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A NONLINEAR WEIGHTS SELECTION IN WEIGHTED SUM FOR CONVEX MULTIOBJECTIVE OPTIMIZATION

Abimbola M. Jubril

Abstract. The weighted sum method of vector objective scalarization is known to generate points on convex Pareto front whose distribution cannot be controlled. This work presents a method of improving the distribution of Pareto points generated by weighted sum method by nonlinear weight selection. Numerical examples are presented to show the effectiveness of the method.

1. Introduction

Most real life engineering design problems usually involve optimizing more than one objective. These objectives are generally conflicting. All the objectives cannot be solved for their minimum values simultaneously, so a compromise has to be reached. This is the nature of the multiobjective optimization problems. Since such optimization problems involve more than one objective, the objective function is expressed as a vector and the problem becomes a vector optimization or a multiobjective problem (MOP). Such problems can be expressed as:

$$(1.1)\qquad \begin{array}{ll} \min_{x\in X} & f(x) = [f_1(x), f_2(x), \ldots, f_p(x)] \\ \text{s.t.} & X = \{x \in \mathbf{R}^n : g(x) \leqslant 0, h(x) = 0\} \end{array}$$

where X is the feasible region in the decision space.

The vector optimization problem is generally solved by reducing it to a scalar optimization problem. This involves an aggregation of the components of the vector objective function into a single objective function. This process maps the objective space onto a real line. This scalar optimization problem is expected to be equivalent to the vector optimization problem. Different scalarization methods have been reported in literature. Some of which includes the weighted sum method [26], ε-constraints method [7], hierarchical approach [25], weighted metrics methods [12], goal attainment method [11].

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The various methods of generating solutions to MOP can be classified into two approaches, namely: *a priori articulation* of preference, and *a posteriori* articulation of preference. The *a priori* articulation of preference involves quantitative definition of preference function in the form of value or utility function. This completely orders the objective space. This allows the resulting scalar objective function to be solved for a single Pareto optimal solution that represents the preference of the decision maker. The *a posteriori* articulation of preference involves generating a set of Pareto optimal solutions by solving a parameterized scalar objective function formed from components of the vector objective function. By varying the parameter of the resulting scalar objective, a set of solutions from which a preferable choice can be made is generated. Although the parameter may not give a direct translation of preferences, it captures well the relative importance of the objectives between one another.

This paper focuses on the most common scalarization method, the weighted sum (WS) method. Although this method can be used with both approaches, its use with *a posteriori* articulation of preference is considered in this paper. It involves a linear or convex combination of the objectives $f_i(x)$, $i = 1, \ldots, p$. Each of the objective $f_i(x)$ is multiplied by a normalized weight factor w_i and the product added to give the scalar objective $\phi(x, w_i)$ as:

$$\phi(x, w) = \sum_{i=1}^{p} w_i f_i(x) \tag{1.2}$$

where p is the number of the objectives, $\sum_{i=1}^{p} w_i = 1$ and $w_i > 0, i = 1, \ldots, p$.

This scalar objective optimization problem has been shown to be equivalent to the vector optimization problem [26]. The WS method is the commonly used scalarization method because of its simplicity, ease of use, and direct translation of weight into the relative importance of the objectives [14]. Its drawbacks are also well-known and discussed in literature [3]. These include the followings:

- It misses solution points on the non-convex part of the Pareto surface;
- Its diversity can not be controlled, therefore even distribution of weights does not translate to uniform distribution of the solution points;
- The distribution of solution points is highly dependent on the relative scaling of the objective.

Generally, two important properties are used in assessing the performance of multiobjective algorithms, namely: *convergence* and *diversity* properties. The ideal Pareto front is usually not known, therefore, every Pareto front generated by any algorithm is considered an estimate of the ideal front. The convergence property of an algorithm measures the relative closeness of a generated solution set to the ideal Pareto front while the diversity property measures the *extent* of the coverage, and how *uniformly* the solutions are distributed [5]. The two properties are rarely satisfied simultaneously by any algorithm that solves multiobjective optimization problem. In spite of these drawbacks, the WS method is known to have very good convergence characteristic. It has been shown in [26, 14, 21, 17] that the scalar formulation (1.2) of the vector optimization problem (1.1) is both necessary and sufficient condition for Pareto optimality for convex MOP. The WS method therefore continues to provide single solution that reflects the preferences represented by a set of weights (when used with *a priori articulation* of preference), and to provide multiple solution points, with *a posteriori articulation* of preference.

