

**SHARP ESTIMATES AND BOUNDEDNESS FOR MULTILINEAR
 INTEGRAL OPERATORS**

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Abstract. In this paper, the sharp estimates for the multilinear operators associated to the Littlewood-Paley operator and Marcinkiewicz operator are proved. As the application, the (L^p, L^q) and Morrey spaces boundedness for the multilinear operators are obtained.

Keywords: Multilinear operator; Littlewood-Paley operator; Marcinkiewicz operator; Morrey space; BMO.

1. Introduction and Theorems

In this paper, we study some integral operators as follows.

Fix $0 \leq \delta < n$, we denote that $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Suppose that m_j are the positive integers ($j = 1, \dots, l$), $m_1 + \dots + m_l = m$ and A_j are the functions on R^n ($j = 1, \dots, l$). Let

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha.$$

Definition 1. Let $\lambda > (3n + 2 - 2\delta)/n$, $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

- (1) $\int_{R^n} \psi(x) dx = 0$,
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

The multilinear Littlewood-Paley operator is defined by

$$g_\lambda^A(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |F_t^A(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

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where

$$F_t^A(f)(x, y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, z)}{|x - z|^m} \psi_t(y - z) f(z) dz$$

and $\psi_t(x) = t^{-n+\delta} \psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\lambda(f)(x) = \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [17]).

Let H be the Hilbert space $H = \left\{ h : \|h\| = (\int \int_{R_+^{n+1}} |h(y, t)|^2 dy dt / t^{n+1})^{1/2} < \infty \right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x, y)$ may be viewed as a mapping from $(0, +\infty)$ to H , and it is clear that

$$g_\lambda^A(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} F_t^A(f)(x, y) \right\|$$

and

$$g_\lambda(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} F_t(f)(y) \right\|.$$

Definition 2. Let $\lambda > \max(1, 2n/(n + 2 - 2\delta))$, $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma$. We denote that $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x - y| < t\}$ and the characteristic of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\lambda^A(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |F_t^A(f)(x, y)|^2 \frac{dy dt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x, y) = \int_{|y-z| \leq t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f(z) dz.$$

Set

$$F_t(f)(y) = \int_{|y-z| \leq t} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f(z) dz.$$

We also define that

$$\mu_\lambda(f)(x) = \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+3}} \right)^{1/2},$$

which is the Marcinkiewicz integral operator (see [18]).

Let H be the Hilbert space

$$H = \left\{ h : \|h\| = \left(\int \int_{R_+^{n+1}} |h(y, t)|^2 dy dt / t^{n+3} \right)^{1/2} < \infty \right\},$$

then for each fixed $x \in R^n$, $F_t^A(f)(x, y)$ may be viewed as a mapping from $(0, +\infty)$ to H , and it is clear that

$$\mu_\lambda^A(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} F_t^A(f)(x, y) \right\|$$

and

$$\mu_\lambda(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} F_t(f)(y) \right\|.$$

Note that when $m = 0$, g_λ^A and μ_λ^A are just the multilinear commutators (see [10-12]). When $m > 0$, g_λ^A and μ_λ^A are non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3-7]). In [9], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [13-15], authors prove a sharp estimate for the multilinear commutator. The purpose of this paper is to prove the sharp inequalities for the multilinear integral operators g_λ^A and μ_λ^A . As the application, the (L^p, L^q) and Morrey spaces boundedness for the multilinear operators are obtained.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f , the sharp function of f is defined by

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [16])

$$f^\#(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. For $1 \leq p < \infty$ and $0 \leq \delta < n$, let

$$M_{\delta, p}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-p\delta/n}} \int_Q |f(y)|^p dy \right)^{1/p},$$

we write that $M_\mu(f) = M_{n\mu, 1}(f)$, which is the fractional maximal operator.

Fixed $\sigma > 0$. For $1 \leq p < \infty$, let

$$\|f\|_{L^{p,\sigma}} = \sup_{x \in R^n, d>0} \left(\frac{1}{d^\sigma} \int_{B(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $B(x,d) = \{y \in R^n : |x-y| < d\}$. The Morrey spaces is defined by(see [2][8])

$$L^{p,\sigma}(R^n) = \{f \in L^1_{loc}(R^n) : \|f\|_{L^{p,\sigma}} < \infty\}.$$

As the Morrey space may be considered an extension of the Lebesgue space, it is natural and important to study the boundedness of the multilinear integral operator on the Morrey Spaces.

Now we state our main results as follows.

Theorem 1. Let $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$.

