(0; f) - \Theta(\infty; f) + \varepsilon)\ T(r, y) + S(r, y)
\end{aligned}$$

i.e.,

$$\frac{2n-4}{n-1} T(r, y) \leq (2 - \Theta(0; f) - \Theta(\infty; f) + \varepsilon)\ T(r, y) + S(r, y), \tag{2.5}$$

where $0 < 2\varepsilon < \Theta(0; f) + \Theta(\infty; f) + \Theta(0; g) + \Theta(\infty; g)$.

Again noting that

$$\sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \overline{N}(r, 0; y) \leq \overline{N}(r, 0; g),$$

by the second fundamental theorem we get

$$\begin{aligned}
& (2n-3)\ T(r, y) \\
\leq\ & \overline{N}(r, \infty; y) + \overline{N}(r, 0; y) + \sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \sum_{k=1}^{n-1} \overline{N}(r, u_k; y) + S(r, y) \\
\leq\ & \overline{N}(r, \infty; y) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + S(r, y) \\
\leq\ & \overline{N}(r, \infty; y) + (n-1)\,(2 - \Theta(0; g) - \Theta(\infty; g) + \varepsilon)\ T(r, y) + S(r, y)
\end{aligned}$$

i.e.,

$$\frac{2n-4}{n-1} T(r, y) \leq (2 - \Theta(0; g) - \Theta(\infty; g) + \varepsilon)\ T(r, y) + S(r, y), \tag{2.6}$$

Adding (2.5) and (2.6) we get

$$\left(\frac{4n-8}{n-1}-4+\Theta(0;f)+\Theta(\infty;f)+\Theta(0;g)+\Theta(\infty;g)-2\varepsilon\right)T(r,y)\leq S(r,y),$$

which is a contradiction.

Hence $f\equiv g$ and this proves the lemma.

Lemma 2.6. [11] If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then

$$N(r,0;f^{(k)}\mid f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\geq k)+S(r,f).$$

3. Proof Proof of Theorem 1.1.

We know from the assumption that the zeros of $z^n+az^{n-1}+b$ are simple and we denote them by w_j, $j=1,2,\ldots n$. Let F, G and H be given by (2.1) and (2.2). Since $E_f(S,m)=E_g(S,m)$ it follows that F, G share $(1,m)$.

Case 1. If possible let us suppose that $H\not\equiv 0$.

Subcase 1.1. $m\geq 1$. While $m\geq 2$, using Lemma 2.6 we note that

$$\begin{aligned}
&\overline{N}_0(r,0;g')+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)\\
\leq\;&\overline{N}_0(r,0;g')+\overline{N}(r,1;G\mid\geq 2)+\overline{N}(r,1;G\mid\geq 3)\\
\leq\;&\overline{N}_0(r,0;g')+\sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g\mid=2)+2\overline{N}(r,\omega_j;g\mid\geq 3)\}\\
\leq\;&N(r,0;g'\mid g\neq 0)+S(r,g)\leq\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+S(r,g).
\end{aligned}\tag{3.1}$$

Hence using (3.1), *Lemmas 2.2* and *2.3* we get from second fundamental theorem for $\varepsilon>0$ that

$$\begin{aligned}
&nT(r,f)\\
\leq\;&\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2)-N_0(r,0;f')\\
&+S(r,f)\\
\leq\;&2\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;nf+a(n-1))\\
&+\overline{N}(r,0;ng+a(n-1))+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;g')\\
&+S(r,f)+S(r,g)\\
\leq\;&2\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right\}+T(r,f)+T(r,g)\\
&+S(r,f)+S(r,g)\\
\leq\;&(10-2\Theta(0;f)-2\Theta(\infty;f)-2\Theta(0;g)-2\Theta(\infty;g)+\varepsilon)\,T(r)+S(r).
\end{aligned}\tag{3.2}$$

In a similar way we can obtain

$$\begin{aligned}(3.3)\qquad & n\,T(r,g)\\ \leq\ & (10-2\Theta(0;f)-2\Theta(\infty;f)-2\Theta(0;g)-2\Theta(\infty;g)+\varepsilon)\,T(r)+S(r).\end{aligned}$$

Combining (3.2) and (3.3) we see that

$$(3.4)\qquad (n-10+2\Theta(0;f)+2\Theta(\infty;f)+2\Theta(0;g)+2\Theta(\infty;g)-\varepsilon)\,T(r)\leq S(r).$$

Since $\varepsilon>0$, (3.4) leads to a contradiction.

