

**CONTINUITY ON BESOV SPACES OF MULTILINEAR  
 COMMUTATOR OF SINGULAR INTEGRAL OPERATOR WITH  
 VARIABLE CALDERÓN-ZYGMUND KERNEL**

Tan Lu, Chuangxia Huang and Lanzhe Liu

**Abstract.** In this paper, we prove the continuity for the multilinear commutator associated to the singular integral operator with variable Calderón-Zygmund kernel on the Besov spaces.

### 1. Introduction

Since the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [10-13]). In [3,11,12], the authors proved that the commutators and multilinear operators generated by the singular integral operators and  $BMO$  functions are bounded on  $L^p(R^n)$  for  $1 < p < \infty$ . In [2,5,9], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on  $L^p(R^n)$  ( $1 < p < \infty$ ) and Triebel-Lizorkin spaces are obtained. In this paper, we will prove the continuity properties for the multilinear commutators related to the singular integral operator with variable Calderón-Zygmund kernel (see [1]) on Besov spaces.

### 2. Preliminaries

First, let us introduce some notations. Throughout this paper,  $Q$  will denote a cube of  $R^n$  with sides parallel to the axes. For a locally integrable function  $f$ , the sharp function of  $f$  is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows,  $f_Q = |Q|^{-1} \int_Q f(x) dx$ . It is well-known that (see [4,13,14])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

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Received June 3, 2012.; Accepted November 17, 2012.

2010 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B25

For  $\beta \geq 0$ , the Besov space  $\dot{\Lambda}_\beta(R^n)$  is the space of functions  $f$  such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left| \Delta_h^{[\beta]+1} f(x) \right| / |h|^\beta < \infty,$$

where  $\Delta_h^k$  denotes the  $k$ -th difference operator (see [9]).

For  $b_j \in \dot{\Lambda}_\beta(R^n)$  ( $j = 1, \dots, m$ ), set

$$\|\vec{b}\|_{\dot{\Lambda}_\beta} = \prod_{j=1}^m \|b_j\|_{\dot{\Lambda}_\beta}.$$

Given some functions  $b_j$  ( $j = 1, \dots, m$ ) and a positive integer  $m$  and  $1 \leq j \leq m$ , we denote by  $C_j^m$  the family of all finite subsets  $\sigma = \{\sigma(1), \dots, \sigma(j)\}$  of  $\{1, \dots, m\}$  of  $j$  different elements. For  $\sigma \in C_j^m$ , set  $\sigma^c = \{1, \dots, m\} \setminus \sigma$ . For  $\vec{b} = (b_1, \dots, b_m)$  and  $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$ , set  $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ ,  $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$  and  $\|\vec{b}_\sigma\|_{\dot{\Lambda}_\beta} = \|b_{\sigma(1)}\|_{\dot{\Lambda}_\beta} \cdots \|b_{\sigma(j)}\|_{\dot{\Lambda}_\beta}$ .

**Definition 2.1.** Let  $0 < p, q \leq \infty$ ,  $\alpha \in R$ . For  $k \in Z$ , set  $B_k = \{x \in R^n : |x| \leq 2^k\}$  and  $C_k = B_k \setminus B_{k-1}$ . Denote by  $\chi_k$  the characteristic function of  $C_k$  and  $\chi_0$  the characteristic function of  $B_0$ .

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha, p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha, p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}} = \left[ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p \right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha, p}(R^n) = \{f \in L_{loc}^q(R^n) : \|f\|_{K_q^{\alpha, p}} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha, p}} = \left[ \sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p + \|f\chi_{B_0}\|_{L^q}^p \right]^{1/p};$$

And the usual modification is made when  $p = q = \infty$ .

**Definition 2.2.** Let  $1 \leq q < \infty$ ,  $\alpha \in R$ . The central Campanato space is defined by (see [15])

$$CL_{\alpha, q}(R^n) = \{f \in L_{loc}^q(R^n) : \|f\|_{CL_{\alpha, q}} < \infty\},$$

where

$$\|f\|_{CL_{\alpha, q}} = \sup_{r>0} |B(0, r)|^{-\alpha} \left( \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(x) - f_{B(0, r)}|^q dx \right)^{1/q}.$$

In this paper, we will study some multilinear commutators as follows (see [1]):

**Definition 2.3.** Let  $K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \rightarrow R$ .  $K$  is said to be a Calderón-Zygmund kernel if

- (a)  $\Omega \in C^\infty(R^n \setminus \{0\})$ ;
- (b)  $\Omega$  is homogeneous of degree zero;
- (c)  $\int_{\Sigma} \Omega(x)x^\alpha d\sigma(x) = 0$  for all multi-indices  $\alpha \in (N \cup \{0\})^n$  with  $|\alpha| = N$ , where  $\Sigma = \{x \in R^n : |x| = 1\}$  is the unit sphere of  $R^n$ .

