FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 27, No 3(2012), 271–282

CERTAIN CURVATURE PROPERTIES OF GENERALIZED SASAKIAN-SPACE-FORMS

U. C. De and Pradip Majhi

Abstract. The aim of the present paper is to study Ricci pseudosymmetric and Weyl semisymmetric generalized Sasakian-space-forms. Quasi-umbilical hypersurfaces of generalized Sasakian-space-forms have also been studied.

1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine the curvature tensor completely. A Riemannian manifold with a constant sectional curvature c is known as a real space-form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold with a constant ϕ-sectional curvature is a Sasakian-space-form and it has a specific form of its curvature tensor. A similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame, Alegre, Blair and Carriazo introduced the notion of generalized Sasakian-space-form in 2004 [2]. Related to this, it should be mentioned that in 1989 Olszak [19] studied generalized complex-space-form and proved its existence. A generalized Sasakian-space-form is defined as follows:
Given an almost contact metric manifold $M(\phi,\xi,\eta,g)$, we say that M is a generalized Sasakian-space-form if there exist three functions $f_1,\ f_2,\ f_3$ on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1.1}$$

Received December 1, 2012.; Accepted January 4, 2013.
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53D15

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [2] the authors cited several examples of generalized Sasakian-space-forms. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form. In [16], Kim studied conformally flat generalized Sasakian-space-form and locally symmetric generalized Sasakian-space-form. He proved some geometric properties of generalized Sasakian-space-form which depend on the nature of the functions f_1, f_2 and f_3. Generalized Sasakian-space-forms have also been studied in ([3], [4], [5], [14], [15]) and many others. In the present paper we study Ricci pseudosymmetric and Weyl semisymmetric generalized Sasakian-space-forms. Quasi-umbilical hypersurface of a generalized Sasakian-space-form have also been considered. The present paper is organized as follows. After preliminaries in section 2, we consider Ricci pseudosymmetric generalized Sasakian-space-forms. Section 4 deals with Weyl semisymmetric generalized Sasakian-space-forms. In this section we prove that for this space forms either $f_1 = f_3$ or the space-form is conformally flat. As a consequence of this result we obtain some important Corollaries. Finally we consider quasi-umbilical hypersurfaces of generalized Sasakian-space-forms and prove that quasi-umbilical hypersurface of a generalized Sasakian-space-form is a generalized quasi-Einstein hypersurface.

2. Priliminaries

An odd dimensional manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([6], [7])

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

The first and one of the remaining three relations in (2.1) imply the other two relations in (2.1). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{2.2}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η), structure, that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{2.4}$$

and

$$g(X, \xi) = \eta(X),$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).
An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.5}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.
Again for an $(2n+1)$-dimensional generalized Sasakian-space-form we have [2]

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{2.6}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y). \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi. \tag{2.8}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y]. \tag{2.9}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X)]. \tag{2.10}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X). \tag{2.11}$$

$$S(\xi,\xi) = 2n(f_1 - f_3). \tag{2.12}$$

$$Q\xi = 2n(f_1 - f_3)\xi. \tag{2.13}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.14}$$

where R, S and r denote the curvature tensor, Ricci tensor of type $(0,2)$ and scalar curvature of the space-form, respectively, and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. We know that [2] the ϕ-sectional curvature of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$.
Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection of (M, g). A Riemannian manifold is called locally symmetric [9] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [18]. The class of Riemannian symmetric manifold is a very natural generalization of the class of manifolds of constant curvature. The same can be extended to the class of semi-Riemannian manifolds, where g is of arbitrary signature.
During the five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as semisymmetric manifolds by Szabó [20], Boeckx, Kowalski and Vanhecke [8], Kowalski [17], conformally symmetric manifolds by Chaki and Gupta [11], recurrent manifolds by Walker [22], conformally recurrent manifold by Adati and Miyazawa [1] and many others.