(1). Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $1 < r < n/\delta$ and $x \in R^n$,

$$(g_\lambda^A(f))^\#(x) \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,r}(f)(x).$$

(2). If $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, then g_λ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is

$$\|g_\lambda^A(f)\|_{L^q} \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \|f\|_{L^p}.$$

(3). If $1 < p < n/\delta$, $0 < \sigma < n - p\delta$, $1/q = 1/p - \delta/(n - \sigma)$, then g_λ^A is bounded from $L^{p,\sigma}(R^n)$ to $L^{q,\sigma}(R^n)$, that is,

$$\|g_\lambda^A(f)\|_{L^{q,\sigma}} \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \|f\|_{L^{p,\sigma}}.$$

Theorem 2. Let $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$.

(1). Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $1 < r < n/\delta$ and $x \in R^n$,

$$(\mu_\lambda^A(f))^\#(x) \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,r}(f)(x).$$

(2). If $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, then μ_λ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is,

$$\|\mu_\lambda^A(f)\|_{L^q} \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \|f\|_{L^p}.$$

(3). If $1 < p < n/\delta$, $0 < \sigma < n - p\delta$, $1/q = 1/p - \delta/(n - \sigma)$, then μ_λ^A is bounded from $L^{p,\sigma}(R^n)$ to $L^{q,\sigma}(R^n)$, that is,

$$\|\mu_\lambda^A(f)\|_{L^{q,\sigma}} \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \|f\|_{L^{p,\sigma}}.$$

2. SoProof of Theorem

To prove the theorems, we need the following lemmas.

Lemma 1.([5]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where \tilde{Q} is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2.([1]) Suppose that $1 \leq r < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then

$$\|M_{\delta,r}(f)\|_{L^q} \leq C\|f\|_{L^p}.$$

Lemma 3.([2][8]) Let $1 < p < \infty$ and $0 < \sigma < n$. Then the following estimates hold:

- (a). $\|M(f)\|_{L^{p,\sigma}} \leq C\|f^\#\|_{L^{p,\sigma}}$.
- (b). $\|M_\mu(f)\|_{L^{q,\sigma}} \leq C\|f\|_{L^{p,\sigma}}$ for $0 < \mu < (n - \sigma)/np$ and $1/q = 1/p - n\eta/(n - \sigma)$.

Lemma 4. Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then g_λ and μ_λ are all bounded from $L^p(R^n)$ to $L^q(R^n)$.

Proof. For g_λ , by Minkowski's inequality and the conditions of ψ , we get

$$\begin{aligned} g_\lambda(f)(x) &\leq \int_{R^n} |f(z)| \left(\int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |\psi_t(y - z)|^2 \frac{dy dt}{t^{1+n}} \right)^{1/2} dz \\ &\leq C \int_{R^n} |f(z)| \left(\int_0^\infty \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} \frac{t^{-2n+2\delta}}{(1 + |y - z|/t)^{2n+2-2\delta}} \frac{dy dt}{t^{1+n}} \right)^{1/2} dz \\ &\leq C \int_{R^n} |f(z)| \left[\int_0^\infty \left(t^{-n} \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} \frac{dy}{(t + |y - z|)^{2n+2-2\delta}} \right) t dt \right]^{1/2} dz, \end{aligned}$$

noting that $2t + |y - z| \geq 2t + |x - z| - |x - y| \geq t + |x - z|$ when $|x - y| \leq t$ and $2^{k+1}t + |y - z| \geq 2^{k+1}t + |x - z| - |x - y| \geq |x - z|$ when $|x - y| \leq 2^{k+1}t$, we get, recall that $\lambda > (3n + 2)/n$,

$$t^{-n} \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} \frac{dy}{(t + |y - z|)^{2n+2-2\delta}}$$