Various efforts have been directed at analyzing, understanding and removing these drawbacks. The problem of solution dependency on the relative scaling of the function objectives is generally handled by the process of normalization. Various normalization methods have presented in literature (see [13] and references therein).

In solving its problem of poor diversity of solution points, it was observed in [5, 14] that the uneven distribution is due to nonlinear relationship between weights and the objective functions. This results in clustering of solution points in the objective space. In [3], a form of relationship between objective functions in which an even distribution of the solution points may be generated by an even distribution of the weights was derived. This relationship is very complex and only very few Pareto fronts fit into this form. Some other methods which are not based on the WS method that overcome this demerit have been developed [4].

The necessary conditions for any multiobjective algorithm to capture points on the Pareto surface (convex or non-convex) are studied and given in [15]. This has been related to the curvature of the scalar objective function relative to that of Pareto surface, and determined by the Hessian of the function of the difference between the Pareto surface and scalar objective. It was also noted that the ability of the resulting aggregate function objective to capture points on the Pareto surface is attributed to the ease with which the curvature of the scalar objective can be varied. This was observed not possible with the WS objective. In [20], the WS method with trust region algorithm was developed with the weights adaptively determined in a black-box simulation optimization context. This was able to find some points on the nonconvex Pareto front with good convergence. However, many of the points generated are dominated points. Another effort was presented in [10]. This reduces the problem into sub-problems. Each sub-problems consists of a small segment of the feasible region in the objective space. The bounds defining these segments introduce new constraints within which the candidate Pareto solution point is searched.

Other efforts have also been directed at understanding how selection of the weights can lead to its better performance. In this direction, a quasi-random weighted criteria method was developed in [2]. The method generates weights using a quasi-random sequences that covers a hypervolume evenly and efficiently, and consequently covers the Pareto set evenly. This method is stochastic and involves a large number of computations. In [6], nonlinear selection of weights to determine an optimal point that is not close to any of the extreme points was considered. Some of the recent efforts in this direction are those presented in [18, 22]. In [18], the weight surface, partitioned into sub-simplices, are mapped to the objective function space

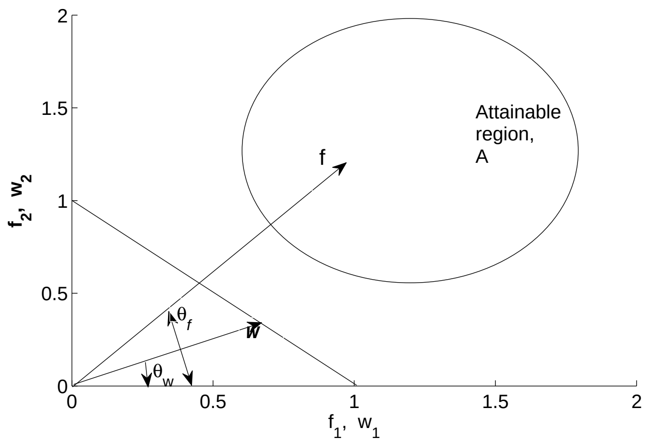


FIG. 2.1: Weighted sum method representation

through the scalarization function. Using branch and bound method, the largest simplex is determined and further divided into sub-simplices. This is continues until the whole space is searched. A symbolic algebra method of selecting the weights was presented in [22, 23].

Most of the research effort at improving the performance of the WS method can be summarized as introduction and control of curvature of the aggregate objective function by raising the power of the component objective functions (e.g. introduction of trust region algorithm, global criterion method, weighted compromise method, exponential weighted method) [20, 15, 1]. In this paper, the nonlinear weight selection in [6] is extended and generalized to provide a means of varying the curvature of the weight surface while the scalar objective is still linear in the component objective functions and the weight constraints are also satisfied. The Pareto front in our consideration is assumed convex.

2. Problem Statement and Main Result

2.1. Linear weights

In this section, the standard WS method formulation and some of the factors responsible for its poor distribution of Pareto points are highlighted. Let the attainable set of objectives in the objective space be denoted by

$$\mathcal{A} = \{f_1(x), \dots, f_p(x) : g(x) \leqslant 0, h(x) = 0\},$$

is as shown in Fig. 2.1. Consider the weight vector $w = (w_1, \dots, w_p)^T \in \mathbf{R}^p$, the vector objective function $f(x) = (f_1(x), \dots, f_p(x))^T \in \mathbf{R}^p$, and the map $\phi(f, w)$:

$\mathbf{R}^p \times \mathbf{R}^p \mapsto \mathbf{R}$. The WS method derives the scalar objective $\phi(x, w)$, through a convex combination of the objectives $f_i(x)$, $i = 1, \ldots, p$. Thus, with p number of the objectives, the equivalent scalar objective $\phi(f, w)$ is given as

$$\begin{aligned} \phi(f, w) &= \sum_{i=1}^{p} w_i f_i(x) \\ &= w^T f(x) \end{aligned} \tag{2.1}$$

and

$$\sum_{i=1}^{p} w_i = 1,\ w_i \geqslant 0, i = 1, \ldots, p. \tag{2.2}$$

This transforms the vector optimization to a scalar form:

$$\begin{array}{ll} \min & \phi(f, w) \\ \text{s.t.} & x \in X \end{array} \tag{2.3}$$

This process maps the p-dimensional objective space onto the positive real line $\mathbf{R}$, and all the non-dominated points are mapped to the same point on the real line. This transformation completely orders the objective function space. Also, the pre-image of the set of Pareto optimal points forms the isoperformance surface in the decision space (i.e. points of same scalar objective value).

More specifically, consider the bi-objective problem with $p = 2$; equations (2.1) and (2.2), respectively, reduce to

$$\phi(f, w) = w_1 f_1(x) + w_2 f_2(x) \tag{2.4}$$

and

$$w_1 + w_2 = 1,\ w_1, w_2 \geqslant 0. \tag{2.5}$$

If the vectors are expressed in polar coordinate forms as shown in Fig. 2.1, (2.4) can be written as

$$\phi(f, w) = |w||f| \cos\theta \tag{2.6}$$

where $f(x) = |f|\angle\theta_f$, $w = |w|\angle\theta_w$ and $\theta = \theta_f - \theta_w$. The minimum points (equivalently, the Pareto points) of the map can be found at points where $|\theta|$ takes its maximum, i.e., $|\theta_{max}| = \pi/2$. Note that for every weight vector w, θ_w and $|w|$ are determined, and the scalar optimization problem reduces to minimizing $f(x)$ in the direction $\theta_f = \theta_w \pm \pi/2$. If the weight vector is parameterized by λ, such that $w_1 = \lambda$ and $w_2 = 1 - \lambda$, then the slope of w is given as

$$\tan\theta_w = \frac{w_2}{w_1} = \frac{1-\lambda}{\lambda} \tag{2.7}$$

and the slope sensitivity as

$$\frac{d}{d\lambda} \tan\theta_w = -\frac{1}{\lambda^2}. \tag{2.8}$$

It would be observed that a change in the value of λ from 0 to 0.1 results in a change in the value of slope from ∞ to 9.0. This implies that slope values between

∞ and 9.0 are missed; therefore, all solution points in that section of the trade-off curve will not be captured. The slope sensitivity can be considered as the ability of the scalarization process to capture solution points on the Pareto surface. For values of λ close to 0, the slope is very sensitive to very small change in λ and for values of λ close to 1, the sensitivity is close to 1. This was noted in [16] to be the source of deficiencies of linear weight selection. This results in the omission of Pareto points in the first, and the clustering of the points in the second. This motivates the idea of reducing these effects by controlling the slope sensitivity.

2.2. Improving distribution through Non-linear Weight Selection

In this section, the effects of the curvature of the weight space on the selectivity of the solution points are analyzed. In Section 2.1., it was observed that the weight vector in the standard WS method is constrained along the line define by the simplex $w_2 = 1 - w_1$ or the hyperplane $Z = \{w|\mathbf{1}^T w = 1\}$, where $\mathbf{1} \in \mathbf{R}^2$ is a vector of all ones. This hyperplane does not have any curvature. Parameterizing the weight vector by λ, such that $w_1 = \lambda$, as λ varies from 0 to 1, $|w|$ varies between 1 and $1/\sqrt{2}$.

However, if the weight vector is constrained to the unit sphere as [6]

$$|w|^2 = w_1^2 + w_2^2 = 1, \text{ then,} \tag{2.9}$$

$$\phi(f, w) = |f| \cos(\theta_f - \theta_w) \tag{2.10}$$

reduces the problem to finding the minimum value of $f(x)$ in the direction of w. The slope becomes

$$\tan\theta_w = \frac{\sqrt{1-\lambda^2}}{\lambda} \tag{2.11}$$

and the slope sensitivity can be written as

$$\frac{d}{d\lambda}\tan\theta_w = -\frac{1}{\lambda^2}\underbrace{\frac{1}{\sqrt{1-\lambda^2}}}_{u}. \tag{2.12}$$

A slight improvement over the standard WS method can be observed when the sensitivities are compared. This is due to the factor u in (2.12), and more solution points can be found. It can be noted by this formulation that a curvature is introduced into the weight space, but the curvature cannot be controlled. Also, the weight constraint, $w_1 + w_2 = 1$, is not satisfied, except at the extreme points; hence, it cannot be considered as a WS method.