Let $R(X,Y)$ and $X \wedge Y$ be endomorphisms of the Lie algebra $\Xi(M)$ of vector fields on M defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

$$(X \wedge Y)Z = g(Z,Y)X - g(Z,X)Y,$$

respectively, where $X,\ Y,\ Z \in \Xi(M)$. We extend the endomorphisms $R(X,Y)$ and $X \wedge Y$ to the derivations $R(X,Y).$ and $(X \wedge Y).$ of the algebra of the tensor fields on M, assuming that they commute with contractions and $R(X,Y).f = 0$, $(X \wedge Y).f = 0$ for every smooth function f on M.

For a $(0,k)-$ tensor field T on M, $k \geq 1$, we define the tensor fields ∇T, $R.T$ and $Q(g,T)$ by the formulas [21]

$$\begin{aligned}(\nabla T)(X_1,...,X_k;X) &= X(T(X_1,X_2,...,X_k)) - T(\nabla_X X_1,X_2,...,X_k)\\ &\quad -... - T(X_1,...,X_{k-1},\nabla_X X_k)),\end{aligned}$$

$$\begin{aligned}(R.T)(X_1,...,X_k;X,Y) &= (R(X,Y).T)(X_1,...,X_k)\\ &= -T(R(X,Y)X_1,X_2,...,X_k) - ...\\ &\quad -T(X_1,...,X_{k-1},R(X,Y)X_k),\end{aligned}$$

$$\begin{aligned}Q(g,T)(X_1,...,X_k;X,Y) &= -((X \wedge Y).T)(X_1,...,X_k)\\ &= T((X \wedge Y)X_1,X_2,...,X_k) + ...\\ &\quad +T(X_1,...,X_{k-1},(X \wedge Y)X_k),\end{aligned}$$

respectively, where X_1,...,X_k, X, $Y \in \Xi(M)$.

We define $G(X,Y)Z = g(Y,Z)X - g(X,Z)Y$ and consider the subsets U_R, U_S of a Riemannian manifold M by $U_R = \{x \in M : R - \frac{\kappa}{(n-1)n}G \neq 0 \quad at \quad x\}$ and $U_S = \{x \in M : S - \frac{\kappa}{n}g \neq 0 \quad at \quad x\}$ respectively. Evidently, we have $U_S \subset U_R$.

A Riemannian manifold is said to be pseudosymmetric if at every point of M the tensor $R.R$ and $Q(g,R)$ are linearly dependent. This is equivalent to

$$R.R = f_R Q(g,R)$$

on U_R, where f_R is some function on U_R. Clearly, every semisymmetric manifold is pseudosymmetric but the converse is not true [21].

A Riemannian manifold M is said to be Ricci pseudosymmetric if $R.S$ and $Q(g,S)$ on M are linearly dependent. This is equivalent to

$$R.S = f_S Q(g,S)$$

holds on U_S, where f_S is a function defined on U_S.

For a $(2n+1)$-dimensional $(n>1)$ Riemannian manifold the Weyl conformal curvature tensor is defined by

$$C(X,Y)Z \quad = \quad R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y$$

$$
\begin{aligned}
&+g(Y,Z)QX - g(X,Z)QY] \\
&+\frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \qquad (2.15)
\end{aligned}
$$

Let (M^{2n}, g) be a hypersurface of $(\tilde{M}^{2n+1}, \tilde{g})$. If A is the $(1,1)$ tensor corresponding to the normal valued second fundamental tensor H, then we have [10] Let (M^{2n}, g) be a hypersurface of $(\tilde{M}^{2n+1}, \tilde{g})$. If A is the $(1,1)$ tensor corresponding to the normal valued second fundamental tensor H, then we have [10]

$$g(A_\rho(X), Y) = \tilde{g}(H(X,Y), \rho), \qquad (2.16)$$

where ρ is the unit normal vector field and X, Y are tangent vector fields.
Let H_ρ be the symmetric $(0,2)$ tensor associated with A_ρ in the hypersurface defined by

$$g(A_\rho(X), Y) = H_\rho(X,Y). \qquad (2.17)$$

A hypersurface of a Riemannian manifold $(\tilde{M}^{2n+1}, \tilde{g})$ is called quasi-umbilical [10] if its second fundamental tensor has the form

$$H_\rho(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y), \qquad (2.18)$$

