SHARP WEIGHTED BOUNDEDNESS FOR VECTOR-VALUED MULTILINEAR SINGULAR INTEGRAL OPERATOR

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Abstract. In this paper, the sharp inequalities for some vector-valued multilinear singular integral operators are obtained. As the applications, we get the weighted $L^p(p > 1)$ norm inequalities and $L \log L$ type estimate for the vector-valued multilinear operators.

Keywords: Vector-valued multilinear operator; Singular integral operator; Sharp estimate; BMO; $A_p$-weight.

1. Preliminaries and Results

As the development of singular integral operators and their commutators, multilinear singular integral operators have been well studied. In this paper, we will study some vector-valued multilinear singular integral operators as following.

Fix $\varepsilon > 0$. Let $S$ and $S'$ be Schwartz space and its dual and $T : S \rightarrow S'$ be a linear operator. If there exists a locally integrable function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)\,dy$$

for every bounded and compactly supported function $f$, where $K$ satisfies:

$$|K(x, y)| \leq C|x - y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leq C|y - z|^\varepsilon |x - z|^{-n - \varepsilon}$$

when $2|y - z| \leq |x - z|$. Let $m_j$ be the positive integers ($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $\mathbb{R}^n$ ($j = 1, \cdots, l$). For $1 < s < \infty$, the vector-valued multilinear operator related to $T$ is defined by

$$|T_A(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T_A(f_i)(x)|^s\right)^{1/s},$$

Received September 25, 2012; Accepted October 25, 2012.
2010 Mathematics Subject Classification. Primary 26D10; Secondary 31B10,35A23

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where
\[ T_A(f_i)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^t R_{m_j + 1}(A_j; x, y)}{|x - y|^m} K(x, y) f_i(y) \, dy \]
and
\[ R_{m_j + 1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(x - y)^\alpha; \]
We also denote
\[ \langle T(f_i)(x) \rangle_s = \left( \sum_{i=1}^\infty |T(f_i)(x)|^s \right)^{1/s}\]
and
\[ \langle f(x) \rangle_s = \left( \sum_{i=1}^\infty |f_i(x)|^s \right)^{1/s}. \]
Suppose that \( |T|_s \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) and weak \( (L^1, L^1) \)-bounded.

Note that when \( m = 0 \), \( T_A \) is just the vector-valued multilinear commutator of \( T \) and \( A \) (see [13]). While when \( m > 0 \), \( T_A \) is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors (see [1-5]). In [7], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [12], Perez and Trujillo-Gonzalez prove a sharp estimate for some multilinear commutator when \( A_j \in O_{\mathcal{C}^{r \text{exp}L}} \). The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear singular integral operators. As the applications, we obtain the weighted \( L^p(p > 1) \) norm inequalities and \( L \log L \) type estimate for the vector-valued multilinear operators.

First, let us introduce some notations. Throughout this paper, \( Q \) will denote a cube of \( \mathbb{R}^n \) with sides parallel to the axes. For any locally integrable function \( f \), the sharp function of \( f \) is defined by
\[ f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy, \]
where, and in what follows, \( f_Q = |Q|^{-1} \int_Q f(x) \, dx \). It is well-known that (see [7])
\[ f^\#(x) \approx \sup_{Q \ni x \in C} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy. \]
We say that \( f \) belongs to \( BMO(\mathbb{R}^n) \) if \( f^\# \) belongs to \( L^\infty(\mathbb{R}^n) \) and \( \|f\|_{BMO} = \|f^\#\|_{L^\infty} \). For \( 0 < r < \infty \), we denote \( f_{r^\#} \) by
\[ f_{r^\#}(x) = \left( |f^\#(x)|^r \right)^{1/r}. \]
Let \( M \) be the Hardy-Littlewood maximal operator, that is
\[ M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| \, dy. \]
For \( k \in \mathbb{N} \), we denote by \( M^k \) the operator \( M \) iterated \( k \) times, i.e., \( M^1(f)(x) = M(f)(x) \) and \( M^k(f)(x) = M(M^{k-1}(f))(x) \) for \( k \geq 2 \).
Let $\Phi$ be a Young function and $\tilde{\Phi}$ be the complementary associated to $\Phi$, we denote the $\Phi$-average by, for a function $f$