However, setting $w_1 = \lambda^2$ and $w_2 = 1 - \lambda^2$ will constrain λ to a unit circle while satisfying the weight constraint. The slope and its sensitivity can be written as

$$\tan\theta_w = \frac{1-\lambda^2}{\lambda^2} \tag{2.13}$$

and

$$\frac{d}{d\lambda}\tan\theta_w = -\frac{1}{\lambda^2}\frac{2}{\lambda}, \tag{2.14}$$

respectively. The slope sensitivity is improved by the factor $\frac{2}{\lambda}$.

To provide control of the curvature of the weight surface and consequently that of the slope sensitivity, a weight selection such that λ is constrained on a hypersurface is proposed. Additional variables $n \in \mathbf{N}$ and $0 < k_j \in \mathbf{R}$, $j = 1, \ldots, p$, which can be manipulated are also introduced.

One of the merits of these additional parameters is the increased degree of freedom for the decision-maker (DM) to explore the Pareto surface. It can also be observed that the an even variation of λ is presented to the DM, this variation is transformed by the relative values of these parameters into a nonlinear one that tries to match the trade-off surface so that better distribution of the Pareto points can be achieved. However, since the shape of the Pareto surface is not known a priori, the parameters will have to be tuned to achieve optimal performance. Increment in the values of both parameters would be observed to lead to reduction of the step length of the transformed weight variation, therefore both parameters should not be tuned simultaneously.

For p number of objectives, the weight hypersurface Z_n is defined as

$$Z_n = \left\{ w : \sum_{j=1}^{p} w_j = 1; w_j = (\lambda_j/k_j)^n \right\} \tag{2.15}$$

This approach is similar to that considered for the weighted compromise method in [1]. The difference is that the constraint $\sum_{j=1}^{p} w_j = 1$ is here still maintained. This ensures a linear relationship between the slope of the Pareto surface and the ratio of the weights

$$\frac{df_j}{df_i} = \frac{w_i}{w_j} \tag{2.16}$$

For clear illustration, consider a bi-objective case, $p = 2$. Setting $\lambda_1 = \lambda$ and $k_1 = 1$ in (2.15), and writing other λ_j, $j \neq 1$ in terms of λ, the weight space Z_2 is given as

$$\begin{aligned} Z_2 &= \left\{ w : w_1 + w_2 = 1; w_j = \frac{\lambda_j^2}{k_j^2}; \right\} \\ &= \left\{ \lambda_1, \lambda_2 : \frac{\lambda_1^2}{k_1^2} + \frac{\lambda_2^2}{k_2^2} = 1 \right\} \\ &= \left\{ \lambda_1, \lambda_2 : \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 1 \right\} \end{aligned} \tag{2.17}$$

which describes an ellipsoid whose major and minor axes are k_1 and k_2. The slope becomes

$$\tan\theta_w = \frac{k_2^2 - \lambda^2}{\lambda^2} \tag{2.18}$$

and the slope sensitivity is given by

$$\frac{d}{d\lambda}\tan\theta_w = -2\frac{k_2^2}{\lambda^3}. \tag{2.19}$$

This nonlinear weight selection provides control of the slope sensitivity through parameter k_2 and n. The additional free parameters can be manipulated in the search of the solution points, and facilitate the control of the slope of the weight factor such that clustered points can be spread out. There is also a reduction in the loss of computational effort of the method.

2.3. Normalization

The WS method of scalarization maps the vector objective space onto the positive real line. It thus requires that the each of the non-dominated solution points be mapped to the same point on the real line. This implies that the scalar objective $\phi(x, w)$ takes its minimum value for all non dominated solution. The WS method does not ensure this when any of the objective function value largely dominates the other(s). This also happens if the feasible objective space $\mathcal{A}$ does not touch the orthant containing it. This implies that the feasible region does not extends to touch the vertical axis in the objective function space. In this state, the weight vector along the horizontal cannot be orthogonal to any $f(x)$ in the feasible region. To avoid this, a scaling of the attainable region, or axes transformation, is necessary. This is called normalization. In the following considerations let $f_{i_{max}}$ and $f_{i_{min}}$ be the maximum and the minimum values of $f_i(x)$ on the Pareto surface. The common normalization methods in the literature include the following. The lower-bound normalization approach consists of dividing each of the component objective function by the minimum attainable value of that functions. This may come in different forms such as [13]

