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COMMON FIXED POINT THEOREMS IN G^* MENGER PM-SPACES

M. Alamgir Khan, Sumitra and Sunny Chauhan

Abstract. The aim of this paper is to do probable modifications in the definition of generalized Menger PM-space introduced by Chugh, Kumar and Vats [Common fixed point theorems in generalized Menger PM-space. Math. Sci. Res. J. 7(2) (2003), 41–48] and prove some common fixed point theorems for weakly commuting mappings.

1. Introduction

There have been a number of generalizations of metric spaces. One of such generalizations is generalized metric space (or D-metric space) initiated by Dhage [3] in 1992. Dealing with D-metric space, Ahmad, Ashraf and Rhoades [1], Dhage [3, 4], Dhage, Pathan and Rhoades [5], Rhoades [12] and others made a significant contribution in fixed point theory of D-metric space.

Unfortunately, Dhage's theory of D-metric space was fundamentally flawed and almost all theorems in D-metric space are not valid (see [9, 10, 11]). Hence the need aroused to address these flaws in D-metric space. Therefore, several mathematicians started work in this direction. In 2004, Mustafa and Sims [7] first studied the theory of D-metric spaces and came up with a new generalization of metric spaces, which they called a G-metric space. Afterwards, in 2006, an attempt to remove the fundamental flaws of D-metric space, Sedghi and Shobe [13, 14] introduced the notion of D^*-metric space by modifying the definition of D-metric space.

As fixed point theory is the hottest area of research these days, therefore, when Dhage introduced the concept of D-metric space, a spate of papers came on it and several other structures were defined in framework of D-metric spaces. One such structure was generalized probabilistic metric space (briefly, GPM-space) studied by Chugh, Kumar and Vats [2]. Since the theory of GPM-spaces is mainly based on D-metric space, so the need arises to do probable modifications in its definition and other concepts. We give some examples to support newly defined definitions. Some common fixed point theorems for weakly commuting mappings are also obtained.

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2. Preliminaries

Definition 2.1. A t-norm is a function $\triangle : [0,1] \times [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each coordinate and $\triangle(a,1,1) = a$ for all $a \in [0,1]$.

Definition 2.2. Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X,F) is said to be a GPM-space if F is a mapping from $X \times X \times X \to D$, where the value of F at $(x,y,z) \in X \times X \times X$ is represented by $F_{x,y,z}$ or $F(x,y,z)$, for all $x,y,z,a \in X$ and $t_1,t_2,t_3 \in \mathbb{R}^+$ such that

1. $F(x,y,z;0) = 0$;
2. For distinct $x,y \in X$, there exists a point $z \in X$ such that $F(x,y,z;t) < 1$;
3. $F(x,y,z;t) = 1 \Leftrightarrow x = y = z$;
4. $F(x,y,z;t) = F\,(p\{x,y,z\};t)$, where p is a permutation function.
5. If $F(x,y,a;t_1) = F(x,a,z;t_2) = F(a,y,z;t_3) = 1$ then $F(x,y,z;\,(t_1+t_2+t_3)) = 1$.

Definition 2.3. A generalized Menger PM-space (briefly GMPM-space) is an ordered triplet $(X,F,\triangle)$, where (X,F) is a GPM-space and $\triangle$ is a t-norm satisfying the following condition:

(6) $F\,(x,y,z;(t_1+t_2+t_3)) \geq \triangle\,(F(x,y,a;t_1), F(x,a,z;t_2), F(a,y,z;t_3))$,
for all $x,y,z,a \in X$ and $t_1,t_2,t_3 \in \mathbb{R}^+$.

Using the concept of G-metric space introduced by Mustfa and Sims [8], D^*-metric space and M-fuzzy metric space introduced by Sedghi and Shobe [13, 14], we suggest some modifications in the definition of GMPM-space studied by Chugh, Kumar and Vats [2] and call it G^*MPM-space.

Definition 2.4. A G^*MPM-space is an ordered triplet $(X,F,\triangle)$, where X is a non-empty set, F is a mapping from $X \times X \times X \to D$ and $\triangle$ is a t-norm. The function $F(x,y,z)$ is assumed to satisfy the properties (1)-(4) of Definition 2.2 and the following conditions:

$(5)^*$ If $F(x,y,a;t_1) = 1 = F(a,z,z;t_2)$ then, $F(x,y,z;t_1+t_2) = 1$.