where ω is the 1-form, the vector field corresponding to the 1-form ω is a unit vector field and α, β are scalars. If $\alpha = 0$ (resp. $\beta = 0$ or $\alpha = \beta = 0$) holds, then it is called cylindrical (resp. umbilical or geodesic).
Now from (2.16), (2.17) and (2.18) we obtain

$$\tilde{g}(H(X,Y), \rho) = \alpha g(X,Y)\tilde{g}(\rho,\rho) + \beta\omega(X)\omega(Y)\tilde{g}(\rho,\rho),$$

which implies that

$$H(X,Y) = \alpha g(X,Y)\rho + \beta\omega(X)\omega(Y)\rho, \qquad (2.19)$$

since ρ is the only unit normal vector field.
We have the following equation of Gauss [10] for any vector fields X, Y, Z, W tangent to the hypersurface

$$
\begin{aligned}
\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) - g(H(X,W), H(Y,Z)) \\
&\quad + g(H(X,Z), H(Y,W)), \qquad (2.20)
\end{aligned}
$$

where $\tilde{R}(X,Y,Z,W) = \tilde{g}(R(X,Y)Z,W)$ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.
A non-flat Riemannian manifold is called a generalized quasi-Einstein manifold [13] if its Ricci tensor S satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \qquad (2.21)$$

where α, β, γ are certain non-zero scalars and A, B are non-zero 1-forms. The unit vector fields U and V corresponding to the 1-forms A, B are defined by

$$g(X,U) = A(X), \qquad g(X,V) = B(X)$$

respectively and the vector fields U,V are orthogonal i.e. $g(U,V) = 0$. The vector fields U and V are called the generators of the manifold. If $\gamma = 0$, then the manifold reduces to a quasi-Einstein manifold.

3. Ricci pseudosymmetric generalized Sasakian-space-forms

For a $(2n+1)$-dimensional Ricci pseudosymmetric generalized Sasakian-space-forms we have

$$R.S = f_S Q(g, S). \tag{3.1}$$

Now, (3.1) can be explicitly written as

$$(R(X,Y).S)(U,V) = -f_S[S((X \wedge_g Y)U, V) + S(U, (X \wedge_g Y)V)], \tag{3.2}$$

where the endomorphism $X \wedge_g Y$ is defined by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{3.3}$$

Using (3.3) in (3.2) yields

$$\begin{aligned} -S(R(X,Y)U,V) - S(U,R(X,Y)V) &= f_S[S(Y,V)g(X,U) \\ &\quad -S(X,V)g(Y,U) + S(U,Y)g(X,V) \\ &\quad -S(U,X)g(Y,V)]. \end{aligned} \tag{3.4}$$

Putting $X = \xi$, $U = \xi$ in (3.4) we obtain

$$\begin{aligned} -S(R(\xi,Y)\xi,V) - S(\xi,R(\xi,Y)V) &= f_S[S(Y,V)g(\xi,\xi) \\ &\quad -S(\xi,V)g(Y,\xi) + S(\xi,Y)g(\xi,V) \\ &\quad -S(\xi,\xi)g(Y,V)]. \end{aligned} \tag{3.5}$$

Again using (2.9), (2.10), (2.11), (2.12) in (3.5) we get

$$\{(f_1 - f_3) - f_S\}\{S(Y,V) - 2n(f_1 - f_3)g(Y,V)\} = 0. \tag{3.6}$$

Therefore either $f_S = (f_1 - f_3)$ or $S(Y,V) = 2n(f_1 - f_3)g(Y,V)$.
If the second condition holds, then $R.S = 0$. Therefore we can state the following:

Theorem 3.1. *A (2n+1)-dimensional Ricci pseudosymmetric generalized Sasakian-space-form is Ricci semisymmetric provided $f_S \neq f_1 - f_3$.*

4. Weyl semisymmetric generalized Sasakian-space-forms

Let us consider the Weyl semisymmetric generalized Sasakian-space-form $M(f_1, f_2, f_3)$. Then we have

$$(R(X,Y).C)(U,V)W = 0. \tag{4.1}$$

This implies

$$\begin{aligned} R(X,Y)C(U,V)W - C(R(X,Y)U,V)W &- C(U,R(X,Y)V)W \\ &- C(U,V)R(X,Y)W = 0. \end{aligned} \tag{4.2}$$

