FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 27 No 3 (2012), 321–336

SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE HIGHER ORDER PARTIAL DERIVATIVES ARE CO-ORDINATED CONVEX

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Abstract. In this paper we point out some inequalities of Hermite-Hadamard type for double integrals of functions whose partial derivatives of higher order are co-ordinated convex.

1. Introduction

The following definition is well known in literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

holds for all x, $y \in I$ and $\lambda \in [0, 1]$.

Many important inequalities have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality. This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, a, $b \in I$ with $a < b$. The inequalities in (1.1) are in reversed order if f a concave function.

The inequalities (1.1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality (1.1), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2, 3, 7, 10, 18, 19] and the references therein.

Received July 21, 2012.; Accepted November 9, 2012.
2010 *Mathematics Subject Classification.* 26A33, 26A51, 26D07, 26D10, 26D15

Let us consider now a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A modification for convex functions on Δ, known as co-ordinated convex functions, was introduced by S. S. Dragomir [4, 5] as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b], y \in [c,d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.1. [11] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality

$$\begin{aligned} &f(tx + (1-t)y, su + (1-s)w) \\ &\qquad \leq (x,u) + t(1-s) f(x,w) + s(1-t) f(y,u) + (1-t)(1-s) f(y,w), \end{aligned}$$

holds for all $t, s \in [0,1]$ and $(x,u),(y,w) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates but converse may not be true [4, 5].

The following Hermite-Hadamrd type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [4]:

Theorem 1.1. *[4] Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ, then*

(1.2)

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\, dydx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b [f(x,c) + f(x,d)]\, dx + \frac{1}{d-c}\int_c^d [f(a,y) + f(b,y)]\, dy\right] \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

In what follows Δ° is the interior of Δ and $L(\Delta)$ is the space of integrable functions over Δ.

The following result will be very useful to establish our one of the results in section 2:

Theorem 1.2. [9] *Let* $f : \Delta \to \mathbb{R}$ *be a continuous mapping such that the partial derivatives* $\frac{\partial^{k+l} f(.,.)}{\partial x^k \partial y^l}$, $k = 0, 1, \ldots, n-1$, $l = 0, 1, \ldots, m-1$ *exist on* Δ° *and are continuous on* Δ, *then*

$$\int_a^b \int_c^d f(t,s)\,dsdt = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k(x)\,Y_l(y)\,\frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} +$$
$$(-1)^m \sum_{k=0}^{n-1} X_k(x) \int_c^d S_m(y,s)\,\frac{\partial^{k+m} f(x,s)}{\partial x^k \partial s^m} ds$$
$$+ (-1)^n \sum_{l=0}^{m-1} Y_l(y) \int_a^b K_n(x,t)\,\frac{\partial^{n+l} f(t,y)}{\partial t^n \partial y^l} dt$$
$$+ (-1)^{m+n} \int_a^b \int_c^d K_n(x,t)\,S_m(y,s)\,\frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} dsdt,$$

where

$$\left\{ \begin{array}{l} K_n(x,t) := \left\{ \begin{array}{l} \frac{(t-a)^n}{n!}, t \in [a,x] \\ \frac{(t-b)^n}{n!}, t \in (x,b] \end{array} \right. \\ \\ S_m(y,s) := \left\{ \begin{array}{l} \frac{(s-c)^m}{m!}, s \in [c,y] \\ \frac{(s-d)^m}{m!}, s \in (y,d] \end{array} \right. \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} X_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \\ \\ Y_l(y) = \frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!} \end{array} \right. ,$$

for $(x,y) \in \Delta$.

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1]-[4]-[6], [11]-[17], [20]. Alomari et al. [1]-[6], proved several Hermite-Hadamard type inequalities for co-ordinated s-convex functions. Dragomir [4, 5], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [6], also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated to the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [11]-[13], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex function, product of two co-ordinated convex mappings and for co-ordinated h-convex mappings. Özdemir et. al [14]-[17], proved Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions.

