ON $M_2$ SURFACES OF BIHARMONIC $B$-GENERAL HELICES
ACCORDING TO BISHOP FRAME IN HEISENBERG GROUP
Heis$^3$

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Abstract. In this paper, we study $M_2$ surfaces of biharmonic $B$-general helices according to Bishop frame in the Heisenberg group Heis$^3$. Finally, we characterize the $M_2$ surfaces of biharmonic $B$-general helices in terms of Bishop frame in the Heisenberg group Heis$^3$.

1. Introduction

Harmonic maps $f : (M, g) \rightarrow (N, h)$ between manifolds are the critical points of the energy

\[ E(f) = \int_M e(f) v_g, \]

where $v_g$ is the volume form on $(M, g)$ and

\[ e(f)(x) := \frac{1}{2} ||df(x)||^2_{T_x M \otimes f^* T N} \]

is the energy density of $f$ at the point $x \in M$.

Critical points of the energy functional are called harmonic maps.

In this paper, we study $M_2$ surfaces of biharmonic $B$-general helices according to Bishop frame in the Heisenberg group Heis$^3$. Finally, we characterize the $M_2$ surfaces of biharmonic $B$-general helices in terms of Bishop frame in the Heisenberg group Heis$^3$.
2. The Heisenberg Group Heis\(^3\)

Heisenberg group Heis\(^3\) can be seen as the space \(\mathbb{R}^3\) endowed with the following multiplication:

\[
(x, y, z)(x', y', z') = (x + x', y + y', z + z' - \frac{1}{2}xy + \frac{1}{2}x'y).
\]

Heis\(^3\) is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric \(g\) is given by

\[
g = dx^2 + dy^2 + (dz - x dy)^2.
\]

The Lie algebra of Heis\(^3\) has an orthonormal basis

\[
e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},
\]

for which we have the Lie products

\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0
\]

with

\[
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\]

3. Biharmonic \(\mathbb{B}\)-General Helices with Bishop Frame In The Heisenberg Group Heis\(^3\)

Let \(\gamma : I \longrightarrow \text{Heis}^3\) be a non geodesic curve on the Heisenberg group Heis\(^3\) parameterized by arc length. Let \(\{T, N, B\}\) be the Frenet frame fields tangent to the Heisenberg group Heis\(^3\) along \(\gamma\) defined as follows:

\(T\) is the unit vector field \(\gamma'\) tangent to \(\gamma\), \(N\) is the unit vector field in the direction of \(\nabla_T T\) (normal to \(\gamma\)), and \(B\) is chosen so that \(\{T, N, B\}\) is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

\[
\nabla_T T = \kappa N,
\]

\[
\nabla_T N = -\kappa T + \tau B,
\]

\[
\nabla_T B = -\tau N,
\]

where \(\kappa\) is the curvature of \(\gamma\) and \(\tau\) is its torsion and

\[
g(T, T) = 1, \quad g(N, N) = 1, \quad g(B, B) = 1,
\]

\[
g(T, N) = g(T, B) = g(N, B) = 0.
\]
In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has a vanishing second derivative. The Bishop frame is expressed as

\[
\nabla_T T = k_1 M_1 + k_2 M_2, \\
\nabla_T M_1 = -k_1 T, \\
\nabla_T M_2 = -k_2 T,
\]

where

\[
g(T, T) = 1, \quad g(M_1, M_1) = 1, \quad g(M_2, M_2) = 1; \\
g(T, M_1) = g(T, M_2) = g(M_1, M_2) = 0.
\]

Here, we shall call the set \{T, M_1, M_2\} as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures, where $\theta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$. Thus, Bishop curvatures are defined by

\[
k_1 = \kappa(s) \cos \theta(s), \\
k_2 = \kappa(s) \sin \theta(s).
\]

With respect to the orthonormal basis \{e_1, e_2, e_3\} we can write

\[
T = T^1 e_1 + T^2 e_2 + T^3 e_3, \\
M_1 = M_1^1 e_1 + M_1^2 e_2 + M_1^3 e_3, \\
M_2 = M_2^1 e_1 + M_2^2 e_2 + M_2^3 e_3.
\]