Let  $K(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \rightarrow R$ .  $K$  is said to be a variable Calderón-Zygmund kernel if

- (d)  $K(x, \cdot)$  is a Calderón-Zygmund kernel for a.e.  $x \in R^n$ ;
- (e)  $\max_{|\gamma| \leq 2n} \left\| \frac{\partial^{|\gamma|}}{\partial y^{|\gamma|}} \Omega(x, y) \right\|_{L^\infty(R^n \times \Sigma)} = M < \infty$ .

Suppose  $b_j$  ( $j = 1, \dots, m$ ) are the fixed locally integrable functions on  $R^n$ . Let  $T$  be the singular integral operator with variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where  $K(x, x - y) = \frac{\Omega(x, x - y)}{|x - y|^n}$  and that  $\Omega(x, y)/|y|^n$  is a variable Calderón-Zygmund kernel. The multilinear commutator of singular integral with variable Calderón-Zygmund kernel is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x, x - y) f(y) dy.$$

Note that when  $b_1 = \dots = b_m$ ,  $T_{\vec{b}}$  is just the  $m$  order commutator (see [11-12]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-3,5-12,16]). Our main purpose is to establish the continuity properties for the multilinear commutators related to the singular integral operator with variable Calderón-Zygmund kernel on Besov spaces.

We begin with the following lemmas (see [9]).

**Lemma 2.1.** For  $0 < \beta < 1, 1 \leq p \leq \infty$ , we have

$$\begin{aligned} \|b\|_{\dot{\Lambda}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p} \\ &\approx \sup_Q \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - c| dx \\ &\approx \sup_Q \inf_c \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |b(x) - c|^p dx \right)^{1/p}. \end{aligned}$$

**Lemma 2.2.** (see [15]) For  $\alpha < 0, 0 < q < \infty$ , we have

$$\|f\|_{\dot{K}_q^{\alpha}} \approx \sup_{\mu \in Z} 2^{\mu\alpha} \|f\chi_{B_\mu}\|_{L^q}.$$

**Lemma 2.3.** Let  $0 < \eta < n$ ,  $1 < p < n/\eta$ . Suppose  $b \in \dot{\Lambda}_\beta(R^n)$ , then

$$|b_{2^{k+1}B} - b_B| \leq C \|b\|_{\dot{\Lambda}_\beta} k |2^{k+1}B|^{\beta/n} \text{ for } k \geq 1.$$

*Proof.*

$$\begin{aligned} |b_{2^{k+1}B} - b_B| &\leq \sum_{j=0}^k |b_{2^{j+1}B} - b_{2^jB}| \leq \sum_{j=0}^k \frac{1}{|2^jB|} \int_{2^jB} |b(y) - b_{2^{j+1}B}| dy \\ &\leq C \sum_{j=0}^k \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^p dy \right)^{1/p} \\ &\leq C \|b\|_{\dot{\Lambda}_\beta} \sum_{j=0}^k |2^{j+1}B|^{\beta/n} \leq C \|b\|_{\dot{\Lambda}_\beta} k |2^{k+1}B|^{\beta/n}. \end{aligned}$$

□

**Lemma 2.4.** (see [6-7]) Let  $0 \leq \beta < 1$ ,  $1 < r < n/\beta$ ,  $1/r - 1/s = \beta/n$  and  $b_j \in \dot{\Lambda}_\beta(R^n)$  for  $j = 1, \dots, m$ . Then  $T_{\vec{b}}(f)(x)$  is bounded from  $L^r(R^n)$  to  $L^s(R^n)$ .

### 3. Theorems and Proofs

**Theorem 3.1.** Let  $0 < \beta < 1/m$  and  $b_j \in \dot{\Lambda}_\beta(R^n)$  for  $j = 1, \dots, m$ . Then  $T_{\vec{b}}$  is bounded from  $L^p(R^n)$  to  $\dot{\Lambda}_{m\beta/n-1/p}(R^n)$  for any  $p$  with  $n/m\beta \leq p < \infty$ .

*Proof.* It is only to prove that there exists a constant  $C_0$  such that

$$\frac{1}{|Q|^{1+m\beta/n-1/p}} \int_Q |T_{\vec{b}}(f)(x) - C_0| dx \leq C \|f\|_{L^p}.$$

Fix a cube  $Q$ ,  $Q = Q(x_0, d)$ , we decompose  $f$  into  $f = f_1 + f_2$  with  $f_1 = f\chi_{2Q}$ ,  $f_2 = f\chi_{(R^n \setminus 2Q)}$ .