$(6)^*$ $F(x,y,z;t_1+t_2) \geq \triangle\,[F(x,y,a;t_1), F(a,z,z;t_2)]$.

Remark 2.1. Let $(X,F,\triangle)$ be a G^*MPM-space. Now we prove that for every $t > 0$, $F(x,x,y;t) = F(x,y,y;t)$.

For each $\epsilon > 0$ by triangular inequality, we have

$$F(x,x,y;\epsilon+t) \geq \triangle\left(F(x,x,x;\epsilon), F(x,y,y;t)\right) = \triangle\left(1, F(x,y,y;t)\right) = F(x,y,y;t). \tag{2.1}$$

$$F(y,y,x;\epsilon+t) \geq \triangle\left(F(y,y,y;\epsilon), F(y,x,x;t)\right) = \triangle\left(1, F(y,x,x;t)\right) = F(y,x,x;t). \tag{2.2}$$

Taking limit as $\epsilon \to 0$ in inequalities (2.1)-(2.2), we get

$$F(x,x,y;t) = F(x,y,y;t).$$

Let $(X,F,\triangle)$ be a G^*MPM-space. For $t > 0$, the open ball $B_F(x,r;t)$ with center $x \in X$ and $0 < r < 1$ is defined by

$$B_F(x,r;t) = \{y \in X; F(x,y,y;t) > 1-r\}.$$

A subset A of X is called open set if for each $x \in X$, there exist $t > 0$ and $0 < r < 1$ such that $B_F(x,r;t) \subseteq A$. A sequence $\{x_n\}$ in X converges to a point x if and only if $F(x,x,x_n;t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $F(x_n,x_n,x_m;t) \to 1-\epsilon$, for each $n,m \geq n_0$

Example 2.1. Let $X = [0,1]$ and D^* be the D^* metric on X defined as

$$D^*(x,y,z) = \mid x-y \mid + \mid y-z \mid + \mid z-x \mid,$$

and $\triangle(a,b,c) = \min\{a,b,c\}$. Define $F(x,y,z;t) = \frac{t}{t+D^*(x,y,z)}$, for all $x,y,z \in X$ and $t > 0$. Then $(X,F,\triangle)$ is a G^*MPM-space. As conditions (1)-(4) of Definition 2.2 are obvious.

Let us verify the conditions $(5)^*$ and $(6)^*$.

$(5)^*$ Let $F(x,y,a;t_1) = 1 = F(a,z,z;t_2)$. We claim that $F(x,y,z;t_1+t_2) = 1$.

Now $F(x,y,a;t_1) = 1 \Leftrightarrow D^*(x,y,a) = 0 \Leftrightarrow x = y = a$ and $F(a,z,z;t_2) = 1 \Leftrightarrow D^*(a,z,z) = 0 \Leftrightarrow z = a$. Thus we get $x = y = a = z \Rightarrow F(x,y,z;t_1+t_2) = 1$.

Similarly condition $(6)^*$ can be easily verified, hence the details are avoided.

Example 2.2. Let $X = [0,1]$ and D^* be the D^* metric on X defined as

$$D^*(x,y,z) = \mid x-y \mid + \mid y-z \mid + \mid z-x \mid,$$

and $\triangle(a,b,c) = \min\{a,b,c\}$. Define $F(x,y,z;t) = H\left(t - D^*(x,y,z)\right)$, for all $x,y,z \in X$ and $t > 0$, where H is the distribution function defined by

$$H(t) = \begin{cases} 0, & \text{If } t \le 0; \\ 1, & \text{If } t > 0. \end{cases}$$

Then $(X, F, \triangle)$ is a G^*MPM-space.

Lemma 2.1. *Let $(X, F, \triangle)$ be a GMPM-space. If we define $F : X^3 \times (0, \infty) \to [0, 1]$ by $F(x, y, z; t) = \triangle[F(x, y; t), F(y, z; t), F(z, x; t)]$ for every $x, y, z \in X$ then $(X, F, \triangle)$ is a G^*MPM-space.*

Proof. 1. It is easy to see that for every $x, y, z \in X$, $t > 0$, $F(x, y, z; t) > 0$.