$$\|f\|_{\Phi, Q} = \inf\left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$

and the maximal function associated to $\Phi$ by

$$M_\Phi(f)(x) = \sup_{x \in Q} \|f\|_{\Phi, Q};$$

The Young functions to be using in this paper are $\Phi(t) = \exp(t^r) - 1$ and $\Psi(t) = \exp(t \log^r(t + e))$, the corresponding $\Phi$-average and maximal functions denoted by $\|\cdot\|_{\exp L^r}$, $\|\cdot\|_{L(\log L)^r}$, $M_{\exp L^r}$ and $M_{L(\log L)^r}$. We have the following inequality, for any $r > 0$ and $m \in N$(see [12])

$$M(f) \leq M_{L(\log L)^r}(f), \quad M_{L(\log L)^m}(f) \approx M^{m+1}(f);$$

For $r \geq 1$, we denote

$$\|b\|_{osc_{exp L^r}} = \sup_Q \|b - b_Q\|_{exp L^r, Q},$$

the space $Osc_{exp L^r}$ is defined by

$$Osc_{exp L^r} = \{b \in L^1_{\log}(\mathbb{R}^n) : \|b\|_{osc_{exp L^r}} < \infty\}.$$ 

It has been known that(see [12])

$$\|b - b_Q\|_{exp L^r, 2^k Q} \leq Ck\|b\|_{osc_{exp L^r}}.$$ 

It is obvious that $Osc_{exp L^r}$ coincides with the $BMO$ space if $r = 1$. And $Osc_{exp L^r} \subset BMO$ if $r > 1$. We denote the Muckenhoupt weights by $A_p$ for $1 \leq p < \infty$(see [7]).

Now we state our main results as follows.

**Theorem 1.** Let $1 < s < \infty$, $r_j \geq 1$ and $D^{\alpha}A_j \in Osc_{exp L^r}$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then for any $0 < p < 1$, there exists a constant $C > 0$ such that for any $f = \{f_i\} \in C_0^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\|T_A(f)(x)\|_p^\#(x) \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j}A_j\|_{Osc_{exp L^r}} \right) M_{L(\log L)^{1/r}}(f)(x).$$

**Theorem 2.** Let $1 < s < \infty$, $r_j \geq 1$ and $D^{\alpha}A_j \in Osc_{exp L^r}$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \cdots, l$.

(1) If $1 < p < \infty$ and $w \in A_p$, then

$$\|T_A(f)(s)\|_{L^p(w)} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j}A_j\|_{Osc_{exp L^r}} \right) \|f(s)\|_{L^p(w)}.$$
(2) If $w \in A_1$. Denote $1/r = 1/r_1 + \cdots + 1/r_l$ and $\Phi(t) = t \log^{1/r}(t + e)$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : |T_A(f)(x)|_s > \lambda\}) \leq C \int_{\mathbb{R}^n} \Phi \left( \lambda^{-1} \prod_{j=1}^{l} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j}A_j\|_{\text{Osc}_{\exp L^r_j}} \right) |f(x)|_s \right) w(x)dx. $$

### 2. Some Lemmas

We give some preliminary lemmas.

**Lemma 1.** ([3]) Let $A$ be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for all $\alpha$ with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{Q(x, y)} \int_Q |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $Q$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

**Lemma 2.** ([7, p.485]) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1 = r_1 = 1/p - 1/q$

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E \|f \chi_E\|_{L^r}/\|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets $E$ with $0 < |E| < \infty$. Then

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q - p))^{1/p} \|f\|_{WL^r}.$$  

**Lemma 3.** ([12]) Let $r_j \geq 1$ for $j = 1, \cdots, m$, we denote that $1/r = 1/r_1 + \cdots + 1/r_m$. Then

$$\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_m(x)| g(x) dx \leq \|f\|_{\exp L^{r_1},Q} \cdots \|f\|_{\exp L^{r_m},Q} |g|_{L((\log L)^{1/r},Q)}.$$

### 3. Proof of Theorem

There remains only to prove Theorem 1.