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f)(x).$$

Then,

$$\begin{aligned} &|T_{b_1}(f)(x) - C_0| \\ &= \left| (b_1(x) - (b_1)_{2Q})T(f)(x) \right. \\ &\quad \left. + T(((b_1)_{2Q} - b_1)f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0) \right| \\ &\leq |(b_1(x) - (b_1)_{2Q})T(f)(x)| + |T(((b_1)_{2Q} - b_1)f_1)(x)| \\ &\quad + |T(((b_1)_{2Q} - b_1)f_2)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\ &= A(x) + B(x) + C(x). \end{aligned}$$

For  $A(x)$ , by the  $L^p$ -boundedness of  $T$  with  $1 < p < \infty$ , we obtain, using Hölder's inequality with  $1/p' + 1/p = 1$ ,

$$\begin{aligned} & \frac{1}{|2Q|^{1+\beta/n-1/p}} \int_{2Q} |A(x)| dx \\ &= \frac{1}{|2Q|^{1+\beta/n-1/p}} \int_{2Q} |(b_1(x) - (b_1)_{2Q})T(f)(x)| dx \\ &\leq C \frac{1}{|2Q|^{1+\beta/n-1/p}} \left( \int_{2Q} |(b_1(x) - (b_1)_{2Q})|^{p'} dx \right)^{1/p'} \left( \int_{2Q} |T(f)(x)|^p dx \right)^{1/p} \\ &\leq C \frac{|2Q|^{1/p'}}{|2Q|^{1-1/p}} \frac{1}{|2Q|^{\beta/n}} \left( \frac{1}{|2Q|} \int_{2Q} |(b_1(x) - (b_1)_{2Q})|^{p'} dx \right)^{1/p'} \\ &\quad \times \left( \int_{2Q} |f(x)|^p dx \right)^{1/p} \\ &\leq C \frac{|2Q|^{\beta/n+1/p'}}{|2Q|^{1+\beta/n-1/p}} \|b_1\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} \leq C \|b_1\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}. \end{aligned}$$

For  $B(x)$ , taking  $1 < r < p < \infty$  and  $p = rt$ , by Hölder's inequality, we have

$$\begin{aligned} & \frac{1}{|2Q|^{1+\beta/n-1/p}} \int_{2Q} |B(x)| dx \\ &= \frac{1}{|2Q|^{1+\beta/n-1/p}} \int_{2Q} |T((b_1)_{2Q} - (b_1)f_1)(x)| dx \\ &\leq C \frac{1}{|2Q|^{\beta/n-1/p}} \left( \frac{1}{|2Q|} \int_{R^n} |T((b_1(x) - (b_1)_{2Q})f\chi_{2Q})(x)|^r dx \right)^{1/r} \\ &\leq C \frac{1}{|2Q|^{\beta/n-1/p+1/r}} \left( \int_{2Q} |(b_1(x) - (b_1)_{2Q})f(x)|^r dx \right)^{1/r} \\ &\leq C \frac{1}{|2Q|^{\beta/n-1/p+1/r}} \left( \int_{2Q} |b_1(x) - (b_1)_{2Q}|^{rt'} dx \right)^{1/rt'} \\ &\quad \times \left( \int_{2Q} |f(x)|^{rt} dx \right)^{1/rt} \\ &\leq C \frac{|2Q|^{\beta/n+1/rt'}}{|2Q|^{\beta/n-1/p+1/r}} \|b_1\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} \leq C \|b_1\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}. \end{aligned}$$

For  $C(x)$ , by [1], we know that

$$T_{\vec{b}}(f)(x) = \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} a_{uv}(x) \int_{R^n} \frac{Y_{uv}(x-y)}{|x-y|^{n+m}} \prod_{j=1}^m (b_j(x) - b_j(y)) f(y) dy,$$

where  $g_u \leq Cu^{n-2}$ ,  $\|a_{uv}\|_{L^\infty} \leq Cu^{-2n}$ ,  $|Y_{uv}(x-y)| \leq Cu^{n/2-1}$  and

$$\left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| \leq Cu^{n/2} |x-x_0| / |x_0-y|^{n+1}$$

