FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 27 No 3 (2012), 309–320

SOME PROPERTIES OF A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED SRIVASTAVA-ATTIYA OPERATOR *

Nagat. M. Mustafa and Maslina Darus

Abstract. Making use of an integral operator which is defined by means of a general Hurwitz−Lerch zeta function, we give some properties of the class $Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$. It is worth noting that the usage of Hurwitz−Lerch zeta function in Geometric Function Theory was first made by Srivastava and Attiya in 2007. Indeed, in this present paper, we obtain integral means inequalities, modified Hadamard products and establish some results concerning the partial sums for functions f belonging to the class $Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$.

Keywords: Analytic functions, Hurwitz−Lerch zeta function, Srivastava−Attiya operator, Integral means, Hadamard product, Partial sums.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}). \tag{1.1}$$

With a view to define the Srivastava−Attiya operator, we recall here a general Hurwitz−Lerch-Zeta function, which is defined in ([4],[6]) by the following series:

$$\Phi(z,s,b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

where $(s \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-)$ *when* $(|z| < 1)$, and $(\mathrm{Re}(b) > 1)$ *when* $(|z| = 1)$.

Received June 10, 2012.; Accepted December 5, 2012.
2010 *Mathematics Subject Classification.* Primary 30C45
*The authors were supported in part by LRGS/TD/2011/UKM/ICT/03/02

By making use of the following normalized function:

$$\begin{aligned} G_{s,b}(z) &= (1+b)^s[\Phi(z,s,b) - b^{-s}] \\ &= z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s z^k, \quad (z \in \mathbb{U}). \end{aligned}$$

Srivastava$-$Attiya [4] introduced operator $\mathcal{L}_{s,b} : \mathcal{A} \to \mathcal{A}$ by the following:

$$\mathcal{L}_{s,b} = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s a_k z^k.$$

The operator $\mathcal{L}_{s,b}$ is now well-known in the literature as the Srivastava$-$Attiya operator. Various basic properties of $\mathcal{L}_{s,b}$ are systematically investigated in ([11], [12]).

Owa and Srivastava [2] introduced the operator $\Omega^\alpha : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral as follows:

$$\Omega^\alpha f(z) = \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k \qquad (\alpha \neq 2,3,4,\cdots),$$

where $D_z^\alpha f(z)$ the fractional derivative of f of order α (see [3]).

Let:

$$\begin{aligned} \Phi^*(z,s,b) &= G_{s,b-1}(z) \\ &= z + \sum_{k=2}^{\infty} \frac{b^s}{(k+b-1)^s} a_k z^k. \end{aligned}$$

Using the technique of Owa and Srivastava [2], we introduced the generalized integral operator $(\mathrm{Im}_{s,b}^\alpha f) : \mathcal{A} \to \mathcal{A}$ by the following:

$$\begin{aligned} \mathrm{Im}_{s,b}^\alpha f(z) &= \Gamma(2-\alpha) z^\alpha D_z^\alpha \Phi^*(z,s,b), \quad (\alpha \neq 2,3,4,\cdots) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \left(\frac{b}{k-1+b}\right)^s a_k z^k, \quad (z \in \mathbb{U}), \end{aligned}$$

where $s \in \mathbb{C}$, $b \in \mathbb{C} - \mathbb{Z}_0^-$, and $0 \leq \alpha < 1$.

It can also be shown that this operator is the generalized Srivastava$-$Attiya operator by taking $f(z) = z + \sum\limits_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k$.

Note that : $\mathrm{Im}^{0}_{0,b}\, f(z) = f(z)$.

Special cases of this operator include:

- $\mathrm{Im}^{\alpha}_{0,b}\, f(z) \equiv \Omega^{\alpha} f(z)$ is the Owa and Srivastava operator [2].
- $\mathrm{Im}^{0}_{s,b+1}\, f(z) \equiv \mathcal{L}_{s,b}$ is the Srivastava and Attiya integral operator [4].
- $\mathrm{Im}^{0}_{\sigma,2}\, f(z) \equiv I^{\sigma} f(z)$ is the Jung Kim Srivastava integral operator [5].