$$f_{i_{nom}}(x) = \frac{f_i(x)}{|f_{i_{min}}|} \tag{2.20}$$

or,

$$f_{i_{nom}}(x) = \frac{f_i(x) - f_{i_{min}}}{|f_{i_{min}}|} \tag{2.21}$$

Apart from the fact that the upper value of the function is left unbounded, the approach can lead to computational difficulty if the denominator is close to zero [13]. Another normalization approach is the upper bound approach. This divides each of the component objective function by its maximum attainable value $f_{i_{max}}$ [19] as

$$f_{i_{nom}}(x) = \frac{f_i(x)}{f_{i_{max}}} \tag{2.22}$$

This sets a bound on the maximum value of the function but no restriction on the lower value. The upper-lower bound normalization approach on the other hand provides both lower and upper bound for the normalized function. It is expressed in the form

$$f_{i_{nom}}(x) = \frac{f_i(x) - f_{i_{min}}}{f_{i_{max}} - f_{i_{min}}}, i = 1, 2, \ldots, p \tag{2.23}$$

This is achieved by dividing the objectives by their respective extreme values on the Pareto surface. In this normalization, the values of each of the objective lies between 0 and 1, and the weighting factors also varies between 0 and 1, this ensures that the resulting scalar objective to have a constant value for all points on the Pareto surface. Equation (1.2) becomes

$$\phi_{nom}(x, w) = \sum_{i=1}^{p} w_i f_{inom}(x) \tag{2.24}$$

The normalization also ensures accurate modeling and equivalence of both the vector and the scalarized form of the problem. For example, consider the extreme points of the Pareto front defined by the weight vectors $w = (1, 0)$ and $w = (0, 1)$. For the upper-lower bound normalization, $\phi_{nom}(x, w)|_{w=(1,0)} = f_{1_{min}} = \phi_{nom}(x, w)|_{w=(0,1)} = f_{2_{min}} = 0$. This does not hold for the other normalization methods. Normalization is also known to improve the uniform distribution of the Pareto points [1].

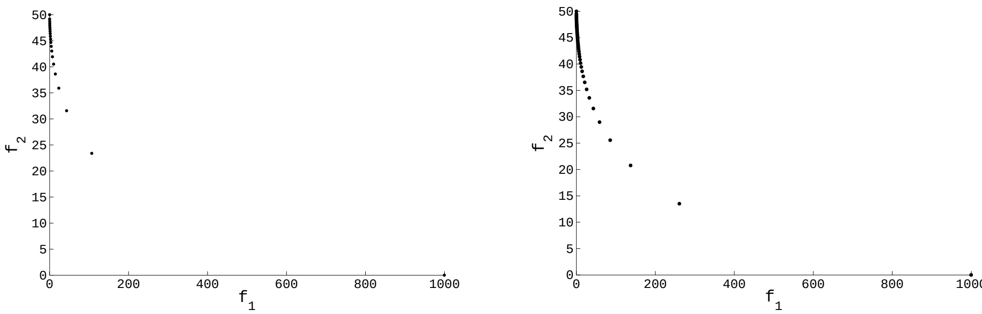
3. Numerical Example

In this section, the evaluation of the effectiveness of the proposed weight selection method is considered. This involves investigating the effect of variations of parameters: k_2 and n, on the distribution of the solution points generated. Only convex polynomial optimization problems (POP) are considered in the following examples. The multiobjective problem is solved by reducing it to a scalar form using the WS method of aggregation. The resulting scalar POP is solved for weight increment of 0.05 between 0 and 1. The resulting POP were solved with Gloptipoly, a freely available MATLAB software that implements POP solution algorithm based on the theory of moments [8]. It solves the resulting SDP using SeDuMi as the default solver [24] and gives the global optimal value, and the global optimizer. Two examples consisting of a two-objective and a three-objective MOPs are considered to illustrate the effectiveness of the method.

3.1. Example 1

Consider the two-objective optimization problem

$$\begin{array}{ll} \min & \begin{bmatrix} f_1(x) = 30x_1^3 + 30x_2^2 \\ f_2(x) = (x_1 - 5)^2 + (x_2 - 5)^2 \end{bmatrix} \\ \text{s.t.} & (x_1 - 5)^2 + x_2^2 - 5^2 \leqslant 0 \\ & (x_1 - 8)^2 + (x_2 + 3)^2 \geqslant 7.7 \end{array} \tag{3.1}$$



(a) $k_2 = 1$, $n = 1$ (21 points) (b) $k_2 = 1$, $n = 1$ (51 points)