**Proof of Theorem 1.** It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant $C_0$, the following inequality holds:

$$\left( \frac{1}{|Q|} \int_Q |T_A(f)(x)|_s - C_0 \right)^{1/p} dx \leq C \prod_{j=1}^{l} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j}A_j\|_{\text{Osc}_{\exp L^r_j}} \right) M_{L((\log L)^{1/r}),Q}(\{f(x)|_s\}(x).$$
Without loss of generality, we may assume \( l = 2 \). Fix a cube \( Q = Q(x_0, d) \) and \( \tilde{x} \in Q \). Let \( \tilde{Q} = 5\sqrt{\tilde{m}}Q \) and \( \tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A_j)_Q x^\alpha \), then \( R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y) = D^\alpha \tilde{A}_j - (D^\alpha A_j)_Q \) for \( |\alpha| = m_j \). We split \( f = g + h = \{g_i\} + \{h_i\} \) for \( g_i = f_i\chi_{\tilde{Q}} \) and \( h_i = f_i\chi_{R^n\setminus \tilde{Q}} \). Write

\[
T_A(f_i)(x) = \int_{R^n} \frac{\Pi_{j=1}^2 R_{m_{j+1}}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) f_i(y) \, dy
\]

\[
= \int_{R^n} \frac{\Pi_{j=1}^2 R_{m_{j+1}}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) h_i(y) \, dy
\]

\[
+ \int_{R^n} \frac{\Pi_{j=1}^2 R_{m_j} (\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) g_i(y) \, dy
\]

\[
- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) K(x, y) g_i(y) \, dy
\]

\[
- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) \, dy
\]

\[
+ \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} (x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) \, dy
\]

then, by Minkowski’ inequality,

\[
\left[ \frac{1}{|Q|} \int_Q \left| T_A(f)(x) \right|_s - \left| T_A(h)(x_0) \right|_s \right]^p dx \leq \left[ \frac{1}{|Q|} \int_Q \left( \sum_{i=1}^\infty \left| T_A(f_i)(x) - T_A(h_i)(x_0) \right| \right)^\frac{p}{s} dx \right]^1_s
\]

\[
\leq \left[ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^\infty \left| \int_{R^n} \frac{\Pi_{j=1}^2 R_{m_{j+1}}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) g_i(y) \, dy \right| \right)^\frac{s}{p} dx \right]^1_s
\]

\[
+ \left[ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^\infty \sum_{|\alpha_1|=m_1} \left| \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) K(x, y) g_i(y) \, dy \right| \right)^\frac{s}{p} dx \right]^1_s
\]

\[
+ \left[ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^\infty \sum_{|\alpha_2|=m_2} \left| \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) \, dy \right| \right)^\frac{s}{p} dx \right]^1_s
\]
For thus, by Lemma 2 and the weak type (1,1) of $|I|$, we get

\[ I := \int_Q \left( \sum_{i=1}^{\infty} |T_A(h_i)(x) - T_A(h_i)(x_0)|^\varepsilon \right)^{1/\varepsilon} dx \]