for  $|x - y| > 2|x_0 - x| > 0$ . Then we get

$$\begin{aligned}
C(x) &= \left| \int_{R^n} (K(x, x-y) - K(x_0, x_0-y))((b_1)_{2Q} - b_1(y))f_2(y)dy \right| \\
&\leq C \int_{(2Q)^c} \left| \frac{\Omega(x, x-y)}{|x-y|^n} - \frac{\Omega(x_0, x_0-y)}{|x_0-y|^n} \right| |(b_1)_{2Q} - b_1(y)| |f(y)| dy \\
&\leq C \sum_{l=1}^{\infty} \int_{2^{l+1}Q \setminus 2^l Q} \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} |a_{uv}(x)| \int_{R^n} \left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| \\
&\quad \times |(b_1)_{2Q} - b_1(y)| |f(y)| dy \\
&\leq C \sum_{u=1}^{\infty} u^{-2n} \cdot u^{n/2} \sum_{l=1}^{\infty} \int_{2^{l+1}Q \setminus 2^l Q} \frac{|x-x_0|}{|x_0-y|^{n+1}} |(b_1)_{2Q} - b_1(y)| |f(y)| dy \\
&\leq C \sum_{u=1}^{\infty} u^{-3n/2} \sum_{l=1}^{\infty} \frac{r}{(2^l r)^{n+1}} \int_{2^{l+1}Q} |(b_1)_{2Q} - b_1(y)| |f(y)| dy \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} \frac{1}{|2^{l+1}Q|} \left( \int_{2^{l+1}Q} |b_1(y) - (b_1)_{2Q}|^{p'} dy \right)^{\frac{1}{p'}} \left( \int_{2^{l+1}Q} |f(y)|^p dy \right)^{\frac{1}{p}} \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} \frac{1}{|2^{l+1}Q|} \left[ \left( \int_{2^{l+1}Q} |b_1(y) - (b_1)_{2^{l+1}Q}|^{p'} dy \right)^{\frac{1}{p'}} \right. \\
&\quad \left. + |(b_1)_{2^{l+1}Q} - (b_1)_{2Q}| |2^{l+1}Q|^{\frac{1}{p'}} \right] \|f\|_{L^p} \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} \frac{1}{|2^{l+1}Q|} \left[ \frac{1}{|2^{l+1}Q|^{\beta/n}} \left( \frac{1}{|2^{l+1}Q|} \int_{2^{l+1}Q} |b_1(y) - (b_1)_{2^{l+1}Q}|^p dy \right)^{\frac{1}{p'}} \right. \\
&\quad \left. \times |2^{l+1}Q|^{\beta/n+\frac{1}{p'}} + |(b_1)_{2^{l+1}Q} - (b_1)_{2Q}| |2^{l+1}Q|^{\frac{1}{p'}} \right] \|f\|_{L^p} \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} \frac{1}{|2^{l+1}Q|} \left[ |2^{l+1}Q|^{\beta/n+\frac{1}{p'}} \|b_1\|_{\dot{\Lambda}_{\beta}} + l \|b_1\|_{\dot{\Lambda}_{\beta}} |2^{l+1}Q|^{\beta/n+\frac{1}{p'}} \right] \|f\|_{L^p} \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} |2^{l+1}Q|^{\beta/n-\frac{1}{p}} \|b_1\|_{\dot{\Lambda}_{\beta}} \|f\|_{L^p} \\
&\leq C |2Q|^{\beta/n-\frac{1}{p}} \|b_1\|_{\dot{\Lambda}_{\beta}} \|f\|_{L^p},
\end{aligned}$$

thus

$$\begin{aligned}
\frac{1}{|2Q|^{1+\beta/n-\frac{1}{p}}} \int_{2Q} C(x) dx &\leq C \frac{1}{|2Q|^{1+\beta/n-\frac{1}{p}}} |2Q|^{\beta/n-\frac{1}{p}} \|b_1\|_{\dot{\Lambda}_{\beta}} \|f\|_{L^p} |2Q| \\
&\leq C \|b_1\|_{\dot{\Lambda}_{\beta}} \|f\|_{L^p}.
\end{aligned}$$

This completes the case  $m = 1$ .

Now, we consider the **Case**  $m \geq 2$ . For  $b = (b_1, \dots, b_m)$ , we have,

$$\begin{aligned}
T_{\vec{b}}(f)(x) &= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})] K(x, x-y) f(y) dy \\
&= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_{\sigma^c} K(x, x-y) f(y) dy \\
&= \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) T(f)(x) + (-1)^m T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} ((b_j(x) - (b_j)_{2Q})_\sigma) T((b_j - (b_j)_{2Q})_{\sigma^c} f)(x),
\end{aligned}$$

thus, recall that  $C_0 = T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2)(x_0)$ ,

$$\begin{aligned}
&|T_{\vec{b}}(f)(x) - T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2)(x_0)| \\
&\leq | \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) T(f)(x) | + | T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_1)(x) | \\
&\quad + | \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ((b_j(x) - (b_j)_{2Q})_\sigma) T((b_j - (b_j)_{2Q})_{\sigma^c} f)(x) | \\
&\quad + | T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2)(x) - T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2)(x_0) | \\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

