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SHARP WEIGHTED BOUNDEDNESS FOR VECTOR-VALUED MULTILINEAR SINGULAR INTEGRAL OPERATOR

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Abstract. In this paper, the sharp inequalities for some vector-valued multilinear singular integral operators are obtained. As the applications, we get the weighted $L^p(p>1)$ norm inequalities and $L\log L$ type estimate for the vector-valued multilinear operators.

Keywords: Vector-valued multilinear operator; Singular integral operator; Sharp estimate; BMO; A_p-weight.

1. Preliminaries and Results

As the development of singular integral operators and their commutators, multilinear singular integral operators have been well studied. In this paper, we will study some vector-valued multilinear singular integral operators as following.

Fix $\varepsilon > 0$. Let S and S' be Schwartz space and its dual and $T : S \to S'$ be a linear operator. If there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$Tf(x) = \int_{R^n} K(x,y)f(y)dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x,y)| \leqslant C|x-y|^{-n}$$

and

$$|K(y,x) - K(z,x)| + |K(x,y) - K(x,z)| \leqslant C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon}$$

when $2|y-z| \leqslant |x-z|$. Let m_j be the positive integers ($j = 1, \cdots, l$), $m_1+\cdots+m_l = m$ and A_j be the functions on R^n ($j = 1, \cdots, l$). For $1 < s < \infty$, the vector-valued multilinear operator related to T is defined by

$$|T_A(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T_A(f_i)(x)|^s\right)^{1/s},$$

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where

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} K(x,y) f_i(y) dy$$

and

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leqslant m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha;$$

We also denote

$$|T(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^s \right)^{1/s} \text{ and } |f(x)|_s = \left(\sum_{i=1}^{\infty} |f_i(x)|^s \right)^{1/s}.$$

Suppose that $|T|_s$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ and weak (L^1, L^1)-bounded.

Note that when $m = 0$, T_A is just the vector-valued multilinear commutator of T and A(see [13]). While when $m > 0$, T_A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors(see [1-5]). In [7], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [12], Perez and Trujillo-Gonzalez prove a sharp estimate for some multilinear commutator when $A_j \in Osc_{expL^{r_j}}$. The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear singular integral operators. As the applications, we obtain the weighted $L^p (p > 1)$ norm inequalities and $L \log L$ type estimate for the vector-valued multilinear operators.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that(see [7])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^\#||_{L^\infty}$. For $0 < r < \infty$, we denote $f_r^\#$ by

$$f_r^\#(x) = [(|f|^r)^\#(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy.$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \geqslant 2$.

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote the Φ-average by, for a function f

$$||f||_{\Phi,Q}=\inf\left\{\lambda>0:\frac{1}{|Q|}\int_Q\Phi\left(\frac{|f(y)|}{\lambda}\right)dy\leqslant 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x)=\sup_{x\in Q}||f||_{\Phi,Q};$$

The Young functions to be using in this paper are $\Phi(t)=exp(t^r)-1$ and $\Psi(t)=tlog^r(t+e)$, the corresponding Φ-average and maximal functions denoted by $||\cdot||_{expL^r,Q}$, M_{expL^r} and $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$. We have the following inequality, for any $r>0$ and $m\in N$(see [12])

$$M(f)\leqslant M_{L(logL)^r}(f),\quad M_{L(logL)^m}(f)\approx M^{m+1}(f);$$

For $r\geq 1$, we denote

$$||b||_{osc_{expL^r}}=\sup_Q||b-b_Q||_{expL^r,Q},$$

the space Osc_{expL^r} is defined by

$$Osc_{expL^r}=\{b\in L^1_{log}(R^n):||b||_{osc_{expL^r}}<\infty\}.$$

It has been known that(see [12])

$$||b-b_{2Q}||_{expL^r,2^kQ}\leqslant Ck||b||_{Osc_{expL^r}}.$$

It is obvious that Osc_{expL^r} coincides with the BMO space if $r=1$. And $Osc_{expL^r}\subset BMO$ if $r>1$. We denote the Muckenhoupt weights by A_p for $1\leqslant p<\infty$(see [7]).

Now we state our main results as follows.

Theorem 1. Let $1<s<\infty$, $r_j\geqslant 1$ and $D^\alpha A_j\in Osc_{expL^{r_j}}$ for all α with $|\alpha|=m_j$ and $j=1,\cdot\cdot\cdot,l$. Denote that $1/r=1/r_1+\cdot\cdot\cdot 1/r_l$. Then for any $0<p<1$, there exists a constant $C>0$ such that for any $f=\{f_i\}\in C_0^\infty(R^n)$ and $x\in R^n$,

$$(|T_A(f)|_s)_p^{\#}(x)\leqslant C\prod_{j=1}^l\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(x).$$

Theorem 2. Let $1<s<\infty$, $r_j\geqslant 1$ and $D^\alpha A_j\in Osc_{expL^{r_j}}$ for all α with $|\alpha|=m_j$ and $j=1,\cdot\cdot\cdot,l$.

