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ON M_2 SURFACES OF BIHARMONIC $\mathfrak{B}$-GENERAL HELICES ACCORDING TO BISHOP FRAME IN HEISENBERG GROUP $\mathbf{Heis}^3$

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Abstract. In this paper, we study $\mathbf{M}_2$ surfaces of biharmonic $\mathfrak{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. Finally, we characterize the $\mathbf{M}_2$ surfaces of biharmonic $\mathfrak{B}$-general helices in terms of Bishop frame in the Heisenberg group Heis^3.

1. Introduction

Harmonic maps $f:(M,g)\longrightarrow(N,h)$ between manifolds are the critical points of the energy

$$E(f)=\int_M e(f)\,v_g, \tag{1.1}$$

where v_g is the volume form on (M,g) and

$$e(f)(x):=\frac{1}{2}\left\|df(x)\right\|^2_{T^*M\otimes f^{-1}TN}$$

is the energy density of f at the point $x\in M$.

Critical points of the energy functional are called harmonic maps.

In this paper, we study $\mathbf{M}_2$ surfaces of biharmonic $\mathfrak{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. Finally, we characterize the $\mathbf{M}_2$ surfaces of biharmonic $\mathfrak{B}$-general helices in terms of Bishop frame in the Heisenberg group Heis^3.

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2. The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}). \tag{2.1}$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\ \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\ \mathbf{e}_3=\frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,\ [\mathbf{e}_2,\mathbf{e}_3]=[\mathbf{e}_3,\mathbf{e}_1]=0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1.$$

3. Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In The Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parameterized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},\end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N})=1,\ g(\mathbf{B},\mathbf{B})=1,\\ g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.\end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has a vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = 1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ g(\mathbf{T},\mathbf{M}_1) &= g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures, where $\theta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\theta(s), \\ k_2 &= \kappa(s)\sin\theta(s). \end{aligned} \tag{3.5}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.6}$$

Theorem 3.1. *$\gamma : I \longrightarrow Heis^3$ is a biharmonic curve with Bishop frame if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' - Ck_1 &= k_1\left[\frac{1}{4} - \left(M_2^3\right)^2\right] - k_2 M_1^3 M_2^3, \\ k_2'' - Ck_2 &= k_1 M_1^3 M_2^3 + k_2\left[\frac{1}{4} - \left(M_1^3\right)^2\right]. \end{aligned} \tag{3.7}$$

To separate a general helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathfrak{B}$-general helix, [10].

4. $\mathbf{M}_2$ Surface of Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In The Heisenberg Group $\mathbf{Heis}^3$

The $\mathbf{M}_2$ surface of $\gamma_{\mathfrak{B}}$ is a ruled surface

$$\mathcal{E}(s,u) = \gamma_{\mathfrak{B}}(s) + u\mathbf{M}_2(s). \tag{4.1}$$

Lemma 4.2. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix with non-zero natural curvatures. Then the $\mathbf{M}_2$ surface of $\gamma_{\mathfrak{B}}$ is*

$$\begin{aligned}
\mathcal{E}(s,u) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +u\cos\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2]\mathbf{e}_1 \\
&\quad +[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +u\cos\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]\mathbf{e}_2 \\
&\quad +[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&\quad [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
&\quad +(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\
&\quad -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]-u\sin\theta+\zeta_4]\mathbf{e}_3,
\end{aligned} \tag{4.2}$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 are constants of integration.

Proof. Using orthonormal basis (2.2) and (3.7), we obtain

$$\begin{aligned}
\mathbf{T} &= (\sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0],\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0], \\
&\quad \cos\theta+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin^2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +\zeta_1\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]),
\end{aligned} \tag{4.3}$$

where ζ_1 is the constant of integration.

$$
\begin{aligned}
\mathbf{T} &= \sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_1+\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_2 \\
&+\cos\theta\mathbf{e}_3. \qquad (4.4)
\end{aligned}
$$

On the other hand, using Bishop formulas (3.3) and (2.1), we have

$$
\mathbf{M}_2=\cos\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_1+\cos\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_2-\sin\theta\mathbf{e}_3. \qquad (4.5)
$$

Using the above equation, we have (4.2), and the theorem is proved.

We need the following lemma.

Lemma 4.2. *Let $\gamma_{\mathfrak{B}}: I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix with non-zero natural curvatures. Then the $\mathbf{M}_2$ surface of $\gamma_{\mathfrak{B}}$ are*

$$
\begin{aligned}
x_{\mathcal{E}}(s,u) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+u\cos\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2], \\
y_{\mathcal{E}}(s,u) &= [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+u\cos\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3], \\
z_{\mathcal{E}}(s,u) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+u\cos\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+u\cos\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]]+\zeta_3] \\
&+[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
&+(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}
\end{aligned}
$$

$$-\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]-u\sin\theta+\zeta_4],$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 *are constants of integration.*

Proof. Using the orthonormal basis we easily have the above system. Hence, the proof is completed.

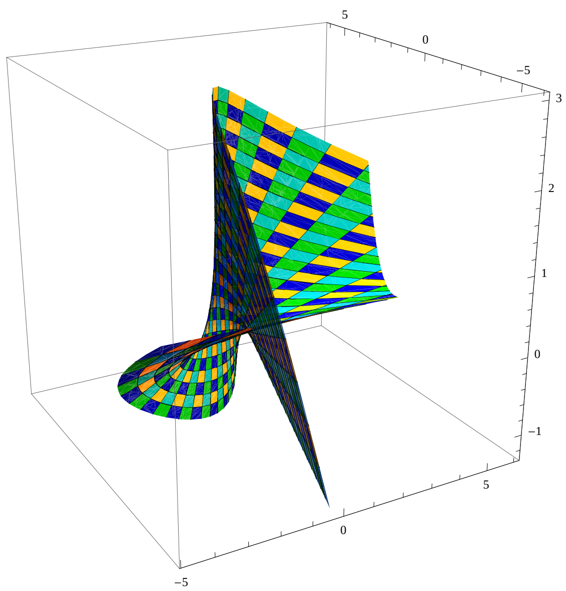

FIG. 4.1: The first illustration.

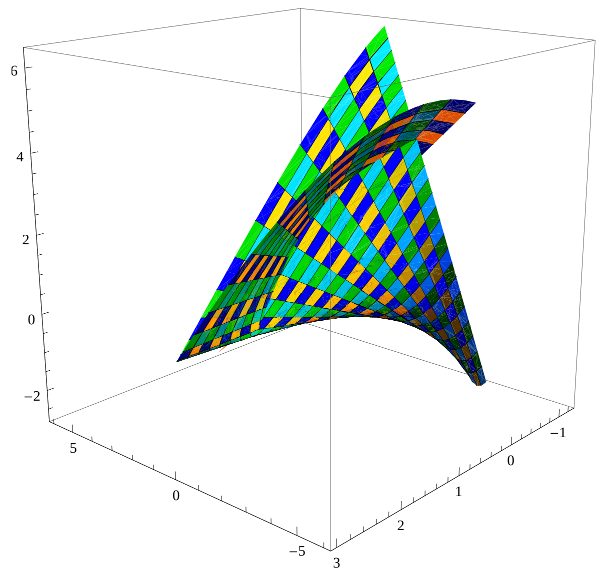

Fig. 4.2: The second illustration

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