For  $I_1(x)$ , by the  $L^p$ -boundedness of  $T$  with  $1 < p < \infty$ , we obtain, using Hölder's inequality with  $1/s_1 + \dots + 1/s_m + 1/p = 1$ ,

$$\begin{aligned}
&\frac{1}{|2Q|^{1+m\beta/n-1/p}} \int_{2Q} |I_1(x)| dx \\
&= \frac{1}{|2Q|^{1+m\beta/n-1/p}} \int_{2Q} \left| (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) T(f)(x) \right| dx \\
&\leq C \frac{1}{|2Q|^{1+m\beta/n-1/p}} \prod_{j=1}^m \left( \int_{2Q} |(b_j(x) - (b_j)_{2Q})|^{s_j} dx \right)^{1/s_j} \\
&\quad \times \left( \int_{2Q} |T(f)(x)|^p dx \right)^{1/p} \\
&\leq C \frac{|2Q|^{m\beta/n+1/s_1+\dots+s_m}}{|2Q|^{1+m\beta/n-1/p}} \prod_{j=1}^m \frac{1}{|2Q|^{m\beta/n}} \left( \frac{1}{|2Q|} \int_{2Q} |(b_1(x) - (b_1)_{2Q})|^{s_j} dx \right)^{1/s_j}
\end{aligned}$$

$$\begin{aligned} & \times \left( \int_{2Q} |f(x)|^p dx \right)^{1/p} \\ & \leq C \frac{|2Q|^{m\beta/n+1/s_1+\dots+s_m}}{|2Q|^{1+m\beta/n-1/p}} \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} \leq C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}. \end{aligned}$$

For  $I_2(x)$ , taking  $1 < r < p < \infty$ ,  $p = rt$  and  $1/t_1 + \dots + 1/t_m + 1/t = 1$ , by Hölder's inequality, we have

$$\begin{aligned} & \frac{1}{|2Q|^{1+m\beta/n-1/p}} \int_{2Q} |I_2(x)| dx \\ & = \frac{1}{|2Q|^{1+m\beta/n-1/p}} \int_{2Q} |T((b_1 - (b_1)_{2Q}) \dots (b_m - (b_m)_{2Q}) f_1)(x)| dx \\ & \leq \frac{1}{|2Q|^{m\beta/n-1/p}} \left( \frac{1}{|2Q|} \int_{R^n} |T((b_1 - (b_1)_{2Q}) \dots (b_m - (b_m)_{2Q}) f \chi_{2Q})(x)|^r dx \right)^{1/r} \\ & \leq \frac{1}{|2Q|^{m\beta/n-1/p+1/r}} \left( \int_{2Q} |((b_1 - (b_1)_{2Q}) \dots (b_m - (b_m)_{2Q})) f(x)|^r dx \right)^{1/r} \\ & \leq \frac{1}{|2Q|^{m\beta/n-1/p+1/r}} \prod_{j=1}^m \left( \int_{2Q} |b_j(x) - (b_j)_{2Q}|^{rt_j} dx \right)^{1/rt_j} \left( \int_{2Q} |f(x)|^{rt} dx \right)^{1/rt} \\ & \leq \frac{|2Q|^{m\beta/n+1/rt_1+\dots+1/rt_m}}{|2Q|^{m\beta/n-1/p+1/r}} \\ & \quad \times \prod_{j=1}^m \frac{1}{|2Q|^{m\beta/n}} \left( \frac{1}{|2Q|} \int_{2Q} |b_j(x) - (b_j)_{2Q}|^{rt_j} d\mu(x) \right)^{1/rt_j} \|f\|_{L^p} \\ & \leq C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}. \end{aligned}$$

For  $I_3(x)$ , taking  $1 < r < p < \infty$ ,  $p = rt$ , and denote  $1 = \sum \frac{1}{t_i}$ , where  $\sigma(i) \in \sigma$ ,  $\frac{1}{q} = \sum \frac{1}{s_k}$  and  $\sigma(k) \in \sigma^c$ , let  $\frac{1}{q} + \frac{1}{t} = 1$ ,  $\lambda_1 + \lambda_2 = m$ , by the boundedness of  $T$  and Hölder's inequality, we have

$$\begin{aligned} & \frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}} \int_{2Q} I_3(x) dx \\ & \leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}} \int_{2Q} |(b_j(x) - (b_j)_{2Q})_\sigma T((b_j - (b_j)_{2Q})_{\sigma^c} f)(x)| dx \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}} \left( \int_{2Q} |(b_j(x) - (b_j)_{2Q})_\sigma|^{r'} dx \right)^{\frac{1}{r'}} \\ & \quad \times \left( \int_{2Q} |T(b_j - (b_j)_{2Q})_{\sigma^c}(f)(x)|^r dx \right)^{\frac{1}{r}} \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}} |2Q|^{\frac{\lambda_1 \beta}{n} - \beta} \prod_i |2Q|^{\sum \frac{1}{rt_i}} \end{aligned}$$