Putting $X = \xi$ in (4.2) we obtain

$$\begin{aligned} R(\xi,Y)C(U,V)W - C(R(\xi,Y)U,V)W &- C(U,R(\xi,Y)V)W \\ &- C(U,V)R(\xi,Y)W = 0. \end{aligned} \tag{4.3}$$

Using (2.10) in (4.3) yields

$$\begin{aligned} (f_1 - f_3)[g(Y,C(U,V)W)\xi &- \{\eta(C(U,V)W)Y + g(Y,U)C(\xi,V)W\} \\ &+ \{\eta(U)C(Y,V)W - g(Y,V)C(U,\xi)W\} \\ &+ \{\eta(V)C(U,Y)W - g(Y,W)C(U,V)\xi\} \\ &+ \eta(W)C(U,V)Y] = 0. \end{aligned} \tag{4.4}$$

Therefore either $f_1 = f_3$. or,

$$\begin{aligned} g(Y,C(U,V)W)\xi &- \eta(C(U,V)W)Y - g(Y,U)C(\xi,V)W \\ &+ \eta(U)C(Y,V)W - g(Y,V)C(U,\xi)W \\ &+ \eta(V)C(U,Y)W - g(Y,W)C(U,V)\xi \\ &+ \eta(W)C(U,V)Y = 0. \end{aligned} \tag{4.5}$$

Taking the inner product with ξ in (4.5) we obtain

$$\begin{aligned} g(Y,C(U,V)W) &- \eta(C(U,V)W)\eta(Y) + g(Y,U)\eta(C(\xi,V)W) \\ &+ \eta(U)\eta(C(Y,V)W) - g(Y,V)\eta(C(U,\xi)W) \\ &+ \eta(V)\eta(C(U,Y)W) - g(Y,W)\eta(C(U,V)\xi) \\ &+ \eta(W)\eta(C(U,V)Y) = 0. \end{aligned} \tag{4.6}$$

Now,

$$\begin{aligned} \eta(C(X,Y)Z) &= g(R(X,Y)Z,\xi) - \frac{1}{(2n-1)}[S(Y,Z)\eta(X) \\ &\quad -S(X,Z)\eta(Y) + g(Y,Z)g(QX,\xi) - g(X,Z)g(QY,\xi)] \\ &\quad + \frac{r}{2n(2n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \end{aligned} \tag{4.7}$$

Using (2.6), (2.7) and (2.11) in (4.7) yields

$$\begin{aligned} \eta(C(X,Y)Z) &= \{(f_1 - f_3) - \frac{4nf_1 + 3f_2 - (2n+1)f_3}{(2n-1)} \\ &\quad + \frac{r}{2n(2n-1)}\}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \end{aligned} \tag{4.8}$$

Putting the value of the scalar curvature r from (2.14) in (4.8), we obtain

$$\eta(C(X,Y)Z) = 0 \tag{4.9}$$

for all X, Y, Z.
In view of (4.9) we have from (4.6)

$$g(C(U,V)W,Y)=0 \tag{4.10}$$

for all U, V, W and Y. This implies

$$C(U,V)W=0. \tag{4.11}$$

Therefore $R(\xi,Y).C=0$ implies either $f_1=f_3$ or, $C=0$. Again if $f_1=f_3$, then from (4.4) it follows that $R(\xi,Y).C=0$ and also $C=0$ implies $R(\xi,Y).C=0$.