While $m=1$, using *Lemma 2.6*, (3.1) changes to

$$\begin{aligned}(3.5)\qquad & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)\\ \leq\ & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_L(r,1;G)+\overline{N}(r,1;F\mid\geq 3)\\ \leq\ & \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\frac{1}{2}\sum_{j=1}^{n}\{N(r,\omega_j;f)-\overline{N}(r,\omega_j;f)\}\\ \leq\ & \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g).\end{aligned}$$

So using (3.5), *Lemmas 2.2* and *2.3* and proceeding as in (3.2) we get from second fundamental theorem for $\varepsilon>0$ that

$$\begin{aligned}(3.6)\qquad & nT(r,f)\\ \leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right\}+\frac{1}{2}\{\overline{N}(r,0;f)\\ & +\overline{N}(r,\infty;f)\}+T(r,f)+T(r,g)+S(r,f)+S(r,g)\\ \leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right\}+2T(r,f)+T(r,g)\\ & +S(r,f)+S(r,g)\\ \leq\ & (11-2\Theta(0;f)-2\Theta(0;g)-2\Theta(\infty;f)-2\Theta(\infty;g)+\varepsilon)\,T(r)+S(r).\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}(3.7)\qquad & nT(r,g)\\ \leq\ & (11-2\Theta(0;f)-2\Theta(0;g)-2\Theta(\infty;f)-2\Theta(\infty;g)+\varepsilon)\,T(r)+S(r).\end{aligned}$$

Combining (3.6) and (3.7) we see that

$$(3.8)\qquad (n-11+2\Theta(0;f)+2\Theta(\infty;f)+2\Theta(0;g)+2\Theta(\infty;g)-\varepsilon)\ T(r)\leq S(r).$$

Since $\varepsilon>0$, (3.8) leads to a contradiction.

Subcase 1.2. $m=0$. Using *Lemma 2.6* we note that

$$
\begin{aligned}
(3.9)\qquad & \overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\le\;& \overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\le\;& \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\ge 2)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\le\;& N(r,0;g^{'}\mid g\neq 0)+\overline{N}(r,1;G\mid\ge 2)+2\overline{N}(r,1;F\mid\ge 2)\\
\le\;& 2\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g)
\end{aligned}
$$

Hence using (3.9), *Lemmas 2.2* and *2.3* we get from the second fundamental theorem for $\varepsilon>0$ that

$$
\begin{aligned}
(3.10)\qquad & n\;T(r,f)\\
\le\;& \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\
&+\overline{N}_E^{(2}(r,1;F)-N_0(r,0;f^{'})+S(r,f)\\
\le\;& 2\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+T(r,f)+T(r,g)\\
&+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g)\\
\le\;& 4\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+3\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\}+T(r,f)+T(r,g)\\
&+S(r,f)+S(r,g)\\
\le\;& (16-3\Theta(0;f)-3\Theta(\infty;f)-3\Theta(0;g)-3\Theta(\infty;g)+\varepsilon)\;T(r)+S(r).
\end{aligned}
$$

In a similar manner we can obtain

$$
\begin{aligned}
(3.11)\qquad & n\;T(r,g)\\
\le\;& (16-3\Theta(0;f)-3\Theta(\infty;f)-3\Theta(0;g)-3\Theta(\infty;g)+\varepsilon)\;T(r)+S(r).
\end{aligned}
$$

Combining (3.10) and (3.11) we see that

$$
(3.12)\qquad (n-16+3\Theta(0;f)+3\Theta(\infty;f)+3\Theta(0;g)+3\Theta(\infty;g)-\varepsilon)\;T(r)\le S(r).
$$

Since $\varepsilon>0$, (3.12) leads to a contradiction.

Case 2. $H\equiv 0$. On integration we get from (2.2)

$$
(3.13)\qquad \frac{1}{F-1}\equiv\frac{A}{G-1}+B,
$$

where A, B are constants and $A\neq 0$. From (3.13) we obtain

$$
(3.14)\qquad F\equiv\frac{(B+1)G+A-B-1}{BG+A-B}.
$$

Clearly (3.14) together with *Lemma 2.1* yields

$$
(3.15)\qquad T(r,f)=T(r,g)+O(1).
$$

Subcase 2.1. Suppose that $B \neq 0, -1$.