$$\begin{aligned}
& \times \left( \frac{1}{|2Q|} \int_{2Q} |(b_j(x) - (b_j)_{2Q})_\sigma|^{r't_i} dx \right)^{\frac{1}{r't_i}} \\
& \times \left( \int_{2Q} |(b_j - (b_j)_{2Q})_{\sigma^c}(f)(x)|^r dx \right)^{\frac{1}{r}} \\
\leq & C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}} |2Q|^{\frac{\lambda_1\beta}{n} + \sum \frac{1}{r't_i}} \|b_\sigma\|_{\dot{\Lambda}_\beta} \left( \int_{2Q} |(f)(x)|^{rt} dx \right)^{\frac{1}{rt}} \\
& \times \prod_k |2Q|^{\frac{\lambda_2\beta}{n} + \sum \frac{1}{r's_k}} \frac{1}{|2Q|^\beta} \left( \frac{1}{|2Q|} \int_{2Q} |(b_j - (b_j)_{2Q})_{\sigma^c}|^{r's_k} dx \right)^{\frac{1}{r's_k}} \\
\leq & C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|^{1+\frac{m\beta}{n}-\frac{1}{p}}} |2Q|^{\frac{\lambda_1\beta}{n} + \frac{1}{r'q}} \|b_\sigma\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} |2Q|^{\frac{\lambda_2\beta}{n} + \frac{1}{rq}} \|b_{\sigma^c}\|_{\dot{\Lambda}_\beta} \\
\leq & C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

For  $I_4(x)$ , set  $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} + \frac{1}{p} = 1$ , using Hölder's inequality and Lemma 3, we have

$$\begin{aligned}
& \left| T \left( \prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2 \right) (x) - T \left( \prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2 \right) (x_0) \right| \\
\leq & \int_{(2Q)^c} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \|f(y)\| K(x, x-y) - K(x_0, x_0-y) \right| dy \\
\leq & C \int_{(2Q)^c} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| \left| \frac{\Omega(x, x-y)}{|x-y|^n} - \frac{\Omega(x_0, x_0-y)}{|x_0-y|^n} \right| dy \\
\leq & C \sum_{l=1}^{\infty} \int_{2^{l+1}Q \setminus 2^lQ} \left[ \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} |a_{uv}(x)| \int_{R^n} \left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| \right] \\
& \times \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| dy \\
\leq & C \sum_{u=1}^{\infty} u^{-2n} \cdot u^{n/2} \sum_{l=1}^{\infty} \int_{2^{l+1}Q \setminus 2^lQ} \frac{|x-x_0|}{|x_0-y|^{n+1}} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| dy \\
\leq & C \sum_{u=1}^{\infty} u^{-3n/2} \sum_{l=1}^{\infty} \frac{r}{(2^l r)^{n+1}} \int_{2^{l+1}Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| dy \\
\leq & C \sum_{l=1}^{\infty} 2^{-l} \frac{1}{|2^{l+1}Q|} \prod_{j=1}^m \left[ \left( \int_{2^{l+1}Q} |b_j(y) - (b_j)_{2^{l+1}Q}|^{p'} dy \right)^{\frac{1}{p'}} \right. \\
& \quad \left. + |(b_j)_{2^{l+1}Q} - (b_j)_{2Q}| |2^{l+1}Q|^{\frac{1}{p'}} \right] \|f\|_{L^p}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{l=1}^{\infty} 2^{-l} \frac{1}{|2^{l+1}Q|} \prod_{j=1}^m \left[ \frac{1}{|2^{l+1}Q|^{m\beta/n}} \left( \frac{1}{|2^{l+1}Q|} \int_{2^{l+1}Q} |b_j(y) - (b_j)_{2^{l+1}Q}|^{p'} dy \right)^{\frac{1}{p'}} \right. \\
&\quad \times |2^{l+1}Q|^{m\beta/n + \frac{1}{p'}} + |(b_j)_{2^{l+1}Q} - (b_j)_{2Q}| |2^{l+1}Q|^{\frac{1}{p'}} \left. \right] \|f\|_{L^p} \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} |2Q|^{m\beta/n - \frac{1}{p}} \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} \leq C |2Q|^{m\beta/n - \frac{1}{p}} \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{|2Q|^{1+m\beta/n - \frac{1}{p}}} \int_{2Q} I_4(x) dx &\leq C \frac{1}{|2Q|^{1+m\beta/n - \frac{1}{p}}} |2Q|^{m\beta/n - \frac{1}{p}} \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p} |2Q| \\
&\leq C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}.
\end{aligned}$$

This completes the proof of Theorem 1.  $\square$

**Theorem 3.2.** *Let  $0 < \beta < 1/2m$ ,  $1 < q_1 < n/m\beta$ ,  $1/q_2 = 1/q_1 - m\beta/n$ ,  $-n/q_2 - 1/2 < \alpha \leq -n/q_2$  and  $b_j \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  for  $j = 1, \dots, m$ . Then  $T_{\vec{b}}$  is bounded from  $\dot{K}_{q_1}^{\alpha, \infty}(\mathbb{R}^n)$  to  $CL_{-\alpha/n-1/q_2, q_2}(\mathbb{R}^n)$ .*