Therefore we can state the following:

Theorem 4.1. *A (2n+1)-dimensional $(n>1)$ generalized Sasakian-space-form $M(f_1,f_2,f_3)$ satisfies $R(\xi,Y).C=0$ if and only if either $f_1=f_3$ or, the space-form M is conformally flat.*

In [16] Kim proved that for a (2n+1)-dimensional generalized Sasakian-space-form the following statements hold:
(i) If $n>1$, then M is conformally flat if and only if $f_2=0$.
(ii) If M is conformally flat and ξ is a Killing vector field, then M is locally symmetric and has constant ϕ-sectional curvature.
In view of the first part of the above theorem we have the following:

Corollary 4.1. *For a $(2n+1)$-dimensional $(n>1)$ Weyl semisymmetric generalized Sasakian-space-form $M(f_1,f_2,f_3)$ either $f_1=f_3$ or, $f_2=0$.*

Again, in view of the second part of the above theorem we have the following:

Corollary 4.2. *For a $(2n+1)$-dimensional $(n>1)$ Weyl semisymmetric generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with ξ as a Killing vector field, either $f_1=f_3$ or the space-form is locally symmetric and has constant ϕ-sectional curvature.*

In [2] Alegre, Blair and Carriazo proved that if a generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is a Sasakian manifold, then the functions f_1, f_2, f_3 are constant and $f_1-1=f_2=f_3$.
Now, in this case $f_1\neq f_3$. Also $f_2=0$ implies $f_3=0$ and $f_1=1$. Thus from (1.1) we have $R(X,Y)Z=g(Y,Z)X-g(X,Z)Y$, that is, the manifold is of constant curvature 1. Therefore we can state the following:

Corollary 4.3. *A $(2n+1)$-dimensional $(n>1)$ Weyl semisymmetric Sasakian manifold is of constant curvature 1.*

The above Corollary was proved by Chaki and Tarafder [12] in another way.

Remark 4.1. The converse of Corollary 4.3 is also true.

5. Quasi-umbilical hypersurfaces of generalized Sasakian-space-forms

Let us consider a quasi-umbilical hypersurface. Therefore we have from (2.18)

$$H(X,Y) = \alpha g(X,Y)\rho + \beta\omega(X)\omega(Y)\rho, \tag{5.1}$$

since ρ is the only unit normal vector field.
We have the following equation of Gauss [10] for any vector fields X, Y, Z, W tangent to the hypersurface

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &- g(H(X,W),H(Y,Z)) \\ &+ g(H(Y,W),H(X,Z)).\end{aligned} \tag{5.2}$$

Using (5.1) in (5.2) we obtain

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) \\ &- g([\alpha g(X,W)\rho + \beta\omega(X)\omega(W)\rho],[\alpha g(Y,Z)\rho + \beta\omega(Y)\omega(Z)\rho]) \\ &+ g([\alpha g(Y,W)\rho + \beta\omega(Y)\omega(W)\rho],[\alpha g(X,Z)\rho + \beta\omega(X)\omega(Z)\rho]).\end{aligned} \tag{5.3}$$

Therefore using (1.1) in (5.3) yields

$$\begin{aligned}&f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ +\ &f_2\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\} \\ +\ &f_3\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ +\ &g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\} \\ =\ &R(X,Y,Z,W) - \alpha^2 g(X,W)g(Y,Z) - \alpha\beta g(X,W)\omega(Y)\omega(Z) \\ -\ &\alpha\beta g(Y,Z)\omega(X)\omega(W) + \alpha^2 g(Y,W)g(X,Z) \\ +\ &\alpha\beta g(Y,W)\omega(X)\omega(Z) + \alpha\beta g(X,Z)\omega(Y)\omega(W).\end{aligned} \tag{5.4}$$

Contracting the above equation (5.4) over X, W we obtain

$$\begin{aligned}S(Y,Z) &= \{2nf_1 - f_3 + 3f_2 + 2n\alpha^2 + \alpha\beta\}g(Y,Z) \\ &+\{-3f_2 - (2n-1)f_3\}\eta(Y)\eta(Z) + (2n-1)\alpha\beta\omega(Y)\omega(Z).\end{aligned} \tag{5.5}$$

Therefore in view of (2.21), M is a generalized quasi-Einstein hypersurface. Thus we can state the following:

Theorem 5.1. *A quasi-umbilical hypersurface of a generalized Sasakian-space-form is a generalized quasi-Einstein hypersurface.*