2. $F(x, y, z; t) = F(x, y; t) = F(y, z; t) = F(z, x; t) = 1 \Leftrightarrow x = y = z$.

3. $F(x, y, z; t) = F(p\{x, y, z\}; t)$, where p is a permutation function.

4. Let $F(x, y, a; t) = 1 = F(a, z, z; s)$.

Now

$$F(x, y, a; t) = \triangle[F(x, y; t), F(y, a; t), F(a, x; t)] = 1 \Leftrightarrow x = y = a, \tag{2.3}$$

and

$$F(a, z, z; s) = \triangle[F(a, z; s), F(z, z; s), F(z, a; s)] = 1 \Leftrightarrow z = a. \tag{2.4}$$

From conditions (2.3) and (2.4), we get $x = y = z = a$. Thus $F(x, y, z; t + s) = 1$.

Let us check Menger's inequality

$$F(x, y, z; t + s) \ge \triangle[F(x, y, a; t), F(a, z, z; s)].$$

Now

$$\begin{aligned} F(x, y, z; t + s) &= \triangle[F(x, y; t + s), F(y, z; t + s), F(z, x; t + s)] \\ &\ge \triangle[F(x, y; t), F(y, a; t), F(a, z; s), F(z, a; s), F(a, x; t)] \\ &= \triangle[F(x, y, a; t), F(a, z; s), F(z, a; s), F(z, z; s)] \\ &= \triangle[F(x, y, a; t), F(a, z, z; s)]. \end{aligned}$$

□

Definition 2.5. Let $(X, F, \triangle)$ be a G^*MPM-space. If we define $F(x, y, z; t) = \frac{t}{t + D^*(x, y, z)}$, where $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ then $F(x, y, z; t)$ is of first type.

On the other hand, if we define $F(x, y; t) = \frac{t}{t + d(x, y)}$ then $F(x, y, z; t) = \triangle[F(x, y; t), F(y, z; t), F(z, x; t)]$ is of second type.

Remark 2.2. Let $(X, F, \triangle)$ be a G^*MPM-space, where F is of second type. Then a sequence $\{x_n\}$ in X converges to a point $x \Leftrightarrow F(x, x, x_n; t) \to 1$ or $\Leftrightarrow F(x, x_n; t) \to 1$ for $F(x, x, x_n; t) = \triangle[F(x, x; t), F(x, x_n; t), F(x, x_n; t)] = \triangle[1, F(x, x_n; t), F(x, x_n; t)]$.

Definition 2.6. Two self mappings A and S of a G^*MPM-space $(X, F, \triangle)$ are called weakly commuting if $F(ASx, SAx, SAx; t) \geq F(Ax, Sx, Sx; t)$ for all $x \in X$.

Clearly a commuting pair is weakly commuting but the converse is not true.

Lemma 2.2. *Let $(X, F, \triangle)$ be a G^*MPM-space, $F(x, y, z; t)$ is non-decreasing with respect to t, for all $x, y, z \in X$.*

Proof. If $F(x, y, z; t+s) \geq \triangle[F(x, y, a; t), F(a, z, z; s)]$, then by putting $a = z$, we get

$$F(x, y, z; t+s) \geq \triangle[F(x, y, z; t), F(z, z, z; s)],$$

and so,

$$F(x, y, z; t+s) \geq F(x, y, z; t).$$

□

Definition 2.7. Let $(X, F, \triangle)$ be a G^*MPM-space, F is said to be a continuous function on $X^3 \times (0, \infty)$ if $\lim_{n\to\infty} F(x_n, y_n, z_n; t_n) = F(x, y, z; t)$, whenever a sequence $\{x_n, y_n, z_n; t_n\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z; t)$ in $X^3 \times (0, \infty)$, that is,

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z,$$

and

$$\lim_{n\to\infty} F(x, y, z; t_n) = F(x, y, z; t).$$

Lemma 2.3. *If $(X, F, \triangle)$ is a G^*MPM-space, then F is said to be a continuous function on $X^3 \times (0, \infty)$.*

Proof. Let $(x_n', y_n', z_n'; t_n')_n$ be a sequence in $X^3 \times (0, \infty)$ and converges to $(x, y, z; t)$, where $x, y, z \in X$ and $t > 0$. Since $F(x_n', y_n', z_n'; t_n')_n$ is a sequence in $[0, 1]$, there is a subsequence $(x_n, y_n, z_n; t_n)_n$ of sequence $(x_n', y_n', z_n'; t_n')_n$ such that the sequence $F(x_n, y_n, z_n; t_n)_n$ converges to some point of $[0, 1]$.