If $A - B - 1 \neq 0$, from (3.14) we obtain

$$\overline{N}(r, \frac{B+1-A}{B+1}; G) = \overline{N}(r, 0; F).$$

From above, *Lemma 2.1* and the second fundamental theorem we obtain

$$\begin{aligned} nT(r,g) &< \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{B+1-A}{B+1}; G) + S(r,g) \\ &\leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,0;g+a) + \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + S(r,g) \\ &\leq 2T(r,f) + 3T(r,g) + S(r,g), \end{aligned}$$

which in view of (3.15) implies a contradiction as $n \geq 6$. Thus $A - B - 1 = 0$ and hence (3.14) reduces to

$$F \equiv \frac{(B+1)G}{BG+1}.$$

From this we have

$$\overline{N}(r, \frac{-1}{B}; G) = \overline{N}(r, \infty; f).$$

Again by Lemma 2.1 and the second fundamental theorem we have

$$\begin{aligned} nT(r,g) &< \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{-1}{B}; G) + S(r,g) \\ &\leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,0;g+a) + \overline{N}(r,\infty;f) + S(r,g\delta) \\ &\leq T(r,f) + 3T(r,g) + S(r,g), \end{aligned}$$

which in view of (3.15) leads to a contradiction since $n \geq 6$.
Subcase 2.2. Suppose that $B = -1$.
From (3.14) we obtain

$$F \equiv \frac{A}{-G+A+1}. \tag{3.16}$$

If $A + 1 \neq 0$, from (3.16) we obtain

$$\overline{N}(r, A+1; G) = \overline{N}(r, \infty; f).$$

So using the same argument as used in the above subcase we can again obtain a contradiction. Hence $A + 1 = 0$ and we have from (3.16) that $FG \equiv 1$ that means $f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2$, which is impossible by *Lemma 2.4.*

Subcase 2.3. Suppose that $B = 0$.
From (3.14) we obtain

$$F \equiv \frac{G+A-1}{A}. \tag{3.17}$$

If $A - 1 \neq 0$, from (3.17) we obtain

$$\overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F).$$

So in the same manner as above we again get a contradiction. So $A = 1$ and hence $F \equiv G$, that is $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$. Now the theorem follows from *Lemma 2.5*.

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REFERENCES

1. M. Fang and X. Hua, *Meromorphic functions that share one finite set CM*. J. Nanjing Univ. Math. Biquarterly, **15** (1)(1998), 15-22.
2. G. Frank and M. Reinders, *A unique range set for meromorphic functions with 11 elements*. Complex Var. Theory Appl. **37** (1)(1998), 185-193.
3. H.Fujimoto, *On uniqueness of meromorphic functions sharing finite sets*. Amer.J.Math., **122** (2000),1175-1203.
4. F. Gross, *Factorization of meromorphic functions and some open problems*. Proc. Conf. Univ. Kentucky, Leixngton, Kentucky(1976); Lecture Notes in Math., 599(1977), 51-69, Springer(Berlin).
5. W.K. Hayman, *Meromorphic Functions*. The Clarendon Press, Oxford (1964).
6. I. Lahiri, *Value distribution of certain differential polynomials*. Int. J. Math. Math. Sci., **28** (2) (2001), 83-91.
7. I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*. Nagoya Math. J., **161** (2001), 193-206.
8. I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*. Complex Variables, **46** (2001), 241-253.
9. I. Lahiri, *A question of gross and weighted sharing of a finite set by meromorphic functions*. Applied Math. E-Notes, **2** (2002), 16-21
10. I. Lahiri and A. Banerjee, *Uniqueness of meromorphic functions with deficient poles*. Kyungpook Math. J., **44** (2004), 575-584.
11. I. Lahiri and S. Dewan, *Value distribution of the product of a meromorphic function and its derivative*. Kodai Math. J., **26** (2003), 95-100.
12. A.Z. Mohon'ko, *On the Nevanlinna characteristics of some meromorphic functions*. Theory of Funct. Funct. Anal. Appl., **14** (1971), 83-87.
13. P. Li and C.C. Yang, *Some further results on the unique range sets of meromorphic functions*. Kodai Math. J., **13** (1995), 437-450.
14. P. Li and C.C. Yang, *On the unique range sets for meromorphic functions*. Proc. Amer. Math. Soc., **124** (1996), 177-185.

15. H.X. Yi, *A question of Gross and the uniqueness of entire functions*. Nagoya Math. J., **138** (1995), 169-177.

16. H.X. Yi, *Unicity theorems for meromorphic or entire functions III*. Bull. Austral. Math. Soc., **53** (1996), 71-82.

17. H.X. Yi, *Meromorphic functions that share one or two values II*. Kodai Math. J., **22** (1999), 264-272.

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