If the unit normal vector field ρ is the characteristic vector field ξ of the almost contact structure on the manifold $\tilde{M}$, that is, $\eta(X) = \omega(X)$ for all X, then equation (5.5) reduces to

$$\begin{aligned}S(Y,Z) &= \{2nf_1 - f_3 + 3f_2 + 2n\alpha^2 + \alpha\beta\}g(Y,Z) \\ &+\{-3f_2 - (2n-1)f_3\}\eta(Y)\eta(Z) + (2n-1)\alpha\beta\eta(Y)\eta(Z),\end{aligned} \tag{5.6}$$

where $g(X,\xi)=\eta(X)$. If the characteristic vector field is normal then $\eta(X)=0$ for all vector field X tangent to the hypersurface. Therefore the equation (5.6) reduces to

$$S(Y,Z)=\{2nf_1-f_3+3f_2+2n\alpha^2+\alpha\beta\}g(Y,Z). \tag{5.7}$$

It follows that M is an Einstein hypersurface. Thus in view of the above result we can state the following:

Theorem 5.2. *A quasi-umbilical hypersurface of a generalized Sasakian-space-form is an Einstein hypersurface provided the associated vector field ρ of the quasi-umbilical hypersurface is the characteristic vector field of an almost contact metric manifold and the characteristic vector field is normal.*

Definition 5.1. For each plane π in the tangent space $T_x(M)$, the sectional curvature is defined by

$$K_M(X\wedge Y)=R(X,Y,Y,X)=g(R(X,Y)Y,X),$$

where X, Y are orthonormal basis, for the plane π, $K_M(X\wedge Y)$ is independent of choice of the orthonormal basis X, Y. If $K_M(X\wedge Y)$ is a constant for all planes π of $T_x(M)$ and for all points x of M, then M is called a space of constant curvature.

Again from (5.2) we have

$$\begin{aligned} R(X,Y,Z,W)=\tilde{R}(X,Y,Z,W) &+ g(H(X,W),H(Y,Z)) \\ &- g(H(Y,W),H(X,Z)). \end{aligned} \tag{5.8}$$

Using (1.1) and (5.1) in (5.6) we obtain

$$\begin{aligned} R(X,Y,Y,X) &= f_1\{g(X,X)g(Y,Y)-\{g(X,Y)\}^2\} \\ &+ 3f_2\{g(X,\phi Y)\}^2 \\ &+ f_3\{2\eta(X)\eta(Y)g(X,Y)-\{\eta(Y)\}^2g(X,X)-\{\eta(X)\}^2g(Y,Y)\} \\ &\quad +\alpha^2\{g(X,X)g(Y,Y)-\{g(X,Y)\}^2\}+\alpha\beta g(X,X)\omega(Y)\omega(Y) \\ &\quad +\alpha\beta g(Y,Y)\omega(X)\omega(X)-2\alpha\beta g(X,Y)\omega(X)\omega(Y). \end{aligned} \tag{5.9}$$

Since X, Y are the unit normal vector fields we have from (5.9)

$$\begin{aligned} R(X,Y,Y,X) &= f_1+3f_2\{g(X,\phi Y)\}^2-f_3(\{\eta(X)\}^2+\{\eta(Y)\}^2) \\ &\quad +\alpha^2+\alpha\beta(\{\omega(X)\}^2+\{\omega(Y)\}^2). \end{aligned} \tag{5.10}$$

Thus we have the following:

Theorem 5.3. *For a quasi-umbilical hypersurface of generalized Sasakian-space-form, the sectional curvature of the plane section spanned by X, Y is given by:*

$$\begin{aligned} K_M(X\wedge Y) &= f_1+3f_2\{g(X,\phi Y)\}^2-f_3(\{\eta(X)\}^2+\{\eta(Y)\}^2) \\ &\quad +\alpha^2+\alpha\beta(\{\omega(X)\}^2+\{\omega(Y)\}^2). \end{aligned}$$

Again if $\beta = 0$ and α is the mean curvature vector, then the quasi-umbilical hypersurface becomes totally umbilical hypersurface.
In [4] Alegre and Carriazo proved that a connected, totally umbilical submanifold of a generalized Sasakian-space-form with $f_2 \neq 0$ is either an invariant or anti-invariant submanifold. Therefore from Theorem 5.2, we have the following:

Corollary 5.1. *For a connected, totally umbilical hypersurface of a generalized Sasakian-space-form with $f_2 \neq 0$, the sectional curvature is given by*

$$\begin{aligned} K_M(X \wedge Y) &= f_1 - f_3(\{\eta(X)\}^2 + \{\eta(Y)\}^2) \\ &\quad +\alpha^2. \end{aligned}$$

REFERENCES

1. T. Adati and T. Miyazawa: *On a Riemannian space with recurrent conformal curvature*, Tensor N. S. **18**(1967), 348-354.
2. P. Alegre, D. E. Blair and A. Carriazo: *Generalized Sasakian-space-forms*, Israel J.Math. **141**(2004), 157-183.
3. P. Alegre and A. Carriazo: *Structures on generalized Sasakian-space-forms*, Diff. Geom. Appl. **26**(2008), 656-666.
4. P. Alegre and A. Carriazo: *Submanifolds of generalized Sasakian-space-forms*, Taiwanese J. Math. **13**(2009), 923-941.
5. P. Alegre and A. Carriazo: *Generalized Sasakian-space-forms and conformal changes of the metric*, Results Math. **59**(2011), 485-493.
6. D. E. Blair: *Lecture notes in Mathematics*, 509, Springer-Verlag Berlin(1976).
7. D. E. Blair: *Riemannian Geometry of contact and symplectic manifolds*, Birkhauser, Boston, 2002.
8. E. Boeckx, O. Kowalski and L. Vanhecke: *Riemannian manifolds of conullity two*, Singapore World Sci. Publishing, 1996.
9. E. Cartan: *Sur une classe remarqable d' espaces de Riemannian*, Bull. Soc. Math. France., **54**(1962), 214-264.
10. B. Y. Chen: *Geometry of submanifolds*, Marcel Dekker. Ine. New York, 1973.
11. M. C. Chaki and B. Gupta: *On conformally symmetric spaces*, Indian J. Math. **5**(1963), 113-295.
12. M. C. Chaki and M. Tarafdar: *On a type of Sasakian manifolds*, Soochow J. of Math. **16**(1990), 23-28.
13. U. C. De and S. Mallick: *On the existence of generalized quasi-Einstein manifolds*, Archivum Mathematicum (Brno), **47**(2011), 279-291.
14. U. C. De and A. Sarkar: *On the projective curvature tensor of generalized Sasakian-space-forms*, Quaestiones Mathematicae, **33**(2010), 245-252.

15. U. C. De and A. Sarkar: *Some results on generalized Sasakian-space-forms*, Thai J. Math. **8**(2010), 1-10.
16. U. K. Kim: *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note Mat. **26**(2006), 55-67.
17. O. Kowalski: *An explicit classification of 3- dimensional Riemannian spaces satisfying R(X,Y).R=0*, Czechoslovak Math. J. **46**(121)(1996), 427-474.
18. B. O'Neill: *Semi-Riemannian geometry with applications to relativity*, Academy Press, New York-London, 1983.
19. Z. Olszak: *On the existence of generalized complex space forms*, Israel J. Math. **65**(1989), 214-218.
20. Z. I. Szabó: *Structure theorems on Riemannian spaces satisfying R(X,Y).R=0, the local version*, J. Diff. Geom. **17**(1982), 531-582.
21. L. Verstraelen: *Comments on pseudosymmetry in the sense of Ryszard Deszcz, In: Geometry and Topology of submanifolds*, VI. River Edge, NJ: World Sci. Publishing, 1994, 199-209.
22. A. G. Walker: *On Ruses spaces of recurrent curvature*, Proc. London Math. Soc. **52**(1951), 36-64.

U. C. De
Department of Pure Mathematics
University of Calcutta
35, Ballygaunge Circular Road
Kolkata -700019, West Bengal, India
uc_de@yahoo.com

Pradip Majhi
Department of Mathematics
University of North Bengal
Raja Rammohunpur, Darjeeling
Pin-734013, West Bengal, India
mpradipmajhi@gmail.com