Fix $\delta > 0$ such that $\delta < \frac{t}{2}$.

Then there exists $n_0 \in \mathbb{N}$ such that $| t - t_n | < \delta$ for every $n \geq n_0$. Hence

$$\begin{aligned}
F(x_n, y_n, z_n; t_n) &\geq F(x_n, y_n, z_n; t-\delta) \\
&\geq \triangle\left[F\left(x_n, y_n, z; t - \frac{4\delta}{3}\right), F\left(z, z_n, z_n; \frac{\delta}{3}\right)\right] \\
&\geq \triangle\left[\begin{array}{c} F\left(x_n, z, y; t - \frac{5\delta}{3}\right), F\left(y, y_n, y_n; \frac{\delta}{3}\right), \\ F\left(z, z_n, z_n; \frac{\delta}{3}\right) \end{array}\right] \\
&\geq \triangle\left[\begin{array}{c} F\left(z, y, x; t - 2\delta\right), F\left(x, x_n, x_n; \frac{\delta}{3}\right), \\ F\left(y, y_n, y_n; \frac{\delta}{3}\right), F\left(z, z_n, z_n; \frac{\delta}{3}\right) \end{array}\right],
\end{aligned}$$

and

$$\begin{aligned} F(x,y,z;t+2\delta) &\geq F(x,y,z;t_n+2\delta) \\ &\geq \triangle\left[F\left(x,y,z_n;t_n+\frac{2\delta}{3}\right), F\left(z_n,z,z;\frac{\delta}{3}\right)\right] \\ &\geq \triangle\left[\begin{array}{c} F\left(x,z_n,y_n;t_n+\frac{\delta}{3}\right), F\left(y_n,y,y;\frac{\delta}{3}\right), \\ F\left(z_n,z,z;\frac{\delta}{3}\right)\end{array}\right] \\ &\geq \triangle\left[\begin{array}{c} F\left(x_n,y_n,z_n;t_n\right), F\left(x_n,x,x;\frac{\delta}{3}\right), \\ F\left(y_n,y,y;\frac{\delta}{3}\right), F\left(z_n,z,z;\frac{\delta}{3}\right)\end{array}\right], \end{aligned}$$

for all $n \geq n_0$. Taking limit as $n \to \infty$, we obtain

$$\begin{aligned} \lim_{n\to\infty} F(x_n,y_n,z_n;t_n) &\geq \triangle[F(x,y,z;t-2\delta),1,1,1] \\ &= F(x,y,z;t-2\delta), \end{aligned}$$

and

$$\begin{aligned} F(x,y,z;t+2\delta) &\geq \lim_{n\to\infty} \triangle[F(x_n,y_n,z_n;t_n),1,1,1] \\ &= \lim_{n\to\infty} F(x_n,y_n,z_n;t_n). \end{aligned}$$

By continuity of the function, we have

$$\lim_{n\to\infty} F(x_n,y_n,z_n;t_n) = F(x,y,z;t).$$

Therefore, F is continuous on $X^3 \times (0,\infty)$.

Henceforth, we assume that $\triangle$ is a continuous t-norm on $[0,1]$ such that for every $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that

$$((1-\lambda),(1-\lambda),\ldots,(1-\lambda)) \geq (1-\mu).$$

□

Lemma 2.4. *Let $(X,F,\triangle)$ be a G^*MPM-space. If we define $E_{\lambda,F} : X^3 \to \mathbb{R}$ by*

$$E_{\lambda,F}(x,y,z) = \inf\{t>0 : F(x,y,z;t) > 1-\lambda\},$$

for every $\lambda \in (0,1)$ then

1. *For each $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that*

$$\begin{aligned} E_{\mu,F}(x_1,x_2,x_n) &\leq E_{\lambda,F}(x_1,x_1,x_2) + E_{\lambda,F}(x_2,x_2,x_3) + \ldots \\ &\quad + E_{\lambda,F}(x_{n-1},x_{n-1},x_n), \end{aligned}$$

for any $x_1,x_2,\ldots,x_n \in X$.

2. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is convergent in* G^**MPM-space* $(X, F, \triangle) \Leftrightarrow E_{\lambda,F}(x_n, x_n, x) \to 0$. *Also the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is Cauchy with* $E_{\lambda,F}$.

Proof. 1. For every $\mu \in (0,1)$, there is a $\lambda \in (0,1)$ such that

$$\triangle\left((1-\lambda),(1-\lambda),\dots,(1-\lambda)\right) \geq (1-\mu).$$

By triangular inequality, we have
$F\left(x_1, x_1, x_n; E_{\lambda,F}(x_1,x_1,x_2) + E_{\lambda,F}(x_2,x_2,x_3) + \ldots + E_{\lambda,F}(x_{n-1},x_{n-1},x_n) + n\delta\right)$
$\geq \triangle[F\left(x_1,x_1,x_n; E_{\lambda,F}(x_1,x_1,x_2)+\delta\right) + \ldots + F\left(x_{n-1},x_{n-1},x_n; E_{\lambda,F}(x_{n-1},x_{n-1},x_n)+\delta\right)]$
$\geq \triangle\left((1-\lambda)\dots,(1-\lambda)\right) \geq (1-\mu)$,
for every $\delta > 0$, it implies

$$E_{\mu,F}(x_1,x_2,x_n) \leq E_{\lambda,F}(x_1,x_1,x_2) + E_{\lambda,F}(x_2,x_2,x_3) + \ldots + E_{\lambda,F}(x_{n-1},x_{n-1},x_n).$$

2. If F is continuous, then

$$E_{\lambda,F}(x,x,y) = \inf\{t > 0 : F(x,x,y;t) > 1-\lambda\}.$$

Hence we have

$$F(x_n,x,x;\eta) > 1-\lambda \Leftrightarrow E_{\lambda,F}(x_n,x,x) < \eta, \text{ for every } \eta > 0.$$

□

Lemma 2.5. *Let* $(X, F, \triangle)$ *be a* G^**MPM-space. If* $F(x_n, x_n, x_{n+1}; t) \geq F(x_0, x_0, x_1; k^n t)$, *for some* $k > 1$ *and for every* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *is a Cauchy sequence in* X.

Proof. For every $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned}
E_{\lambda,F}(x_n,x_n,x_{n+1};t) &= \inf\{t > 0 : F(x_n,x_n,x_{n+1};t) > 1-\lambda\}\\
&\leq \inf\{t > 0 : F(x_0,x_0,x_1;k^n t) > 1-\lambda\}\\
&= \inf\left\{\frac{t}{k^n} > 0 : F(x_0,x_0,x_1;t) > 1-\lambda\right\}\\
&= \frac{1}{k^n}\inf\{t > 0 : F(x_0,x_0,x_1;t) > 1-\lambda\}\\
&= \frac{1}{k^n}E_{\lambda,F}(x_0,x_0,x_1;t).
\end{aligned}$$

By Lemma 2.4, for each $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned}
E_{\mu,F}(x_1,x_2,x_n) &\leq E_{\lambda,F}(x_1,x_1,x_2) + E_{\lambda,F}(x_2,x_2,x_3) + \ldots\\
&\quad + E_{\lambda,F}(x_{n-1},x_{n-1},x_n)\\
&\leq \frac{1}{k^n}E_{\lambda,F}(x_0,x_0,x_1;t) + \frac{1}{k^{n+1}}E_{\lambda,F}(x_0,x_0,x_1;t) + \ldots\\
&\quad + \frac{1}{k^{m-1}}E_{\lambda,F}(x_0,x_0,x_1;t)\\
&= E_{\lambda,F}(x_0,x_0,x_1;t)\sum_{j=n}^{m-1}\frac{1}{k^j} \to 0.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in X. $\square$

3. Common fixed point theorems

Let Φ denotes a family of mappings such that each $\phi \in \Phi$, $\phi : [0,1] \to [0,1]$ is continuous and $\phi(s) > s$ for every $s \in [0,1)$.