By using the following lemma:

Lemma 1.1. [20, Lemma 1] *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b] \times [c,d]$ *in* $\mathbb{R}^2$ *with* $a < b$, $c < d$. *If* $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, *then the* *following equality holds:*

$$\begin{aligned}
(1.3)\quad & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
& \qquad\qquad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
& -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{d-c}\int_c^d [f(a,y)\,dy + f(b,y)]\,dy\right] \\
= & \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 (1-2t)(1-2s)\frac{\partial^2 f(ta+(1-t)b, sc+(1-s)d)}{\partial t\partial s}dtds.
\end{aligned}$$

Sarikaya, et. al [20], proved the following Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions:

Theorem 1.3. *[20, Theorem 2, Page 4] Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is convex on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}
(1.4)\quad & \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
& \left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \le \frac{(b-a)(d-c)}{16} \\
& \times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right),
\end{aligned}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{d-c}\int_c^d [f(a,y)\,dy + f(b,y)]\,dy\right].$$

Theorem 1.4. *[20, Theorem 3, Page 6-7] Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is convex on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}
(1.5)\quad & \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
& \left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \\
& \times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left[f(x,c)+f(x,d)\right]dx + \frac{1}{d-c}\int_c^d \left[f(a,y)\,dy + f(b,y)\right]dy\right]$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.5. *[20, Theorem 4, Page 8-9] Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, is convex on the co-ordinates on Δ, then one has the inequalities:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \leq \frac{(b-a)(d-c)}{16} \times \left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}, \tag{1.6}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left[f(x,c)+f(x,d)\right]dx + \frac{1}{d-c}\int_c^d \left[f(a,y)\,dy + f(b,y)\right]dy\right].$$

We also quote the following result from [13] to be used in the sequel of the paper:

Theorem 1.6. *[13, Theorem 4, page 8] Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ and $q\geq 1$, then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy - \frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right| \leq \frac{(b-a)(d-c)}{16} \times \left(\frac{\left|\frac{\partial^2}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}. \tag{1.7}$$

2. Main Results

In this section we establish new Hermite-Hadamard type inequalities for double integrals of functions whose partial derivatives of higher order are co-ordinated convex functions.

To make the presentation easier and compact to understand; we make some symbolic representation:

$$\begin{aligned} A^{'} &= \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left[f\left(x,c\right)+f\left(x,d\right)\right]dx + \frac{1}{d-c}\int_c^d \left[f\left(a,y\right)+f\left(b,y\right)\right]dy\right] \\ &+ \frac{1}{2}\sum_{l=2}^{m-1}\frac{\left(l-1\right)\left(d-c\right)^l}{2\left(l+1\right)!}\left[\frac{\partial^l f\left(a,c\right)}{\partial y^l}+\frac{\partial^l f\left(b,c\right)}{\partial y^l}\right] \\ &+ \frac{1}{2}\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}\left[\frac{\partial^k f\left(a,c\right)}{\partial x^k}+\frac{\partial^k f\left(a,d\right)}{\partial x^k}\right] \\ &- \frac{1}{b-a}\sum_{l=2}^{m-1}\frac{\left(l-1\right)\left(d-c\right)^l}{2\left(l+1\right)!}\int_a^b \frac{\partial^l f\left(x,c\right)}{\partial y^l}dx \\ &- \frac{1}{d-c}\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}\int_c^d \frac{\partial^k f\left(a,y\right)}{\partial x^k}dy \\ &- \sum_{k=2}^{n-1}\sum_{l=2}^{m-1}\frac{\left(k-1\right)\left(l-1\right)\left(b-a\right)^k\left(d-c\right)^l}{4\left(k+1\right)!\left(l+1\right)!}\frac{\partial^{k+l} f\left(a,c\right)}{\partial x^{k+l}}. \end{aligned}$$

$$B_{(n,m)} = \left|\frac{\partial^{n+m} f\left(a,c\right)}{\partial t^n \partial s^m}\right|; \quad C_{(n,m)} = \left|\frac{\partial^{n+m} f\left(a,d\right)}{\partial t^n \partial s^m}\right|.$$

$$D_{(n,m)} = \left|\frac{\partial^{n+m} f\left(b,c\right)}{\partial t^n \partial s^m}\right|; \quad E_{(n,m)} = \left|\frac{\partial^{n+m} f\left(b,d\right)}{\partial t^n \partial s^m}\right|.$$

It is obvious that for $m=n=1$ and $m=n=2$, $A^{'}=A$.