(**1**). If $1<p<\infty$ and $w\in A_p$, then

$$|||T_A(f)|_s||_{L^p(w)}\leqslant C\prod_{j=1}^l\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)|||f|_s||_{L^p(w)}.$$

(**2**). If $w \in A_1$. Denote $1/r = 1/r_1 + \cdots + 1/r_l$ and $\Phi(t) = t log^{1/r}(t+e)$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$w(\{x \in R^n : |T_A(f)(x)|_s > \lambda\}) \leqslant C \int_{R^n} \Phi\left[\lambda^{-1} \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right) |f(x)|_s\right] w(x)dx.$$

2. Some Lemmas

We give some preliminary lemmas.

Lemma 1.([3]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leqslant C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.([7, p.485]) Let $0 < p < q < \infty$ and for any function $f \geqslant 0$. We define that, for $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leqslant N_{p,q}(f) \leqslant (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 3.([12]) Let $r_j \geqslant 1$ for $j = 1, \cdots, m$, we denote that $1/r = 1/r_1 + \cdots + 1/r_m$. Then

$$\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_m(x) g(x)| dx \leqslant ||f||_{expL^{r_1},Q} \cdots ||f||_{expL^{r_m},Q} ||g||_{L(logL)^{1/r},Q}.$$

3. Proof of Theorem

There remains only to prove Theorem 1.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q ||T_A(f)(x)|_s - C_0|^p dx\right)^{\frac{1}{p}} \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right) M_{L(logL)^{1/r}}(|f|_s)(x).$$

Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{A}_j(x)=A_j(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha$, then $R_m(A_j;x,y)=R_m(\tilde{A}_j;x,y)$ and $D^\alpha\tilde{A}_j=D^\alpha A_j-(D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha|=m_j$. We split $f=g+h=\{g_i\}+\{h_i\}$ for $g_i=f_i\chi_{\tilde{Q}}$ and $h_i=f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
T_A(f_i)(x) &= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy\\
&= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)h_i(y)dy\\
&+ \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy\\
&- \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy\\
&- \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy\\
&+ \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy,
\end{aligned}$$

then, by Minkowski' inequality,

$$\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q\left||T_A(f)(x)|_s-|T_{\tilde{A}}(h)(x_0)|_s\right|^p dx\right]^{\frac{1}{p}}\\
&\leqslant\left[\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^\infty|T_A(f_i)(x)-T_{\tilde{A}}(h_i)(x_0)|^s\right)^{\frac{p}{s}}dx\right]^{\frac{1}{p}}\\
&\leqslant\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^\infty\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy\right|^s\right)^{\frac{p}{s}}dx\right]^{\frac{1}{p}}\\
&+v\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^\infty\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy\right|^s\right)^{\frac{p}{s}}dx\right]^{\frac{1}{p}}\\
&+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^\infty\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy\right|^s\right)^{\frac{p}{s}}dx\right]^{\frac{1}{p}}
\end{aligned}$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy\right|^s\right)^{\frac{p}{s}}dx\right]^{\frac{1}{p}}$$

$$+\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}|T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)|^s\right)^{\frac{p}{s}}dx\right]^{\frac{1}{p}}$$

$$:= \quad I_1+I_2+I_3+I_4+I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j;x,y)\leqslant C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}},$$

thus, by Lemma 2 and the weak type (1,1) of $|T|_s$, we obtain

$$\begin{aligned}
I_1 &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)\left(\frac{1}{|Q|}\int_Q|T(g)(x)|_s^p dx\right)^{1/p}\\
&= C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)|Q|^{-1}\frac{|||T(g)|_s\chi_Q||_{L^p}}{|Q|^{1/p-1}}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)|Q|^{-1}|||T(g)|_s||_{WL^1}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)|Q|^{-1}|||g|_s||_{L^1}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M(|f|_s)(\tilde{x})\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_2, note that $||\chi_Q||_{expL^{r_2},Q}\leqslant C$, similar to the proof of I_1 and by using Lemma 3, we get

$$\begin{aligned}
I_2 &\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_s^p dx\right)^{1/p}\\
&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}\sum_{|\alpha_1|=m_1}|Q|^{-1}|||T(D^{\alpha_1}\tilde{A}_1g)(x)|_s\chi_Q||_{WL^1}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)||g(x)|_s dx\\
&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}||\chi_Q||_{expL^{r_2},Q}\\
&\times \sum_{|\alpha_1|=m_1}||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL^{r_1},\tilde{Q}}|||f|_s||_{L(logL)^{1/r},\tilde{Q}}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x});$$