Also, the authors [1] have recently introduced a new subclass of analytic functions with negative coefficients, and stated the following:

For $(0 \leq \delta < 1)$, $(0 < \beta \leq 1)$ and $(\frac{1}{2} < \gamma \leq 1)$ if $\delta = 0$, and $(\frac{1}{2} < \gamma \leq \frac{1}{2\delta})$if $\delta \neq 0$, we let $Q^{\alpha}_{s,b}(\delta, \beta, \gamma)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left| \frac{(\mathrm{Im}^{\alpha}_{s,b}\, f(z))' - 1}{2\gamma((\mathrm{Im}^{\alpha}_{s,b}\, f(z))' - \delta) - ((\mathrm{Im}^{\alpha}_{s,b}\, f(z))' - 1)} \right| < \beta.$$

We further let

$$Q^{*\alpha}_{s,b}(\delta, \beta, \gamma) = Q^{\alpha}_{s,b}(\delta, \beta, \gamma) \cap T, \tag{1.2}$$

where

$$T := \left\{ f \in \mathcal{A} : f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad where\ \ a_k \geq 0\ \ for\ all\ \ k \geq 2 \right\},$$

is a subclass of $\mathcal{A}$ introduced and studied by Silverman [9].

In [1], it was also shown that the sufficient condition for a function f to be in the class $Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$.

Theorem 1.1. *Let the function f be defined by (1.2). Then $f \in Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty} k\left[1 + \beta(2\gamma - 1)\right] \left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \right) \left| \left(\frac{b}{k-1+b} \right)^{s} \right| |a_k| \leq 2\beta\gamma(1-\delta). \tag{1.3}$$

The result is sharp.

2. Integral Means Inequalities

In order to prove the results regarding integral means inequalities, we need the concept of subordination between analytic functions and also the following lemma.

Lemma 2.1. *[8]*

If f, g are analytic in $\mathbb{U}$,such that $f \prec g$, then

$$\int_0^{2\pi} |f(z)|^y d\theta \leq \int_0^{2\pi} |g(z)|^y d\theta, \quad (z = re^{i\theta}, 0 < r < 1, y > 0).$$

Theorem 2.2. *Let $f \in Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$. Then for $z = re^{i\theta}$, $0 < r < 1$, we have*

$$\int_0^{2\pi} |f(re^{i\theta})|^y d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^y d\theta, \quad (0 < r < 1, y > 0),$$

where the function $f_2(z)$ defined by

$$f_2(z) = z - \frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|} z^2. \tag{2.1}$$

Proof: Let $f \in Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$ and satisfying (1.3), and $f_2(z)$ be given by (2.1). We must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{\infty} a_k z^{k-1}\right| d\theta \leq \int_0^{2\pi} \left|1 - \frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|} z\right|^y d\theta.$$

By Lemma 2.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 - \frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|} z.$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} = 1 - \frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|} \omega(z). \tag{2.2}$$

From (2.2), we obtain

$$|\omega(z)| \leq |z| \sum_{k=2}^{\infty} \frac{2\beta\gamma(1-\delta)}{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|} |a_k| \leq |z| < 1.$$

This completes the proof of the theorem.

3. Modified Hadamard Products

Let the functions $f_j(z)(j=1;2)$ be defined by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k, \quad for \quad all \quad (a_{k,j} \geq \quad 0, z \in \mathbb{U}). \tag{3.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k.$$

Using the techniques of Schild and Silverman [7], we prove the following results.

Theorem 3.1. *For functions $f_j(z)(j=1;2)$ defined by (3.1), let $f_1(z) \in Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$, $f_2(z) \in Q^{*\alpha}_{s,b}(\delta,\mu,\gamma)$. Then $(f_1 * f_2)(z) \in \xi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta,\beta,\mu,\gamma))$, where*

$$\xi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta,\beta,\gamma) =$$
$$\frac{2\gamma(1-\delta)\beta\mu}{2\gamma(1-\delta)\beta\mu - 4\beta\mu\gamma^2(1-\delta) + 2\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|(1+\mu(2\gamma-1))(1+\beta(2\gamma-1))}.$$