We quote the following lemma from [7], which will help us establish our main results:

Lemma 2.1. [7, Lemma 2.1] *Suppose $f: I^{\circ}\subset\mathbb{R}\to\mathbb{R}$, a, $b\in I^{\circ}$ with $a<b$. If $f^{(n)}$ exists on I° and $f^{(n)}\in L\left(a,b\right)$ for $n\geq 1$, then we have the identity:*

$$\begin{aligned} &\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}f^{(k)}(a) \\ &\qquad = \frac{\left(b-a\right)^n}{2n!}\int_0^1 t^{n-1}\left(n-2t\right)f^{(n)}(a+\left(1-t\right)b)dt. \end{aligned}$$

Lemma 2.2. *Let $f : \Delta \to \mathbb{R}$ $a < b$; $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial x^n \partial y^m}$ exists on Δ° and $\frac{\partial^{m+n} f}{\partial x^n \partial y^m} \in L(\Delta)$, for $m, n \in \mathbb{N}$, $m, n \geq 1$, then*

$$\begin{aligned}(2.1) \quad & \frac{(b-a)^n (d-c)^m}{4n!m!} \int_0^1 \int_0^1 t^{n-1} s^{m-1} (n-2t)(m-2s) \\ & \quad \times \frac{\partial^{n+m} f(ta + (1-t)b, cs + (1-s)d)}{\partial t^n \partial s^m} dtds + A' = \\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx.\end{aligned}$$

Proof. For $n = m = 1$, the lemma coincides with Lemma 1.1.

Consider the case, for $m, n \geq 2$, then

$$\begin{aligned}(2.2) \quad & \frac{(b-a)^n (d-c)^m}{4n!m!} \int_0^1 \int_0^1 t^{n-1} s^{m-1} (n-2t)(m-2s) \\ & \quad \times \frac{\partial^{n+m} f(ta + (1-t)b, cs + (1-s)d)}{\partial t^n \partial s^m} dtds \\ & = \frac{(d-c)^m}{2m!} \int_0^1 s^{m-1} (m-2s) \left[\int_0^1 \frac{(b-a)^n}{2n!} t^{n-1} (n-2t) \right. \\ & \quad \left. \times \frac{\partial^{n+m} f(ta + (1-t)b, cs + (1-s)d)}{\partial t^n \partial s^m} dt \right] ds\end{aligned}$$

An application of Lemma 2.1 with respect to the first argument yields:

$$\begin{aligned}(2.3) \quad & \frac{(b-a)^n (d-c)^m}{4n!m!} \int_0^1 \int_0^1 t^{n-1} s^{m-1} (n-2t)(m-2s) \\ & \quad \times \frac{\partial^{n+m} f(ta + (1-t)b, cs + (1-s)d)}{\partial t^n \partial s^m} dtds \\ & = \frac{(d-c)^m}{4m!} \int_0^1 s^{m-1} (m-2s) \frac{\partial^m f(a, cs + (1-s)d)}{\partial s^m} ds \\ & + \frac{(d-c)^m}{4m!} \int_0^1 s^{m-1} (m-2s) \frac{\partial^m f(b, cs + (1-s)d)}{\partial s^m} ds \\ & \quad - \frac{(d-c)^m}{2m!(b-a)} \int_a^b \int_0^1 s^{m-1} (m-2s) \\ & \quad \times \frac{\partial^m f(x, cs + (1-s)d)}{\partial s^m} dsdx - \frac{(d-c)^m}{2m!} \\ & \times \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} \int_0^1 s^{m-1} (m-2s) \frac{\partial^{k+m} f(a, cs + (1-s)d)}{\partial x^k \partial s^m} ds.\end{aligned}$$

Now repeated application of Lemma 2.1 with respect to the second argument yields:

$$\text{(2.4)}\quad \frac{(d-c)^m}{4m!}\int_0^1 s^{m-1}(m-2s)\frac{\partial^m f(a,cs+(1-s)d)}{\partial s^m}ds$$
$$=\frac{f(a,c)+f(a,d)}{4}-\frac{1}{2(d-c)}\int_c^d f(a,y)\,dy-\frac{1}{2}\sum_{l=2}^{m-1}\frac{(k-1)(d-c)^l}{2(l+1)!}\frac{\partial^l f(a,c)}{\partial y^l}.$$