Theorem 3.1. *Let A, S and T be self mappings of a complete G^* MPM-space $(X, F, \triangle)$ and S, T be continuous mappings on X satisfying the following conditions:*

1. *the pairs (A, S) and (A, T) are weakly commuting and $A(X) \subseteq S(X) \cap T(X)$,*
2. *there exists a constant $k > 1$ such that*
$$F(Ax, Ay, Az; t) \geq \phi\left(\max\left\{\begin{array}{c} F(Sx, Ty, Tz; kt), F(Sx, Ax, Ax; kt), \\ F(Ty, Ax, Ax; kt), F(Ty, Ay, Ay; kt) \end{array}\right\}\right),$$
for all $x, y, z \in X$, $\phi \in \Phi$, and $t > 0$.

Then A, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ such that $A(X) \subseteq S(X)$, there exists $x_1 \in X$ such that $Ax_0 = Sx_1$. Also since $A(X) \subseteq T(X)$, there is another point $x_2 \in X$ such that $Ax_0 = Tx_2$. Inductively, we can choose x_{2n+1} and x_{2n+2} in X such that

$y_{2n} = Sx_{2n+1} = Ax_{2n}$, $Tx_{2n+2} = Ax_{2n+1} = y_{2n+1}$, for $n = 0, 1, \ldots$.

$$\begin{aligned} F(Ax_{2n}, Ax_{2n+1}, Ax_{2n+1}; t) &\geq \phi\left(\max\left\{\begin{array}{c} F(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}; kt), \\ F(Sx_{2n}, Ax_{2n}, Ax_{2n}; kt), \\ F(Tx_{2n+1}, Ax_{2n}, Ax_{2n}; kt), \\ F(Tx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}; kt) \end{array}\right\}\right) \\ F(y_{2n}, y_{2n+1}, y_{2n+1}; t) &\geq \phi\left(\max\left\{\begin{array}{c} F(y_{2n-1}, y_{2n}, y_{2n}; kt), \\ F(y_{2n-1}, y_{2n}, y_{2n}; kt), \\ F(y_{2n}, y_{2n}, y_{2n}; kt), \\ F(y_{2n}, y_{2n+1}, y_{2n+1}; kt) \end{array}\right\}\right). \end{aligned}$$

If $F(y_{2n}, y_{2n+1}, y_{2n+1}; kt) \geq F(y_{2n-1}, y_{2n}, y_{2n}; kt)$, then we get

$$\begin{aligned} F(y_{2n}, y_{2n+1}, y_{2n+1}; t) &\geq \phi\left(F(y_{2n}, y_{2n+1}, y_{2n+1}; kt)\right) \\ &> F(y_{2n}, y_{2n+1}, y_{2n+1}; kt), \end{aligned}$$

which is a contradiction. Therefore, we have

$$\begin{aligned} F(y_{2n}, y_{2n+1}, y_{2n+1}; t) &\geq \phi\left(F(y_{2n-1}, y_{2n}, y_{2n}; kt)\right) \\ &> F(y_{2n-1}, y_{2n}, y_{2n}; kt) \\ &\geq F(y_{2n-2}, y_{2n-1}, y_{2n-1}; k^2 t) \\ &\vdots \\ &\geq F(y_0, y_0, y_1; k^n t). \end{aligned}$$

Since F is of first or second type, by Remark 2.2 and Lemma 2.5, $\{Ax_n\}$ is a Cauchy sequence. By completeness of X, $\{Ax_n\}$ converges to a point $p \in X$, clearly the subsequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ of $\{Ax_n\}$ also converges to $p \in X$.

Since the mappings A and S are weakly commuting, $F(ASx_n, SAx_n, SAx_n; t) \geq F(Ax_n, Sx_n, Sx_n; t)$ and so $\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n = Sp$ (as S is continuous).

First we show that $Sp = p$. Suppose that $Sp \neq p$, then using (2) we get

$$F(ASx_n, Ax_n, Ax_n; t) \geq \phi\left(\max\left\{\begin{array}{c} F(SSx_n, Tx_n, Tx_n; kt), \\ F(SSx_n, ASx_n, ASx_n; kt), \\ F(Tx_n, ASx_n, ASx_n; kt), \\ F(Tx_n, Ax_n, Ax_n; kt) \end{array}\right\}\right).$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} F(Sp, p, p; t) &\geq \phi\left(\max\left\{\begin{array}{c} F(Sp, p, p; kt), F(Sp, Sp, Sp; kt), \\ F(p, Sp, Sp; kt), F(p, p, p; kt) \end{array}\right\}\right) \\ &= \phi(F(Sp, p, p; kt)) \\ &> F(Sp, p, p; kt), \end{aligned}$$

which is a contradiction. Thus p is a fixed point of S. Similarly we can show that p is a fixed point of A.