**Theorem 3.1.** $\gamma : I \to \Heis^3$ is a biharmonic curve with Bishop frame if and only if

\[
k_1^2 + k_2^2 = \text{constant} = C \neq 0, \\
k_1'' - Ck_1 = k_1 \left[ \frac{1}{4} - (M_2^3)^2 \right] - k_2 M_1^3 M_2^3, \\
k_2'' - Ck_2 = k_1 M_1^3 M_2^3 + k_2 \left[ \frac{1}{4} - (M_1^3)^2 \right].
\]

To separate a general helix according to Bishop frame from that of Frenet–Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $B$-general helix, [10].
4. M2 Surface of Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In
The Heisenberg Group $\text{Heis}^3$

The $M_2$ surface of $\gamma_{\mathfrak{B}}$ is a ruled surface

\begin{equation}
E(s, u) = \gamma_{\mathfrak{B}}(s) + uM_2(s).
\end{equation}

**Lemma 4.2.** Let $\gamma_{\mathfrak{B}} : I \rightarrow \text{Heis}^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix

with non-zero natural curvatures. Then the $M_2$ surface of $\gamma_{\mathfrak{B}}$ is

\begin{equation}
E(s, u) = \left[ \frac{\sin \theta}{(\kappa_1^2 + \kappa_2^2)} \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0 \right]
+ u \cos \theta \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0 + \zeta_2 e_1
\end{equation}

\begin{equation}
+ u \cos \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0 + \zeta_3 e_2
\end{equation}

\begin{equation}
+ u \cos \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0 + \zeta_2 e_1
\end{equation}

\begin{equation}
+ u \cos \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0 + \zeta_3 e_2
\end{equation}

\begin{equation}
+ (\cos \theta) s + \frac{\sin^2 \theta}{(\kappa_1^2 + \kappa_2^2)} \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} - \frac{\sin 2 \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0}{4(\kappa_1^2 + \kappa_2^2)}
\end{equation}

\begin{equation}
+ \zeta_1 \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0 - u \sin \theta + \zeta_4 e_3,
\end{equation}

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration.

**Proof.** Using orthonormal basis (2.2) and (3.7), we obtain

\begin{equation}
T = (\sin \theta \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0), \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0),
\end{equation}

\begin{equation}
\cos \theta + \frac{\sin^2 \theta}{(\kappa_1^2 + \kappa_2^2)} \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0)
\end{equation}

\begin{equation}
+ \zeta_1 \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0)
\end{equation}

where $\zeta_1$ is the constant of integration.
\[ T = \sin \theta \cos(\frac{k^2_1 + k^2_2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0)e_1 + \sin \theta \sin(\frac{k^2_1 + k^2_2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0)e_2 \]

(4.4) \quad + \cos \theta e_3. 

On the other hand, using Bishop formulas (3.3) and (2.1), we have

\[ \text{M}_2 = \cos \theta \cos(\frac{k^2_1 + k^2_2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0)e_1 + \cos \theta \sin(\frac{k^2_1 + k^2_2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0)e_2 - \sin \theta e_3. \]

(4.5)

Using the above equation, we have (4.2), and the theorem is proved.

We need the following lemma.

**Lemma 4.2.** Let \( \gamma_B : I \to \text{Heis}^3 \) be a unit speed biharmonic \( \mathfrak{B} \)-general helix with non-zero natural curvatures. Then the \( \text{M}_2 \) surface of \( \gamma_B \) are

\[
\begin{align*}
\text{x}(s,u) &= \sin \theta \cos(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + u \cos \theta \cos(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + \zeta_2, \\
y(s,u) &= -\sin \theta \cos(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + u \cos \theta \sin(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + \zeta_3, \\
z(s,u) &= \sin \theta \cos(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + u \cos \theta \sin(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + \zeta_3 + [-\sin \theta \cos(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + \zeta_3] \\
&\quad + \sin \theta \cos(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + \zeta_2 \\
&\quad + [-\sin \theta \cos(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0 + \zeta_2] \\
&\quad + \cos \theta s + \frac{\sin \theta}{(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}}} - \frac{2 \sin 2(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}} s + \zeta_0}{4(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{3}{2}}} \\
\end{align*}
\]
− \frac{\zeta_1 \sin \theta}{(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}}} \cos\left[(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0\right] - u \sin \theta + [\zeta_4],

where \( \zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \) are constants of integration.

**Proof.** Using the orthonormal basis we easily have the above system. Hence, the proof is completed.
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Fig. 4.2: The second illustration

REFERENCES


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