$$\text{(2.5)}\quad \frac{(d-c)^m}{4m!}\int_0^1 s^{m-1}(m-2s)\frac{\partial^m f(b,cs+(1-s)d)}{\partial s^m}ds$$
$$=\frac{f(b,c)+f(b,d)}{4}-\frac{1}{2(d-c)}\int_c^d f(b,y)\,dy-\frac{1}{2}\sum_{l=2}^{m-1}\frac{(l-1)(d-c)^l}{2(l+1)!}\frac{\partial^l f(b,c)}{\partial y^l}.$$

$$\text{(2.6)}\quad \frac{(d-c)^m}{2m!(b-a)}\int_a^b\int_0^1 s^{m-1}(m-2s)\frac{\partial^m f(x,cs+(1-s)d)}{\partial s^m}dsdx$$
$$=\frac{1}{2(b-a)}\int_a^b f(x,c)\,dx+\frac{1}{2(b-a)}\int_a^b f(x,d)\,dx$$
$$-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx$$
$$-\frac{1}{b-a}\sum_{l=2}^{m-1}\frac{(l-1)(d-c)^l}{2(l+1)!}\int_a^b\frac{\partial^l f(x,c)}{\partial y^l}dx,$$

and

$$\text{(2.7)}\quad \frac{(d-c)^m}{2m!}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}$$
$$\times\int_0^1 s^{m-1}(m-2s)\frac{\partial^{k+m} f(a,cs+(1-s)d)}{\partial x^k\partial s^m}ds$$
$$=\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\frac{\partial^k f(a,c)}{\partial x^k}$$
$$+\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\frac{\partial^k f(a,d)}{\partial x^k}$$
$$-\frac{1}{d-c}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\int_c^d\frac{\partial^k f(a,y)}{\partial x^k}dy$$
$$-\sum_{k=2}^{n-1}\sum_{l=2}^{m-1}\frac{(k-1)(l-1)}{4(k+1)!}\frac{(b-a)^k(d-c)^l}{(l+1)!}\frac{\partial^{k+l} f(a,c)}{\partial x^{k+l}}.$$

Use (2.4)-(2.7) in (2.3) to get (2.1). This completes the proof of the lemma. □

Theorem 2.1. *Let $f : \Delta \to \mathbb{R}$ $a < b$; $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^n \partial s^m}$ exists on Δ° and $\frac{\partial^{m+n} f}{\partial t^n \partial s^m} \in L(\Delta)$. If $\left|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right|$ is convex on the co-ordinates on Δ, for $m, n \in \mathbb{N}$, $m, n \geq 2$, then*

$$\begin{aligned}(2.8)\quad &\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\ &\qquad\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A'\right|\\ &\leq \frac{(b-a)^n(d-c)^m}{4(n+2)!(m+2)!}\left[\left(n^2-2\right)\left\{\left(m^2-2\right)B_{(n,m)}+mC_{(n,m)}\right\}\right.\\ &\qquad\left.+n\left\{\left(m^2-2\right)D_{(n,m)}+mE_{(n,m)}\right\}\right].\end{aligned}$$

Proof. Suppose $m, n \geq 2$. By Lemma 2.2, we have:

$$\begin{aligned}(2.9)\quad &\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\ &\qquad\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A'\right|\\ &\leq \frac{(b-a)^n(d-c)^m}{4n!m!}\int_0^1\int_0^1 t^{n-1}s^{m-1}(n-2t)(m-2s)\\ &\qquad\times\left|\frac{\partial^{n+m} f(ta+(1-t)b, cs+(1-s)d)}{\partial t^n \partial s^m}\right| dtds\end{aligned}$$

By convexity of $\left|\frac{\partial^{m+n} f}{\partial t^n \partial s^m}\right|$ on the co-ordinates on Δ

$$\begin{aligned}(2.10)\quad &\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\ &\qquad\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A'\right|\\ &\leq \frac{(b-a)^n(d-c)^m}{4n!m!}\left[\left|\frac{\partial^{n+m} f(a,c)}{\partial t^n \partial s^m}\right|\int_0^1\int_0^1 t^n s^m (n-2t)(m-2s)\,dsdt\right.\\ &\quad+\left|\frac{\partial^{n+m} f(a,d)}{\partial t^n \partial s^m}\right|\int_0^1\int_0^1 t^n s^{m-1}(1-s)(n-2t)(m-2s)\,dsdt\\ &\quad+\left|\frac{\partial^{n+m} f(b,c)}{\partial t^n \partial s^m}\right|\int_0^1\int_0^1 t^{n-1} s^m (1-t)(n-2t)(m-2s)\,dsdt\\ &\quad\left.+\left|\frac{\partial^{n+m} f(b,d)}{\partial t^n \partial s^m}\right|\int_0^1\int_0^1 \left(t^{n-1}-t^n\right)(n-2t)\left(s^{m-1}-s^m\right)(m-2s)\,dsdt\right]\end{aligned}$$