Now, let us estimate $I_1$, $I_2$, $I_3$, $I_4$ and $I_5$, respectively. First, for $x \in Q$ and $y \in \mathcal{Q}$, by Lemma 1, we get

\[ R_{m,j}(\hat{A}; x, y) \leq C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{osc_{expL^r_j}}, \]

thus, by Lemma 2 and the weak type (1,1) of $|T|_s$, we obtain

\[ I_1 \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{osc_{expL^r_j}} \right) \left( \frac{1}{|Q|} \int_Q |T(g)(x)|^p dx \right)^{1/p} \]
\[ = C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{osc_{expL^r_j}} \right) |Q|^{-1} \frac{||T(g)||_s |\chi_Q||_L^p}{|Q|^{1/p-1}} \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{osc_{expL^r_j}} \right) |Q|^{-1} ||g||_{L^1} \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{osc_{expL^r_j}} \right) M(||f||_s)(\bar{x}) \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{osc_{expL^r_j}} \right) M_L(\log L)^{1/r}(||f||_s)(\bar{x}). \]

For $I_2$, note that $||\chi_Q||_{expL^r_2,Q} \leq C$, similar to the proof of $I_1$ and by using Lemma 3, we get

\[ I_2 \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{osc_{expL^r_2}} \sum_{|\alpha_1|=m_1} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \hat{A}_1 g)(x)|^p dx \right)^{1/p} \]
\[ \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{osc_{expL^r_2}} \sum_{|\alpha_1|=m_1} |Q|^{-1} ||T(D^{\alpha_1} \hat{A}_1 g)(x)||_s |\chi_Q||_{L^1} \]
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\[
\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{Osc_{exp} L^2} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_{\mathbb{R}^n} |D^{\alpha_1} \tilde{A}_1(x)||g(x)||dx \\
\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{Osc_{exp} L^2} \|\chi_Q\|_{exp L^\infty, \tilde{Q}} \\
\times \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} A_1 - (D^{\alpha_1} A_1)\|_{exp L^\infty, \tilde{Q}} \|f|s\|_{L(\log L)^{1/r}, \tilde{Q}} \\
\leq C \sum_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Osc_{exp} L^{j}} \right) M_{L(\log L)^{1/r}} (\|f|s\|)(\tilde{x}).
\]

For \( I_3 \), similar to the proof of \( I_2 \), we get

\[
I_3 \leq C \sum_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Osc_{exp} L^{j}} \right) M_{L(\log L)^{1/r}} (\|f|s\|)(\tilde{x});
\]

Similarly, for \( I_4 \), by using Lemma 3, we get

\[
I_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)||x_s||dx \right)^{1/p} \\
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \|T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)||x_s||_{W^1} \\
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_{\mathbb{R}^n} |D^{\alpha_1} \tilde{A}_1(x)D^{\alpha_2} \tilde{A}_2(x)||g(x)||dx \\
\leq C \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} A_1 - (D^{\alpha_1} A_1)\|_{exp L^\infty, \tilde{Q}} \\
\times \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2 - (D^{\alpha_2} A_2)\|_{exp L^\infty, \tilde{Q}} \|f|s\|_{L(\log L)^{1/r}, \tilde{Q}} \\
\leq C \sum_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Osc_{exp} L^{j}} \right) M_{L(\log L)^{1/r}} (\|f|s\|)(\tilde{x}).
\]

For \( I_5 \), we write

\[
T_3(h_i)(x) - T_3(h_i)(x_0) = \int_{\mathbb{R}^n} \left( \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y)h_i(y)dy \\
+ \int_{\mathbb{R}^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0 - y|^m} K(x, y)h_i(y)dy \\
+ \int_{\mathbb{R}^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0 - y|^m} K(x, y)h_i(y)dy
\]
By Lemma 1, we know that, for $x \in Q$ and $y \in 2^{k+1}\hat{Q} \setminus 2^k\hat{Q}$,

$$|R_{m_j}(\tilde{A}_j; x, y)| \leq C|x - y|^{m_j} \sum_{|\alpha_j| = m_j} \left( \|D^{\alpha_j} A\|_{Osc_{exp, \zeta, j}} + \| (D^{\alpha_j} A) \tilde{Q}(x, y) - (D^{\alpha_j} A) \tilde{Q} \| \right)$$

$$\leq CK|x - y|^{m_j} \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A\|_{Osc_{exp, \zeta, j}}.$$  