Also the mappings A and T are weakly commuting, that is,

$$F(ATx_n, TAx_n, TAx_n; t) \geq F(Ax_n, Tx_n, Tx_n; t),$$

then we get $\lim_{n\to\infty} ATx_n = \lim_{n\to\infty} TAx_n = Tp$ (as T is continuous).

Now we we claim that p is also a fixed point of T. Suppose that $Tp \neq p$, then using (2) we get

$$F(Ap, ATx_n, ATx_n; t) \geq \phi\left(\max\left\{\begin{array}{c} F(Sp, TTx_n, TTx_n; kt), F(Sp, Ap, Ap; kt), \\ F(TTx_n, Ap, Ap; kt), F(TTx_n, ATx_n, ATx_n; kt) \end{array}\right\}\right).$$

Taking limit as $n \to \infty$, it yields

$$\begin{aligned} F(p, Tp, Tp; t) &\geq \phi\left(\max\left\{\begin{array}{c} F(p, Tp, Tp; kt), F(p, p, p; kt), \\ F(Tp, p, p; kt), F(Tp, Tp, Tp; kt) \end{array}\right\}\right) \\ &= \phi(F(p, Tp, Tp; kt)) \\ &> F(p, Tp, Tp; kt), \end{aligned}$$

which is a contradiction. Hence, $p = Tp$. Therefore p is a common fixed point of A, S and T.

Uniqueness of the common fixed point is an easy consequence of condition (2). $\square$

Example 3.1. Let (X, D^*) be a D^*-metric space, where $X = [0,1]$, $D^*(x,y,z) = \mid x-y \mid + \mid y-z \mid + \mid z-x \mid$ and $\triangle(a,b,c) = \min(a,b,c)$. Set $F(x,y,z;t) = \frac{t}{t+D^*(x,y,z)}$, for all $x,y,z \in X$ and $t > 0$. Then $(X, F, \triangle)$ is a G^*MPM-space. Define the self mappings A, S and T on X by $A(x) = 1$, $S(x) = x$ and $T(x) = \frac{x+1}{2}$, for all $x \in X$. We have $A(x) = \{1\} \subseteq [0,1] \cap \left[\frac{1}{2}, 1\right] = S(X) \cap T(X)$ and $ATx = TAx = ASx = SAx = 1$. Now we verify the condition $F(ATx, TAx, TAx; t) \geq F(Ax, Tx, Tx; t)$, that is, $F(1,1,1;t) \geq F\left(1, \frac{x+1}{2}, \frac{x+1}{2}; t\right)$, which is always true.

Similarly, $F(ASx, SAx, SAx; t) \geq F(Ax, Sx, Sx; t)$ and so $F(1,1,1;t) \geq F(1,x,x;t)$ again always true. Hence the pairs (A, S) and (A, T) are weakly commuting.

Also, for all $x, y, z \in X$, we have

$$1 = F(Ax, Ay, Az; t) \geq \phi\left(\max\left\{\begin{array}{c} F(Sx, Ty, Tz; kt), F(Sx, Ax, Ax; kt), \\ F(Ty, Ax, Ax; kt), F(Ty, Ay, Ay; kt) \end{array}\right\}\right).$$

Thus all the conditions of Theorem 3.1 are satisfied and 1 is a unique common fixed point of the mappings A, S and T.

Theorem 3.2. *Let A, R, S, T and H be self mappings of a complete G^*MPM-space $(X, F, \triangle)$ and SR, TH be continuous self mappings on X satisfying the following conditions:*

1. *the pairs (A, SR) and (A, TH) are weakly commuting and $A(X) \subseteq SR(X) \cap TH(X)$,*
2. *there exists a constant $k > 1$ such that*

   $$F(Ax, Ay, Az; t) \geq \phi\left(\max\left\{\begin{array}{c} F(SRx, THy, THz; kt), F(SRx, Ax, Ax; kt), \\ F(THy, Ax, Ax; kt), F(THy, Ay, Ay; kt) \end{array}\right\}\right),$$

   for all $x, y, z \in X$, $\phi \in \Phi$, and $t > 0$.
3. *$SR = RS$, $TH = HT$, $AH = HA$ and $AR = RA$.*

Then A, R, S, T and H have a unique common fixed point in X.