$$= \frac{(b-a)^n (d-c)^m}{4n!m!} \left[\frac{\left(n^2-2\right)\left(m^2-2\right)}{(n+1)(n+2)(m+1)(m+2)} \left| \frac{\partial^{n+m} f(a,c)}{\partial t^n \partial s^m} \right| + \frac{\left(n^2-2\right) m}{(n+1)(n+2)(m+1)(m+2)} \left| \frac{\partial^{n+m} f(a,d)}{\partial t^n \partial s^m} \right| + \frac{n\left(m^2-2\right)}{(n+1)(n+2)(m+1)(m+2)} \left| \frac{\partial^{n+m} f(b,c)}{\partial t^n \partial s^m} \right| + \frac{mn}{(n+1)(n+2)(m+1)(m+2)} \left| \frac{\partial^{n+m} f(b,d)}{\partial t^n \partial s^m} \right| \right].$$

This completes the proof of the theorem. □

Theorem 2.2. *Let $f : \Delta \to \mathbb{R}$ $a < b$; $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^n \partial s^m}$ exists on Δ° and $\frac{\partial^{m+n} f}{\partial t^n \partial s^m} \in L(\Delta)$. If $\left| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right|^q$, $q \geq 1$, is convex on the co-ordinates on Δ, for $m, n \in \mathbb{N}$, $m, n \geq 2$, then*

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx - A' \right| \leq \frac{(b-a)^n (d-c)^m (n-1)^{1-1/q} (m-1)^{1-1/q}}{4(n+1)!(m+1)!(n+2)^{1/q}(m+2)^{1/q}} \times \left[\left(m^2-2\right) \left\{ \left(n^2-2\right) B^q_{(n,m)} + nD^q_{(n,m)} \right\} + m \left\{ \left(n^2-2\right) C^q_{(n,m)} + nE^q_{(n,m)} \right\} \right]^{\frac{1}{q}}. \tag{2.11}$$

Proof. Suppose $m, n \geq 2$. By Lemma 2.2 and the power mean inequality, we have

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx - A' \right| \leq \frac{(b-a)^n (d-c)^m}{4n!m!} \left\{ \int_0^1 \int_0^1 t^{n-1} s^{m-1} (n-2t)(m-2s)\, dsdt \right\}^{1-1/q} \times \left\{ \int_0^1 \int_0^1 t^{n-1} s^{m-1} (n-2t)(m-2s) \times \left| \frac{\partial^{n+m} f(ta+(1-t)b, cs+(1-s)d)}{\partial t^n \partial s^m} \right|^q dtds \right\}^{1/q}. \tag{2.12}$$

By the similar arguments used to obtain (2.8) and the fact

$$\int_0^1 \int_0^1 t^{n-1} s^{m-1} (n-2t)(m-2s)\, dsdt = \frac{(n-1)(m-1)}{(n+1)(m+1)},$$

we get (2.11). This completes the proof of the theorem. $\square$

Theorem 2.3. *Let $f : \Delta \to \mathbb{R}$, $a < b$; $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^n \partial s^m}$ exist on Δ° and $\frac{\partial^{m+n} f}{\partial t^n \partial s^m} \in L(\Delta)$. If $\left|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right|^q$, $q \geq 1$, is convex on the co-ordinates on Δ, for $m, n \in \mathbb{N}$, $m, n \geq 1$, then*

$$\begin{aligned}
(2.13)\quad & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, dsdt \right. \\
& - \frac{1}{(b-a)(d-c)} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}} \\
& \times \frac{(b-a)^{k+1}(d-c)^{l+1}}{(k+1)!\,(l+1)!} \frac{\partial^{k+l} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{\partial x^k \partial y^l} \\
& + \frac{(-1)^{m+1}}{(d-c)\, m!} \sum_{k=0}^{n-1} \frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} \int_c^d Q(s) \frac{\partial^{k+m} f\left(\frac{a+b}{2}, s\right)}{\partial x^k \partial s^m} ds \\
& \left. + \frac{(-1)^{n+1}}{(b-a)\, n!} \sum_{l=0}^{m-1} \frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f\left(t, \frac{c+d}{2}\right)}{\partial t^n \partial y^l} dt \right| \\
& \leq \frac{(b-a)^n (d-c)^m \sqrt[q]{B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}}}{2^{n+m+2/q}(n+1)!\,(m+1)!},
\end{aligned}$$