Note that $|x - y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \hat{Q}$, we obtain, by the condition of $K$,

$$|I_5^{(1)}| \leq C \int_{R^n} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{m+n+\varepsilon}} \right)^2 \prod_{j=1}^{2^{m_j}} R_{m_j}(\tilde{A}_j; x, y)|h_i(y)|dy$$

$$\leq C \prod_{j=1}^{2^{m_j}} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{Osc_{exp, \zeta, j}} \right)^2 \times \sum_{k=0}^{\infty} \int_{2^k\hat{Q} \setminus 2^{k+1}\hat{Q}} k^2 \left( \frac{|x - x_0|}{|x_0 - y|^{n+1}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon}} \right) |f_i(y)|dy$$

$$\leq C \prod_{j=1}^{2^{m_j}} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{Osc_{exp, \zeta, j}} \right)^2 \sum_{k=1}^{\infty} k^2(2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\hat{Q}|} \int_{2^k\hat{Q}} |f_i(y)|dy,$$

thus, by Minkowski’ inequality,

$$\left( \sum_{i=1}^{\infty} |I_5^{(1)}|^s \right)^{1/s} \leq C \prod_{j=1}^{2^{m_j}} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{Osc_{exp, \zeta, j}} \right)^s \times \sum_{k=1}^{\infty} k^2(2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\hat{Q}|} \int_{2^k\hat{Q}} |f(y)|_s dy.$$
Similarly, by the formula (see [3]):
\[ R_{m_j}(A; x, y) - R_{m_j}(\tilde{A}; x_0, y) = \sum_{|\beta| < m_j} \frac{1}{|\beta|!} R_{m_j-|\beta|}(D^\beta \tilde{A}; x, x_0)(x - y)^\beta \]
and Lemma 1, we have
\[ |R_{m_j}(A; x, y) - R_{m_j}(\tilde{A}; x_0, y)| \leq C \sum_{|\beta| < m_j} |x - x_0|^{m_j - |\beta|} |x - y|^{\beta} ||D^\beta A||_{\text{osc}_{\text{exp}}L_j}, \]
thus
\[
\left( \sum_{i=1}^{\infty} |I_5^{(2)}|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{\text{osc}_{\text{exp}}L_j} \right) \int_0^\infty \sum_{k=0}^{\infty} k \left| \frac{x - x_0}{|x_0 - y|^{n+1}} \right| |f(y)|_s dy
\]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{\text{osc}_{\text{exp}}L_j} \right) M_{L(\log L)^{1/r}}(|f|_s)(\tilde{x}). \]

Similarly,
\[
\left( \sum_{i=1}^{\infty} |I_5^{(3)}|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{\text{osc}_{\text{exp}}L_j} \right) M_{L(\log L)^{1/r}}(|f|_s)(\tilde{x}). \]

For \( I_5^{(4)} \), similar to the proof of \( I_5^{(1)} \), \( I_5^{(2)} \) and \( I_2 \), we get
\[
\left( \sum_{i=1}^{\infty} |I_5^{(4)}|^s \right)^{1/s} \leq C \sum_{|\alpha_1| = m_1} \int_{R^n \setminus \tilde{Q}} \left| \frac{(x - y)^{\alpha_1} K(x, y)}{|x - y|^m} - \frac{(x_0 - y)^{\alpha_1} K(x_0, y)}{|x_0 - y|^m} \right| dy
\times |R_{m_2}(\tilde{A}_2; x, y)||D^{\alpha_1} \tilde{A}_1(y)||f(y)|_s dy
+ C \sum_{|\alpha_1| = m_1} \int_{R_{m_2}(\tilde{A}_2; x, y) - R_{m_4}(\tilde{A}_2; x_0, y)} \left| \frac{(x_0 - y)^{\alpha_1} K(x_0, y)}{|x_0 - y|^m} \right| |D^{\alpha_1} \tilde{A}_1(y)||f(y)|_s dy
\leq C \sum_{|\alpha_2| = m_2} ||D^{\alpha_2} A_2||_{\text{osc}_{\text{exp}}L_2} \sum_{|\alpha_1| = m_1} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\epsilon_k}) \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)||f(y)|_s dy
\]
\[ \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{osc_{\exp L^{2}}} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-c k}) \]
\[ \times |D^{\alpha_1}A_1 - (D^{\alpha_1}A_1)_Q|_{\exp L^{1}, 2^k \tilde{Q}} ||f|_{L(|logL|^1/r, 2^k \tilde{Q})} || \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{osc_{\exp L^{2}}} \right) M_{L(|logL|^1/r)}(||f|_{s})(\tilde{x}). \]