**Remark 3.1.** Theorem 2 also holds for the nonhomogeneous Herz type Hardy space.

*Proof.* Fix a ball  $B = B(0, w)$ , there exists  $\epsilon_0 \in \mathbf{Z}$  such that  $2^{\epsilon_0-1} \leq w < 2^{\epsilon_0}$ . We choose  $x_0$  such that  $2w < |x_0| < 3w$ . It is only to prove that

$$|B_{\epsilon_0}|^{\alpha + \frac{n}{q_2}} \left( \frac{1}{|B_{\epsilon_0}|} \int_{B_{\epsilon_0}} |T_{\vec{b}}(f)(x) - T_{\vec{b}}(f_2)(x_0)|^{q_2} dx \right)^{\frac{1}{q_2}} \leq C \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}.$$

We write  $f_1 = f \chi_{4B_{\epsilon_0}}$  and  $f_2 = f \chi_{R^n \setminus 4B_{\epsilon_0}}$ , then

$$\begin{aligned}
&|T_{\vec{b}}(f)(x) - T_{\vec{b}}(f_2)(x_0)| \\
&\leq |T_{\vec{b}}(f_1)(x)| + |T_{\vec{b}}(f_2)(x) - T_{\vec{b}}(f_2)(x_0)| \\
&\leq |T_{\vec{b}}(f_1)(x)| \\
&\quad + \left| T \left( \prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right)(x) - T \left( \prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right)(x_0) \right| \\
&\quad + \left| \prod_{j=1}^m (b_j(x) - (b_j)_Q) \right| |T(f_2)(x) - T(f_2)(x_0)|.
\end{aligned}$$

So,

$$|B_{\epsilon_0}|^{\alpha + \frac{n}{q_2}} \left( \frac{1}{|B_{\epsilon_0}|} \int_{B_{\epsilon_0}} |T_{\vec{b}}(f)(x) - T_{\vec{b}}(f_2)(x_0)|^{q_2} dx \right)^{\frac{1}{q_2}}$$

$$\begin{aligned}
&\leq |B_{\epsilon_0}|^{\alpha+\frac{n}{q_2}} \left( \frac{1}{|B_{\epsilon_0}|} \int_{B_{\epsilon_0}} |T_b(f_1)(x)|^{q_2} dx \right)^{\frac{1}{q_2}} \\
&\quad + |B_{\epsilon_0}|^{\alpha+\frac{n}{q_2}} \left( \frac{1}{|B_{\epsilon_0}|} \int_{B_{\epsilon_0}} \left| T \left( \prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right) (x) \right. \right. \\
&\quad \quad \quad \left. \left. - T \left( \prod_{j=1}^m (b_j - (b_j)_Q) f_2 \right) (x_0) \right|^{q_2} dx \right)^{\frac{1}{q_2}} \\
&\quad + |B_{\epsilon_0}|^{\alpha+\frac{n}{q_2}} \left( \frac{1}{|B_{\epsilon_0}|} \int_{B_{\epsilon_0}} \left| \prod_{j=1}^m (b_j(x) - (b_j)_Q) \right| |T(f_2)(x) - T(f_2)(x_0)|^{q_2} dx \right)^{\frac{1}{q_2}} \\
&= E_1 + E_2 + E_3.
\end{aligned}$$

For  $E_1$ , by Lemma 2 and Lemma 4, we get

$$\begin{aligned}
E_1 &\leq C |B_{\epsilon_0}|^{\alpha+\frac{n}{q_2}-\frac{n}{q_1}} \left( \int_{B_{\epsilon_0}} |f_1(x)|^{q_1} dx \right)^{\frac{1}{q_1}} \\
&\leq C |B_{\epsilon_0}|^\alpha \|f \chi_{B_{\epsilon_0}}\|_{L^{q_1}} \leq C \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}.
\end{aligned}$$