where

$$P(t) := \begin{cases} (t-a)^n, t \in \left[a, \frac{a+b}{2}\right] \\ (t-b)^n, t \in \left(\frac{a+b}{2}, b\right] \end{cases} \quad \textit{and} \quad Q(s) := \begin{cases} (s-c)^m, s \in \left[c, \frac{c+d}{2}\right] \\ (s-d)^m, s \in \left(\frac{c+d}{2}, d\right]. \end{cases}$$

Proof. The proof follows directly from Theorem 1.2 by letting $x \mapsto \frac{a+b}{2}$ and $y \mapsto$

$\frac{c+d}{2}$, to obtain

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt-\frac{1}{(b-a)(d-c)} \tag{2.14}$$
$$\times\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^{k+1}(d-c)^{l+1}}{(k+1)!\,(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l}$$
$$+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(s)\frac{\partial^{k+m}f\left(\frac{a+b}{2},s\right)}{\partial x^k\partial s^m}ds$$
$$+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt$$
$$=\frac{(-1)^{m+n}}{(b-a)(d-c)\,m!n!}\int_a^b\int_c^d P(t)Q(s)\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}dsdt.$$

An argument parallel to that of Theorem 2.2 but with (2.14) in place of Lemma 2.2 gives the desired result. □

We now derive results comparable to Theorem 2.1 and Theorem 2.2 with a concavity property instead of convexity property.

Theorem 2.4. *Let $f:\Delta\to\mathbb{R}$ $a<b$; $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial s^m}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial s^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$,$q\geq 1$, is concave on the co-ordinates on Δ, for $m,n\in\mathbb{N}$, $m,\ n\geq 1$, then*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \tag{2.15}$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|$$
$$\leq\frac{(n-1)(m-1)(b-a)^n(d-c)^m}{4(n+1)!\,(m+1)!}$$
$$\times\frac{\partial^{n+m}f\left(\frac{(n^2-2)a+nb}{(n-1)(n+2)},\frac{(m^2-2)c+md}{(m-1)(m+2)}\right)}{\partial t^n\partial s^m}.$$

Proof. By the concavity of $\left|\dfrac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$ on the co-ordinates on Δ and the power mean inequality, the following inequality holds:

$$\left|\frac{\partial^{n+m} f\left(\lambda x+(1-\lambda) y, v\right)}{\partial t^{n} \partial s^{m}}\right|^{q} \geq \lambda\left|\frac{\partial^{n+m} f\left(x, v\right)}{\partial t^{n} \partial s^{m}}\right|^{q}+(1-\lambda)\left|\frac{\partial^{n+m} f\left(y, v\right)}{\partial t^{n} \partial s^{m}}\right|^{q} \geq\left(\lambda\left|\frac{\partial^{n+m} f\left(x, v\right)}{\partial t^{n} \partial s^{m}}\right|+(1-\lambda)\left|\frac{\partial^{n+m} f\left(y, v\right)}{\partial t^{n} \partial s^{m}}\right|\right)^{q},$$

for all $x, y \in [a, b]$ and $\lambda \in [0,1]$ for some fixed $v \in [c, d]$. Similarly

$$\left|\frac{\partial^{n+m} f\left(u, \lambda z+(1-\lambda) w\right)}{\partial t^{n} \partial s^{m}}\right| \geq \lambda\left|\frac{\partial^{n+m} f\left(u, z\right)}{\partial t^{n} \partial s^{m}}\right|+(1-\lambda)\left|\frac{\partial^{n+m} f\left(u, w\right)}{\partial t^{n} \partial s^{m}}\right|,$$

for all $z, w \in [c, d]$ and $\lambda \in [0,1]$ for some fixed $u \in [a, b]$, implying $\left|\frac{\partial^{n+m} f}{\partial t^{n} \partial s^{m}}\right|$ is concave on the co-ordinates on Δ.