Similarly, for I_4, by using Lemma 3, we get

$$\begin{aligned}
I_4 &\leqslant C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2f_1)(x)|_s^p dx\right)^{1/p}\\
&\leqslant C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}|Q|^{-1}|||T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)|_s\chi_Q||_{WL^1}\\
&\leqslant C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)||g(x)|_s dx\\
&\leqslant C\sum_{|\alpha_1|=m_1}||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL^{r_1},\tilde{Q}}\\
&\times \sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2-(D^{\alpha_2}A_2)_{\tilde{Q}}||_{expL^{r_2},\tilde{Q}}|||f|_s||_{L(logL)^{1/r},\tilde{Q}}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)=\int_{R^n}\left(\frac{K(x,y)}{|x-y|^m}-\frac{K(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy\\
&+\int_{R^n}\Big(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x_0,y)\Big)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy\\
&+\int_{R^n}\Big(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\Big)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y)-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)\right]\\
&\times\ D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy\\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y)-\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0,y)\right]\\
&\times\ D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy\\
&+\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\right]\\
&\times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy\\
&=I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}
$$

By Lemma 1, we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_{m_j}(\tilde{A}_j;x,y)| &\leqslant C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}(||D^{\alpha_j}A||_{Osc_{expL^{r_j}}}+|(D^{\alpha_j}A)_{\tilde{Q}(x,y)}-(D^{\alpha_j}A)_{\tilde{Q}}|)\\
&\leqslant Ck|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A||_{Osc_{expL^{r_j}}}.
\end{aligned}
$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, we obtain, by the condition of K,

$$
\begin{aligned}
|I_5^{(1)}| &\leqslant C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)||h_i(y)|dy\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)\\
&\times\quad\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right)|f_i(y)|dy\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f_i(y)|dy,
\end{aligned}
$$

thus, by Minkowski' inequality,

$$
\begin{aligned}
\left(\sum_{i=1}^{\infty}\left|I_5^{(1)}\right|^s\right)^{1/s} &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{Osc_{expL^{r_j}}}\right)\\
&\times\quad\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s dy
\end{aligned}
$$

$$\leqslant \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{Osc_{expL^{r_j}}}\right)M(|f|_s)(\tilde{x})$$
$$\leqslant \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).$$

For $I_5^{(2)}$, by the formula (see [3]):

$$R_{m_j}(\tilde{A};x,y)-R_{m_j}(\tilde{A};x_0,y)=\sum_{|\beta|<m_j}\frac{1}{\beta!}R_{m_j-|\beta|}(D^{\beta}\tilde{A};x,x_0)(x-y)^{\beta}$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{A};x,y)-R_{m_j}(\tilde{A};x_0,y)|\leqslant C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}||D^{\alpha}A||_{Osc_{expL^{r_j}}},$$

thus

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(2)}\right|^s\right)^{1/s}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)$$
$$\cdot\sum_{k=0}^{\infty}\int\limits_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|_s dy$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(3)}\right|^s\right)^{1/s}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).$$

For $I_5^{(4)}$, similar to the proof of $I_5^{(1)}$, $I_5^{(2)}$ and I_2, we get

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(4)}\right|^s\right)^{1/s}\leqslant C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|$$
$$\times\quad |R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy$$
$$+C\sum_{|\alpha_1|=m_1}\int\limits_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy$$
$$\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int\limits_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy$$

$$\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})$$
$$\times \quad |D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL^{r_1},2^k\tilde{Q}}|||f|_s||_{L(logL)^{1/r},2^k\tilde{Q}}$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(5)}\right|^s\right)^{1/s}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).$$

For $I_5^{(6)}$, by using Lemma 3, we obtain

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(6)}\right|^s\right)^{1/s}\leqslant C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\right|$$
$$\times \ |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_s dy$$
$$\leqslant \ C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_s dy$$
$$\leqslant \ C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).$$

Thus

$$|I_5|\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(|f|_s)(\tilde{x}).$$

This completes the proof of Theorem 1.

By Theorem 1 and the L^p-boundedness of $M_{L(logL)^{1/r}}$, we may obtain the conclusions (1)(2) of Theorem 2.

4. Example

In this section we shall apply Theorem 1 and 2 of the paper to the Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator(see [4,7,14]), the vector-valued multilinear operator related to T is defined by

$$|T_A(f)(x)|_r=\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)|^r\right)^{1/r},$$

where

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} K(x,y) f_i(y) dy.$$

Let $r_j \geqslant 1 (j = 1, \cdots, l)$ and $1/r = 1/r_1 + \cdots 1/r_l$. Then

$$(1) \quad (|T^A(f)|_s)_p^{\#}(x) \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) M_{L(logL)^{1/r}}(|f|_s)(x)$$

for any $1 < s < \infty$, $0 < p < 1$ and $f \in C_0^\infty(R^n)$.

$$(2) \quad |||T^A(f)|_s||_{L^p(w)} \leqslant C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) |||f|_s||_{L^p(w)}$$

for any $w \in A_p$ and $1 < s, p < \infty$.

$$(3) \quad w(\{x \in R^n : |T^A(f)(x)|_s > \lambda\})$$
$$\leq C \int_{R^n} \Phi \left[\lambda^{-1} \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) |f(x)|_s \right] w(x) dx$$

for any $w \in A_1$, $1 < s < \infty$ and all $\lambda > 0$.

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