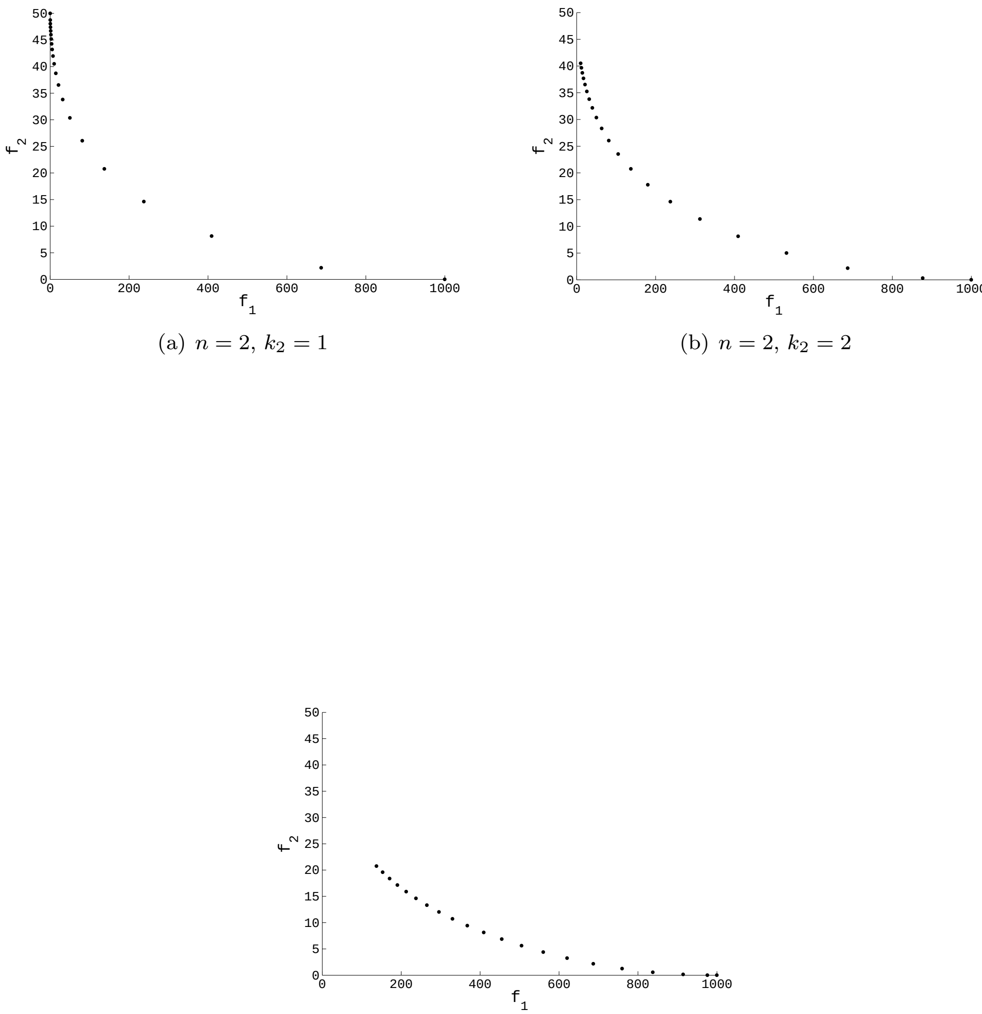
FIG. 3.1: Pareto front using Standard Weighted Sum with even increment in λ

In this example, the standard WS with reduced weight increment in λ is investigated. Also, the effects of variation of k_2 with constant n, and variation of n with constant k_2 will also be considered.

In the first case, the standard WS was used. This was achieved by setting $k_2 = 1$ and $n = 1$ in (2.15), reducing the non-linear weight selection method to the standard WS method. Observe that the only variation allowed in the WS method is the increment in the value of λ from 0 to 1. The MOP was solved first with weight increment of 0.05 to give 21 runs, and then weight increment of 0.02 to give 51 runs. This was to see the effects of increments reduction on the distribution of the solution points. The generated Pareto points for 21 runs and 51 runs are as shown in Fig. 3.1. It would be observed that though reducing this increment increases the number of points, it does not remove the clustering of the solution points.