By the Jensen's inequality we have

(2.16)

$$\begin{aligned}
&\int_{0}^{1} s^{m-1}(m-2s)\left[\int_{0}^{1} t^{n-1}(n-2t)\left|\frac{\partial^{n+m} f\left(ta+(1-t) b, cs+(1-s) d\right)}{\partial t^{n} \partial s^{m}}\right| dt\right] ds \\
&\quad \leq \int_{0}^{1} s^{m-1}(m-2s)\left[\left(\int_{0}^{1} t^{n-1}(n-2t)\, dt\right)\right. \\
&\qquad \times\left.\left|\frac{\partial^{n+m} f\left(\frac{\int_{0}^{1} t^{n-1}(n-2t)(ta+(1-t) b) dt}{\int_{0}^{1} t^{n-1}(n-2t) dt}, cs+(1-s) d\right)}{\partial t^{n} \partial s^{m}}\right|\right] ds \\
&\quad =\frac{n-1}{n+1} \int_{0}^{1} s^{m-1}(m-2s)\left|\frac{\partial^{n+m} f\left(\frac{\left(n^{2}-2\right) a+nb}{(n-1)(n+2)}, cs+(1-s) d\right)}{\partial t^{n} \partial s^{m}}\right| ds \\
&\quad \leq \frac{(n-1)(m-1)}{(n+1)(m+1)}\left|\frac{\partial^{n+m} f\left(\frac{\left(n^{2}-2\right) a+nb}{(n-1)(n+2)}, \frac{\left(m^{2}-2\right) c+md}{(m-1)(m+2)}\right)}{\partial t^{n} \partial s^{m}}\right|.
\end{aligned}$$

Application of lemma 2.2 and (2.16), we get (2.15). This completes the proof of theorem. □

Theorem 2.5. *Let $f : \Delta \to \mathbb{R}$ $a<b$; $c<d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^{n} \partial s^{m}}$ exist on Δ° and $\frac{\partial^{m+n} f}{\partial t^{n} \partial s^{m}} \in L(\Delta)$. If $\left|\frac{\partial^{n+m} f}{\partial t^{n} \partial s^{m}}\right|^{q}$, $q \geq 1$, is concave on the*

co-ordinates on Δ, *for* $m, n \in \mathbb{N}$, $m, n \geq 1$, *then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt-\frac{1}{(b-a)(d-c)}\right.$$
$$\times\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{(k+1)!\,(l+1)!}\frac{(b-a)^{k+1}(d-c)^{l+1}}{2^{k+l+2}}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l}$$
$$+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(s)\frac{\partial^{k+m}f\left(\frac{a+b}{2},s\right)}{\partial x^k\partial s^m}ds$$
$$+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt|$$
$$\leq\frac{(b-a)^n(d-c)^m}{2^{n+m}(n+1)!\,(m+1)!}\left|\frac{\partial^{n+m}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^n\partial y^m}\right|. \tag{2.17}$$

Proof. Similar to proof of Theorem 2.4 by using (2.14). Therefore we omit the details for reader. □

Remark 2.1. On letting $m = n = 2$ in (2.8), (2.11) and (2.13) respectively yield:

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|$$
$$\leq\frac{(b-a)^2(d-c)^2}{144\times 4}\{B_{(2,2)}+C_{(2,2)}+D_{(2,2)}+E_{(2,2)}\}. \tag{2.18}$$

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|$$
$$\leq\frac{(b-a)^2(d-c)^2}{9\times 2^{4+2/q}}\sqrt[q]{B_{(2,2)}^q+C_{(2,2)}^q+D_{(2,2)}^q+E_{(2,2)}^q} \tag{2.19}$$

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.-\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{2(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt\right|$$
$$\leq\frac{(b-a)^2(d-c)^2}{9\times 2^{6+2/q}}\sqrt[q]{B_{(2,2)}^q+C_{(2,2)}^q+D_{(2,2)}^q+E_{(2,2)}^q}. \tag{2.20}$$

It may be noted that the bounds in (2.18), (2.19) and (2.20) are sharper than the bounds of the inequalities proved in Theorem 1.3, Theorem 1.5 and Theorem 1.6 respectively.

3. Acknowledgment

The authors thank to the anonymous referee for his/her very useful and constructive comments which helped the authors to improve the final version of the paper.

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