Proof. By Theorem 3.1, A, SR and TH have a unique common fixed point in X i.e., there exists $p \in X$, such that $A(p) = SR(p) = TH(p) = p$. Now we assert that $Rp = p$. Suppose that $Rp \neq p$, then using (2) we get

$$F(A(Rp), Ap, Ap; t) \geq \phi\left(\max\left\{\begin{array}{c} F(SR(Rp), THp, THp; kt), \\ F(SR(Rp), A(Rp), A(Rp); kt), \\ F(THp, A(Rp), A(Rp); kt), \\ F(THp, Ap, Ap; kt) \end{array}\right\}\right),$$

or, equivalently,

$$\begin{aligned} F(Rp, p, p; t) &\geq \phi\left(\max\left\{\begin{array}{c} F(Rp, p, p; kt), F(Rp, Rp, Rp; kt), \\ F(p, Rp, Rp; kt), F(p, p, p; kt) \end{array}\right\}\right) \\ &= \phi(F(Rp, p, p; kt)) \\ &> F(Rp, p, p; kt), \end{aligned}$$

which is a contradiction. Then we have $R(p) = p$. Hence $S(p) = SR(p) = p$. Similarly, we can obtain $T(p) = TH(p) = p$. This completes the proof. $\square$

Theorem 3.3. *Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of self mappings of a complete G^*MPM-space $(X, F, \triangle)$ and S, T be continuous self mappings on X satisfying the following conditions:*

1. there exists $i_0 \in \mathbb{N}$ such that the pairs (A_{i_0}, S) and (A_{i_0}, T) are weakly commuting and $A_{i_0}(X) \subseteq S(X) \cap T(X)$,

2. $F(A_i x, A_j y, A_k z; t) \geq \phi\left(\max\left\{\begin{array}{l} F(Sx, Ty, Tz; kt), F(Sx, A_i x, A_i x; kt), \\ F(Ty, A_i x, A_i x; kt), F(Ty, A_j y, A_j y; kt) \end{array}\right\}\right)$,
for all $x, y, z \in X$, $\phi \in \Phi$, $k > 1$, $t > 0$ and $i, j, k \in \mathbb{N}$.

Then A_i, S and T have a unique common fixed point in X.

Proof. By Theorem 3.1, S, T and A_{i_0} for some $i = j = k = i_0 \in \mathbb{N}$ have a unique common fixed point in X, that is, there exists $p \in X$, such that $A_{i_0}(p) = S(p) = T(p) = p$.

Suppose that there exists $i \in \mathbb{N}$ such that $i \neq i_0$ and $j = k = i_0$, then using (2) we get

$$F(A_i p, A_{i_0} p, A_{i_0} p; t) \geq \phi\left(\max\left\{\begin{array}{c} F(Sp, Tp, Tp; kt), F(Sp, A_i p, A_i p; kt), \\ F(Tp, A_i p, A_i p; kt), F(Tp, A_{i_0} p, A_{i_0} p; kt) \end{array}\right\}\right).$$

Hence if $A_i p \neq p$, then we have

$$F(A_i p, p, p; t) \geq \phi\left(\max\left\{\begin{array}{c} F(p, p, p; kt), F(p, A_i p, A_i p; kt), \\ F(p, A_i p, A_i p; kt), F(p, p, p; kt) \end{array}\right\}\right),$$

which is a contradiction. Therefore, it follows that $A_i p = p$ for every $i \in \mathbb{N}$. This completes the proof. $\square$

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M. Alamgir Khan
Department of Natural Resources Engineering & Management
University of Kurdistan, Hewler, Iraq
alam3333@gmail.com

Sumitra
Faculty of Science
Department of Mathematics
Jizan University, Saudi Arabia
mathsqueen_d@yahoo.com

Sunny Chauhan
Near Nehru Training Centre
H. No. 274, Nai Basti B-14
Bijnor-246 701, Uttar Pradesh, India
sun.gkv@gmail.com