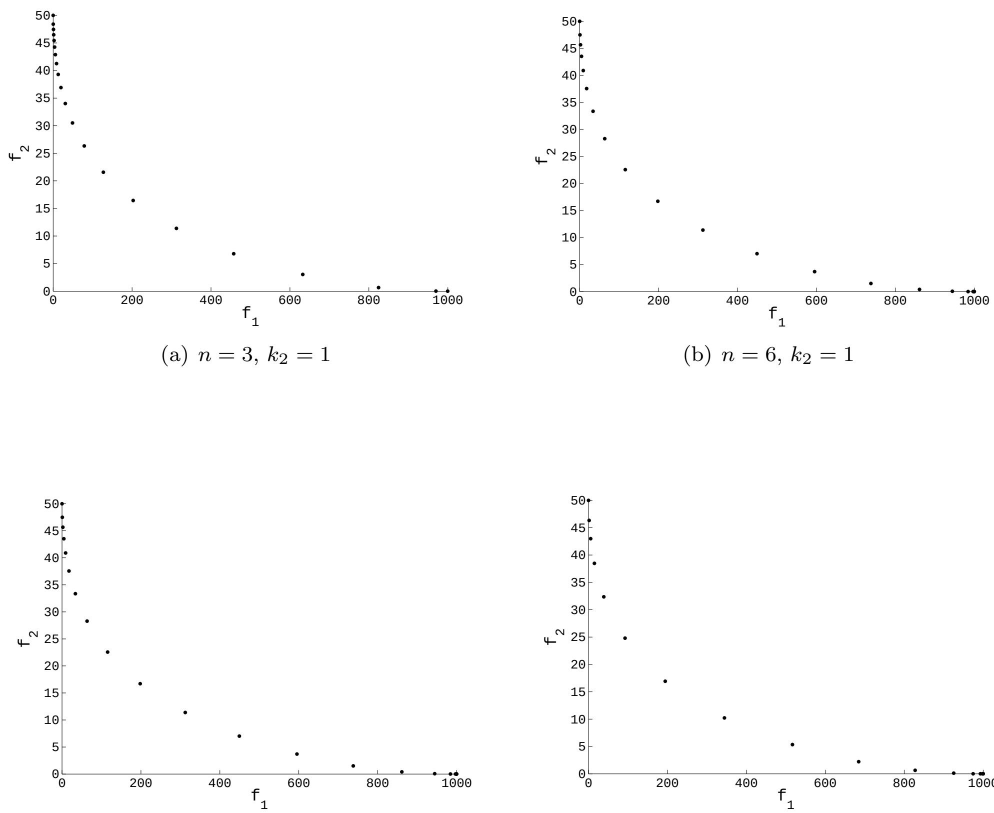
For the second case, the value of n was kept at 2, and the value of k_2 was increased from 1. At $k_2 = 1$, an improvement in the distribution of the Pareto points can be observed compared with the standard WS. As the value of k_2 is increased from 1, a progressive improvement in the distribution of the solution points can be observed. However, it was also noted that with increasing k_2, the solution points around the extreme point defined by $w_1 = 0$ were being missed. This leads to a reduction of the extent of the generated Pareto front. An initial improvement in the distribution of the solution set is observed as k_2 is increased. As the value of k_2 is further increased, an optimal point is reached beyond which an increase in k_2 reduces the distribution quality. Therefore, the value of k_2 can be tuned to obtain an optimal value that give appreciable extent and fairly uniform spacing of the solution points. Fig. 3.2 shows the generated fronts for values of k_2 = 1, 2, and 5.

In the third case, k_2 was set to 1 and the value of n increased from 1, taking only integral values. Improvement in the distribution of solution points was observed as n is increased from 1. It was noted that the extent property was not reduced as observed with the variation in k_2 in the second case. An optimal value of n is



(a) $n = 2$, $k_2 = 1$ (b) $n = 2$, $k_2 = 2$ (c) $n = 2$, $k_2 = 5$

FIG. 3.2: Pareto front using Standard Weighted Sum with different values of k_2



(a) $n = 3$, $k_2 = 1$ (b) $n = 6$, $k_2 = 1$

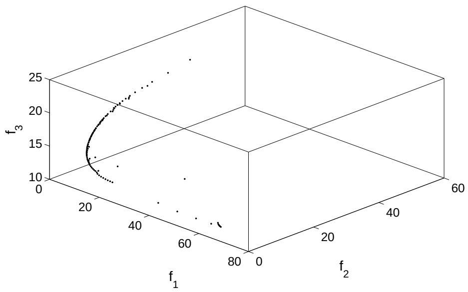
(c) $n = 6$, $k_2 = 1$ (d) $n = 10$, $k_2 = 1$

FIG. 3.3: Pareto front using Standard Weighted Sum with different values of n

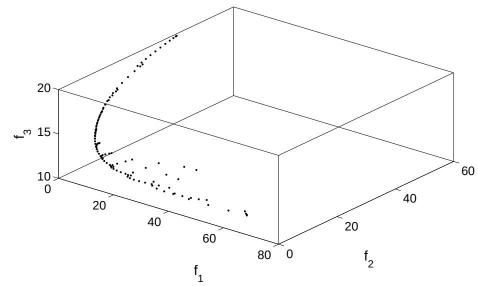
reached beyond which the distribution becomes poorer. Generated Pareto fronts for values of n= 3, 6, and 10 are as shown in Fig. 3.3.