Similarly,
\[ \left( \sum_{i=1}^{\infty} |I_5^{(i)}|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{osc_{\exp L^{2}}} \right) M_{L(|logL|^1/r)}(||f|_{s})(\tilde{x}). \]

For \( I_5^{(6)} \), by using Lemma 3, we obtain
\[ \left( \sum_{i=1}^{\infty} |I_5^{(6)}|^s \right)^{1/s} \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int |(x-y)^{\alpha_1+\alpha_2} K(x,y) - (x_0-y)^{\alpha_1+\alpha_2} K(x_0,y)| |x-y|^m |x_0-y|^m \]
\[ \times |D^{\alpha_1}A_1(y)||D^{\alpha_2}A_2(y)||f(y)||_s \, dy \]
\[ \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} (2^{-k} + 2^{-c k}) \frac{1}{|2^k \tilde{Q}|} \int |D^{\alpha_1}A_1(y)||D^{\alpha_2}A_2(y)||f(y)||_s \, dy \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{osc_{\exp L^{2}}} \right) M_{L(|logL|^1/r)}(||f|_{s})(\tilde{x}). \]

Thus
\[ |I_5| \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{osc_{\exp L^{2}}} \right) M_{L(|logL|^1/r)}(||f|_{s})(\tilde{x}). \]

This completes the proof of Theorem 1.

By Theorem 1 and the \( L^p \)-boundedness of \( M_{L(|logL|^1/r)} \), we may obtain the conclusions (1)(2) of Theorem 2.

4. Example

In this section we shall apply Theorem 1 and 2 of the paper to the Calderón-Zygmund singular integral operator.

Let \( T \) be the Calderón-Zygmund operator (see [4,7,14]), the vector-valued multilinear operator related to \( T \) is defined by
\[ |T_A(f)(x)|_r = \left( \sum_{i=1}^{\infty} |T_A(f_i)(x)|^r \right)^{1/r}, \]
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where

\[ T_A(f_i)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l \frac{R_m + 1(A_j; x, y) K(x, y) f_i(y)}{|x - y|^m} dy. \]

Let \( r_j > 1 (j = 1, \cdots, l) \) and \( 1/r = 1/r_1 + \cdots + 1/r_l \). Then

(1) \[ (|T^A(f)|_s)_p(x) \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^\alpha_j A_j\|_{\text{osc} L^{r_j}} \right) M_{L(\log L)^{1/r}} (|f|_s)(x) \]

for any \( 1 < s < \infty, 0 < p < 1 \) and \( f \in C_0^\infty (\mathbb{R}^n) \).

(2) \[ \||T^A(f)|_s\|_{L^p(w)} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^\alpha_j A_j\|_{\text{osc} L^{r_j}} \right) \|f\|_{L^p(w)} \]

for any \( w \in A_p \) and \( 1 < s, p < \infty \).

(3) \[ w(\{x \in \mathbb{R}^n : |T^A(f)(x)|_s > \lambda\}) \leq C \int_{\mathbb{R}^n} \phi \left( \sum_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^\alpha_j A_j\|_{\text{osc} L^{r_j}} \right) \|f\|_s \right) w(x) dx \]

for any \( w \in A_1, 1 < s < \infty \) and all \( \lambda > 0 \).

REFERENCES


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