For  $E_2$ , similar to the estimates of  $I_4(x)$  in Theorem 1, let  $\frac{1}{p_1} + \dots + \frac{1}{p_m} + \frac{1}{q_1} = 1$ , by Hölder's inequality and the Minkowski's inequality, we obtain

$$\begin{aligned}
&\left| T \left( \prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2 \right) (x) - T \left( \prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2 \right) (x_0) \right| \\
&\leq C \sum_{l=1}^{\infty} \int_{C_{\epsilon_0+l}} \left| \prod_{j=1}^m (b_j - (b_j)_{2Q}) \right| |f(y)| |K(x, x-y) - K(x_0, x_0-y)| dy \\
&\leq C \sum_{l=1}^{\infty} \int_{B_{\epsilon_0+l}/B_{\epsilon_0+l-1}} \left[ \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} |a_{uv}(x)| \int_{R^n} \left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| \right] \\
&\quad \times \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| dy \\
&\leq C \sum_{u=1}^{\infty} u^{-3n/2} \sum_{l=1}^{\infty} \frac{1}{|B_{\epsilon_0+l}|} \int_{B_{\epsilon_0+l}} |f(y)| \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| dy \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} \frac{|B_{\epsilon_0+l}|^{m\beta + \frac{n}{p_1} + \dots + \frac{n}{p_m}}}{|B_{\epsilon_0+l}|} \prod_{j=1}^m \frac{1}{|B_{\epsilon_0+l}|^{m\beta/n}} \\
&\quad \times \left( \frac{1}{|B_{\epsilon_0+l}|} \int_{B_{\epsilon_0+l}} |(b_j(y) - (b_j)_{2Q})|^{p_j} dy \right)^{\frac{1}{p_j}} \left( \int_{B_{\epsilon_0+l}} |f(y)|^{q_1} dy \right)^{\frac{1}{q_1}} \\
&\leq C \sum_{l=1}^{\infty} 2^{-l} |B_{\epsilon_0+l}|^{m\beta - \frac{n}{q_1}} \prod_{j=1}^m \|b_j\|_{\dot{\Lambda}_\beta} \|f \chi_{B_{\epsilon_0}}\|_{L^{q_1}}
\end{aligned}$$

$$\begin{aligned} &\leq C \sum_{l=1}^{\infty} 2^{-l} |B_{\epsilon_0+l}|^{m\beta - \frac{n}{q_1} - \alpha} \|\vec{b}\|_{\dot{\Lambda}_\beta} |B_{\epsilon_0+l}|^\alpha \|f \chi_{B_{\epsilon_0}}\|_{L^{q_1}} \\ &\leq C |B_{\epsilon_0+l}|^{-\frac{n}{q_2} - \alpha} \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}, \end{aligned}$$

thus

$$\begin{aligned} E_2 &\leq C |B_{\epsilon_0}|^{\alpha + \frac{n}{q_2}} |B_{\epsilon_0+l}|^{-\frac{n}{q_2} - \alpha} \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}} \\ &\leq C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}. \end{aligned}$$

For  $E_3$ , with the same method as above, let  $\frac{1}{d_1} + \dots + \frac{1}{d_m} = 1$ , using Lemma 2, we have

$$\begin{aligned} |T(f_2)(x) - T(f_2)(x_0)| &\leq \int_{B_{\epsilon_0}} |K(x, x-y) - K(x_0, x_0-y)| |f(y)| dy \\ &\leq C \frac{1}{|B_{\epsilon_0}|} \|f \chi_{B_{\epsilon_0}}\|_{L^{q_1}} |B_{\epsilon_0}|^{1-\frac{n}{q_1}} \\ &\leq C |B_{\epsilon_0}|^{-\frac{n}{q_1} - \alpha} |B_{\epsilon_0}|^\alpha \|f \chi_{B_{\epsilon_0}}\|_{L^{q_1}} \\ &\leq C |B_{\epsilon_0}|^{-\frac{n}{q_1} - \alpha} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}, \end{aligned}$$

thus

$$\begin{aligned} E_3 &\leq |B_{\epsilon_0}|^{\alpha + \frac{n}{q_2}} |B_{\epsilon_0}|^{-\frac{n}{q_1} - \alpha} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}} |B_{\epsilon_0}|^{-\frac{n}{q_2}} \\ &\quad \times \left( \int_{B_{\epsilon_0}} \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \right|^{q_2} dx \right)^{\frac{1}{q_2}} \\ &\leq C |B_{\epsilon_0}|^{-\frac{n}{q_1}} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}} |B_{\epsilon_0}|^{m\beta + \frac{n}{q_2}} \\ &\quad \times \prod_{j=1}^m \frac{1}{|B_{\epsilon_0}|^{m\beta/n}} \left( \frac{1}{|B_{\epsilon_0}|} \int_{B_{\epsilon_0}} |(b_j(x) - (b_j)_{2Q})|^{q_2 d_j} dx \right)^{\frac{1}{q_2 d_j}} \\ &\leq C \|\vec{b}\|_{\dot{\Lambda}_\beta} \|f\|_{\dot{K}_{q_1}^{\alpha, \infty}}. \end{aligned}$$

This completes the proof of Theorem 2.  $\square$

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Tan Lu, Chuangxia Huang and Lanzhe Liu  
 College of Mathematics  
 Changsha University of Science and Technology  
 Changsha 410077, P. R. of China  
[lanzheliu@163.com](mailto:lanzheliu@163.com)