3.2. Example 2

An evaluation of the method on three-objective problem is considered in this example. The problem was presented in [13].



(a) $n = 1$, $k_2 = 1$



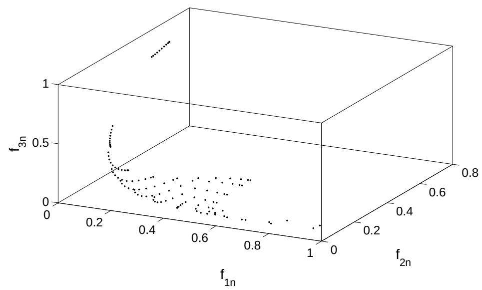
(b) $n = 2$, $k_2 = 1.02$

$$\min \begin{bmatrix} f_1(x) = 25(x_1 - 0.5)^2 + (2x_2 - 2)^2 + 0.1 \\ f_2(x) = [(x_1 - 2.5)^2 + 4(x_2 - 1.8)^2]^2 \\ f_3(x) = (x_1 - 2.0)^4 + 1.5(x_2 - 2.8)^2 + 0.3x_1x_2 + 10 \end{bmatrix}$$

$$\text{s.t.} \quad \begin{aligned} g_1 &= (x_1 - 2.1) - 0.08(2.2 - x_2)^2 \leqslant 0 \\ g_2 &= -x_1 \leqslant 0 \\ g_3 &= -x_2 \leqslant 0 \\ g_4 &= -x_2 - 3.0 \leqslant 0 \end{aligned} \tag{3.2}$$

The MOP was solved using the standard WS method i.e. $k_1 = k_2 = 1$ and $n = 1$. The problem was also solved with the proposed method, first with $k_2 = 1.02$ and $n = 2$ and then with component objectives normalized. The lower-upper bound normalization technique was used. To compare the solutions generated in each situation, 101 runs were made. The Pareto generated for each of the cases are as shown in Fig. 3.4. Looking at Fig. 3.4(a), the clustering of solution points can be observed while many points on the Pareto surface are missed. It is however observed that the extreme points of the solution set are captured. On the other hand with the proposed method, the Pareto point are better distributed as shown in Fig. 3.4(b). An improvement in the distribution of solution points was observed as the parameters were increased from 1. And increment beyond an optimal value, reduces the distribution performance. This optimal value is observed to be different for different problem. It was observed that some of the extreme values are not captured.

Fig. 3.4(c) is the Pareto surface generated with the normalized objective functions. Comparing the Pareto surface with that in Fig. 3.4(b), with the same values of k_2 and n, improvement due to function normalization is obvious. The solution points are uniformly distributed.



(c) $n = 2$, $k_2 = 1.02$ with $f(x)$ normalized

FIG. 3.4: Pareto front using Standard Weighted Sum with different values of n

4. Conclusions

A nonlinear weight selection method proposed in this paper has been shown to provide a means of controlling the distribution of points on the convex Pareto front. One major drawback of the WS method is that it does not provide means of controlling the distribution of points on the Pareto front. This is due its inability to take into consideration the curvature of the Pareto surface to determine its own slope change and also to control its own slope sensitivity. This is because the weight space constraint for the standard WS is defined on a simplex which does not have curvature. The proposed method maps the linear weight space into another weight space constraint which allows its curvature to be controlled through free parameters. Looking at (2.15) with $n = 2$, the weight space constraint Z_n defines an ellipsoid and k_1, k_2 are the axes of the ellipsoid. The relative values of the k_i and n determine the curvature of the weight space constraint, and therefore the slope sensitivity. This gives the decision makers greater degree of freedom to control the distribution of the weight space constraint and consequently the solution points. The nonlinear weight selection thus greatly improves computational efficiency of the WS method by reducing the number of same point with different weight factors. One demerit with the method is the reduction of the extent of the Pareto surface and its inability to capture extreme values of the Pareto surface for values of k_i different from 1.

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Abimbola M. Jubril
Department of Electronic and Electrical Engineering
Faculty of Technology
Obafemi Awolowo University
Ile-Ife, Nigeria
ajubril@oauife.edu.ng