Proof: To prove the theorem, we need to find the largest $\xi = \xi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta,\beta,\mu,\gamma))$ such that

$$\sum_{k=2}^{\infty} \frac{k\left[1+\xi(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\xi\gamma(1-\delta)} a_{k,1} a_{k,2} \leq 1,$$

since

$$\sum_{k=2}^{\infty} \frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)} a_{k,1} \leq 1,$$

and

$$\sum_{k=2}^{\infty} \frac{k\left[1+\mu(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\mu\gamma(1-\delta)} a_{k,2} \leq 1.$$

By the Cauchy-Schwarz inequality, we have;

$$\sum_{k=2}^{\infty} \frac{k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\gamma(1-\delta)} \sqrt{\frac{(1+\beta(2\gamma-1))}{\beta}\frac{(1+\mu(2\gamma-1))}{\mu} a_{k,1} a_{k,2}} \leq 1.$$

Thus, it suffices to show that

$$\frac{(1+\xi(2\gamma-1))}{\xi}a_{k,1}a_{k,2} \leq \sqrt{\frac{(1+\beta(2\gamma-1))}{\beta}\frac{(1+\mu(2\gamma-1))}{\mu}a_{k,1}a_{k,2}}.$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{2\gamma(1-\delta)}{k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}\sqrt{\frac{\beta}{(1+\beta(2\gamma-1))}\frac{\mu}{(1+\mu(2\gamma-1))}}.$$

Consequently, we need only to prove that

$$\frac{2\gamma(1-\delta)}{k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}\sqrt{\frac{\beta}{(1+\beta(2\gamma-1))}\frac{\mu}{(1+\mu(2\gamma-1))}} \leq$$

$$\frac{\xi}{(1+\xi(2\gamma-1))}\sqrt{\frac{(1+\beta(2\gamma-1))}{\beta}\frac{(1+\mu(2\gamma-1))}{\mu}},$$

or, equivalently that

$$\xi \leq \frac{2\gamma(1-\delta)\beta\mu}{2\gamma(1-\delta)\beta\mu-4\beta\mu\gamma^2(1-\delta)+k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|(1+\mu(2\gamma-1))(1+\beta(2\gamma-1))}$$
$$=\psi(k),$$

is an increasing function of k, letting $k=2$, we obtain

$$\psi(2)=\frac{2\gamma(1-\delta)\beta\mu}{2\gamma(1-\delta)\beta\mu-4\beta\mu\gamma^2(1-\delta)+2\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|(1+\mu(2\gamma-1))(1+\beta(2\gamma-1))},$$

which completes the proof. □

Theorem 3.2. *For functions $f_j(z)(j=1;2)$ defined by (3.1), be in the class $Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$ Then the function $h(z)=z-\sum_{k=2}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)z^k$, belongs to the class $\varphi^{*\alpha}_{s,b}(\delta,Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$, where*

$$\varphi^{*\alpha}_{s,b}(\delta,Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)=\frac{4\beta^2\gamma(1-\delta)}{2(1+\beta(2\gamma-1))]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|-4\beta^2\gamma(1-\delta)}.$$

Proof: By virtue of Theorem 1.1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\beta\gamma(1-\delta)}\right]^{2}a_{k,1}^{2}\leq 1, \tag{3.2}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\beta\gamma(1-\delta)}\right]^{2}a_{k,2}^{2}\leq 1. \tag{3.3}$$

It follows from (3.2) and(3.3).

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\beta\gamma(1-\delta)}\right]^{2}(a_{k,1}^{2}+a_{k,2}^{2})\leq 1.$$

Therefore,we need to find the largest $\varphi=\varphi_{s,b}^{*\alpha}(\delta,Q_{s,b}^{*\alpha}(\delta,\beta,\gamma)$

$$\frac{k\left[1+\varphi(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\varphi\gamma(1-\delta)}$$
$$\leq\frac{1}{2}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\beta\gamma(1-\delta)}\right]^{2},$$

that is,

$$\varphi\leq\frac{4\beta^{2}\gamma(1-\delta)}{k(1+\beta(2\gamma-1))^{2}\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|-4\beta^{2}\gamma(1-\delta)}=\chi(k),$$

is an increasing function of k, letting $k=2$, we obtain

$$\chi(2)=\frac{4\beta^{2}\gamma(1-\delta)}{2(1+\beta(2\gamma-1))]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|-4\beta^{2}\gamma(1-\delta)},$$

which completes the proof. □

4. Partial sums

By following the earlier work by Silverman[10] on partial sums of analytic functions, we study the ratio of a function of the form (1.2) to its sequence of partial sums of the form $f_1(z) = z,\ f_n(z) = z + \sum_{k=2}^{n} a_k z^k, \quad (z \in \mathbb{U})$.

We will determine sharp lower bounds for

$$\mathrm{Re}\left\{\frac{f(z)}{f_n(z)}\right\}, \mathrm{Re}\left\{\frac{f_n(z)}{f(z)}\right\}, \mathrm{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\} \quad \textit{and} \quad \mathrm{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\}.$$

Theorem 4.1. *Let $f \in Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$ and satisfying (1.3), then*

$$\mathrm{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq 1 - \frac{1}{c_{n+1}}, \quad (n \in \mathbb{N}, \quad z \in \mathbb{U}), \tag{4.1}$$

and

$$\mathrm{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{1 + c_{n+1}}, \quad (n \in \mathbb{N}, \quad z \in \mathbb{U}), \tag{4.2}$$

where c_n be defined as

$$c_n = \frac{n\left[1 + \beta(2\gamma - 1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)}.$$

The results are sharp for every k with the function given by

$$f(z) = z - \frac{z^{n+1}}{c_{n+1}}, \quad (z \in \mathbb{U}, n \in \mathbb{N}). \tag{4.3}$$

Proof: In order to prove (4.1), it suffices to show that

$$\begin{aligned} c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{n+1}}\right)\right\} &= \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} c_{n+1} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} \\ &= \frac{1 + w(z)}{1 - w(z)}. \end{aligned} \tag{4.4}$$

Then

$$w(z) = \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} c_{n+1} a_k z^{k-1}}.$$

Notice that $w(0) = 0$ and

$$|w(z)| = \frac{c_{n+1} \sum_{k=n+1}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{n} |a_k| |z^{k-1} - \sum_{k=n+1}^{\infty} c_{n+1} |a_k|}.$$

Now $|w(z)| \leq 1$ if and only if

$$c_{n+1} \sum_{k=n+1}^{\infty} |a_k| + \sum_{k=2}^{n} |a_k| \leq 1. \tag{4.5}$$

It suffices to show that the LHS of (4.5) is bounded above by the condition (1.3) which is equivalent to

$$\sum_{k=n+1}^{\infty} (c_k - c_{n+1}) |a_k| + \sum_{k=2}^{n} (c_k - 1) |a_k| \geq 0.$$

To see that the function given by (4.3) gives the sharp result, we observe that for $z = re^{\frac{\pi i}{n}}$,

$$\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{c_{n+1}} \to 1 - \frac{1}{c_{n+1}}, \qquad \textit{when} \quad (z \to 1^-).$$

To prove the second part of this theorem, we write

$$(1 + c_{n+1}) \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{c_{n+1} + 1} \right\} = \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} c_{n+1} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} = \frac{1 + w(z)}{1 - w(z)}, \tag{4.6}$$

we find that

$$w(z) = \frac{\sum_{k=n+1}^{\infty} (1 + c_{n+1}) a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} (1 + c_{n+1}) |a_k z^{k-1}}.$$

Now $|w(z)| \leq 1$ if and only if

$$(1 + c_{n+1}) \sum_{k=n+1}^{\infty} |a_k| + \sum_{k=2}^{n} |a_k| \leq 1.$$

The equality holds in (4.2) for the extremal function f given by (4.3).

This completes the proof. $\square$

Theorem 4.2. *Let $f \in Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$ and satisfying (1.3), then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\} \geq 1 - \frac{n+1}{c_{n+1}}, \quad (z \in \mathbb{U}), \tag{4.7}$$

and

$$\operatorname{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{n+1+c_{n+1}}. \tag{4.8}$$

The results are sharp with the function given by (4.3).

Proof: To prove the result (4.7), define the function $w(z)$ by

$$c_{n+1}\left\{\frac{f'(z)}{f'_n(z)} - (1 - \frac{n+1}{c_{n+1}})\right\} \quad = \quad \frac{1+w(z)}{1-w(z)}.$$

Then

$$w(z) = \frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k}{2 + 2\sum_{k=2}^{n} ka_k z^{k-1} + \sum_{k=n+1}^{\infty} \frac{c_{n+1}}{n+1}k}.$$

Now $|w(z)| \leq 1$ if and only if

$$\left(\frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k|a_k| + \sum_{k=2}^{n} k|a_k| \leq 1.$$

¿From the condition (1.3), it suffices to show that

$$\left(\frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k|a_k| + \sum_{k=2}^{n} k|a_k| \leq c_k|a_k|.$$

This is equivalent to showing that

$$\sum_{k=2}^{n}(c_k - k)|a_k| + \sum_{k=n+1}^{\infty}\frac{(n+1)c_k - kc_{n+1}}{n+1}|a_k| \geq 0.$$

To prove the second part of this theorem, we write

$$\begin{aligned} w(z) &= (n+1+c_{n+1})\left\{\frac{f'_n(z)}{f'(z)} - (\frac{c_{n+1}}{n+1+c_{n+1}})\right\} \\ &= 1 - \frac{(1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} ka_k z^{k-1}}{1+\sum_{k=2}^{n} ka_k z^{k-1}}, \end{aligned}$$

yields

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{(1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k|a_k|}{2-2\sum_{k=2}^{n} k|a_k|-(1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k|a_k|} \leq 1, \quad (z \in \mathbb{U}),$$

if and only if

$$2(1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k|a_k| \leq 2-2\sum_{k=2}^{n} k|a_k|.$$

The bound in (4.8) is sharp for all $n \in \mathbb{N}$ with the extremal function (4.3).

This completes the proof of theorem. □

REFERENCES

1. N.M.Mustafa and M. Darus: *On a subclass of analytic functions with negative cofficients associated to an integral operator involving Hurwitz-Lerch zeta function*, "Vasile Alecsandri" University of Bacau, Faculty of Sviences, Scientific Studies and Research, Series Mathematics and Informatics, **21(2)** (2011), 45-56.
2. S. Owa and H. M. Srivastava: *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math, **39(5)** (1987), 1057-1077.
3. H. M. Srivastava and S. Owa: *An application of the fractional derivative*, Math Japon., **29(3)** (1984), 383-389.
4. H. M. Srivastava and A. A. Attiya: *An integral operator associated with the HurwitzLerch Zeta function and differential subordination*, Integral Transforms Spec. Funct., **18(3)** (2007), 207-216.
5. I.B.Jung, Y.C.Kim, and H.M.Srivastava: *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J.Math. Anal. Appl, **176** (1993), 138-147.
6. H.M. Srivastava and J.Choi: *Series Associated with the Zeta and Related Functions*, (Dordrecht, Boston and London: Kluwer Academic Publishers), (2001).
7. A. Schild and H.M. Srivastava: *Convolutions of Univalent functions with negative coefficients*, Proc. Ann. Univ. Mariae Curie-Skłodowska sect., **A(29)** (1975), 99-107.
8. J. E. Littlewood: *On inequalities in the theory of functions*, Proc. London Math. Soc., **2(23)**(1925), 481-519 .
9. H. Silverman: *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. one-parameter families of integral operators, **51**(1975), 109-116.
10. H. Silverman: *Partial sums of starlike and convex functions*, J. Math. Ana. Appl. **209**(1997), 221-227
11. N. E. Cho, I. H. Kim and H. M. Srivastava: *Sandwich-type theorems for multivalent functions associated with the Srivastava-Attiya operator*, Appl. Math. Comput., **217** (2010), 918-928.

12. G. Murugusundaramoorthy: *Subordination results for spiral-like functions associated with Srivastva- Attiya operator*, Integral Transforms Spec. Funct., **23(2)** (2012), 97-103.

Nagat.M.Mustafa, *Maslina Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia.
nmma1975@yahoo.com, *maslina@ukm.my (corresponding author)