$$\begin{aligned}
&= t^{-n} \int_{|x-y| \leq t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \frac{dy}{(t+|y-z|)^{2n+2-2\delta}} \\
&\quad + t^{-n} \sum_{k=0}^{\infty} \int_{2^k t < |x-y| \leq 2^{k+1} t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \frac{dy}{(t+|y-z|)^{2n+2-2\delta}} \\
&\leq t^{-n} \left[\int_{|x-y| \leq t} \frac{2^{2n+2-2\delta} dy}{(2t+2|y-z|)^{2n+2-2\delta}} + \sum_{k=0}^{\infty} \int_{|x-y| \leq 2^{k+1} t} 2^{-kn\lambda} \frac{2^{(k+2)(2n+2-2\delta)} dy}{(2^{k+2}t+2^{k+2}|y-z|)^{2n+2-2\delta}} \right] \\
&\leq Ct^{-n} \left[\int_{|x-y| \leq t} \frac{dy}{(2t+|y-z|)^{2n+2-2\delta}} + \sum_{k=0}^{\infty} \int_{|x-y| \leq 2^{k+1} t} 2^{-kn\lambda} \frac{2^{k(2n+2-2\delta)} dy}{(t+2^{k+1}t+|y-z|)^{2n+2-2\delta}} \right] \\
&\leq Ct^{-n} \left[\int_{|x-y| \leq t} \frac{dy}{(t+|x-z|)^{2n+2-2\delta}} + \sum_{k=0}^{\infty} \int_{|x-y| \leq 2^{k+1} t} 2^{-kn\lambda} \frac{2^{k(2n+2-2\delta)} dy}{(t+|x-z|)^{2n+2-2\delta}} \right] \\
&\leq Ct^{-n} \left[\frac{t^n}{(t+|x-z|)^{2n+2-2\delta}} + \sum_{k=0}^{\infty} 2^{k(3n+2-n\lambda)} \frac{t^n}{(t+|x-z|)^{2n+2-2\delta}} \right] \\
&\leq \frac{C}{(t+|x-z|)^{2n+2-2\delta}},
\end{aligned}$$

since

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$g_\lambda(f)(x) \leq C \int_{R^n} |f(z)| \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} \right)^{1/2} dz = C \int_{R^n} \frac{|f(z)|}{|x-z|^{n-\delta}} dz;$$

For μ_λ , notice that $|x-z| \leq 2t$, $|y-z| \geq |x-z| - t \geq |x-z| - 3t$ when $|x-y| \leq t$, $|y-z| \leq t$, and $|x-z| \leq t(1+2^{k+1}) \leq 2^{k+2}t$, $|y-z| \geq |x-z| - 2^{k+3}t$ when $|x-y| \leq 2^{k+1}t$, $|y-z| \leq t$, we obtain

$$\begin{aligned}
\mu_\lambda(f)(x) &\leq \int_{R^n} \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \left(\frac{|\Omega(y-z)||f(z)|}{|y-z|^{n-\delta-1}} \right)^2 \chi_{\Gamma(z)}(y,t) \frac{dydt}{t^{n+3}} \right]^{1/2} dz \\
&\leq C \int_{R^n} |f(z)| \left[\int_0^\infty \int_{|x-y| \leq t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-3t)^{2n-2\delta-2}} \frac{dydt}{t^{n+3}} \right]^{1/2} dz \\
&\quad + C \int_{R^n} |f(z)| \left[\int_0^\infty \sum_{k=0}^{\infty} \int_{2^k t < |x-y| \leq 2^{k+1} t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \frac{\chi_{\Gamma(z)}(y,t)t^{-n-3}dydt}{(|x-z|-2^{k+3}t)^{2n-2\delta-2}} \right]^{1/2} dz \\
&\leq C \int_{R^n} \frac{|f(z)|}{|x-z|^{1/2}} \left[\int_{|x-z|/2}^\infty \frac{dt}{(|x-z|-3t)^{2n-2\delta}} \right]^{1/2} dz \\
&\quad + C \int_{R^n} \frac{|f(z)|}{|x-z|^{1/2}} \left[\sum_{k=0}^{\infty} \int_{2^{-2-k}|x-z|}^\infty 2^{-kn\lambda} (2^k t)^n t^{-n} \frac{2^k dt}{(|x-z|-2^{k+3}t)^{2n-2\delta}} \right]^{1/2} dz \\
&\leq C \int_{R^n} \frac{|f(z)|}{|x-z|^{n-\delta}} dz + C \int_{R^n} \frac{|f(z)|}{|x-z|^{n-\delta}} dz \left[\sum_{k=0}^{\infty} 2^{kn(1-\lambda)} \right]^{1/2} \\
&= C \int_{R^n} \frac{|f(z)|}{|x-z|^{n-\delta}} dz.
\end{aligned}$$

Thus, the lemma follows from [1].

Proof of Theorem 1(1). It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|} \int_Q |g_\lambda^A(f)(x) - C_0| dx \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,r}(f)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x, y) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, z)}{|x-z|^m} \psi_t(y-z) f(z) dz \\ &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, z)}{|x-z|^m} \psi_t(y-z) f_2(z) dz \\ &\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z)}{|x-z|^m} \psi_t(y-z) f_1(z) dz \\ &\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, z) (x-z)^{\alpha_1}}{|x-z|^m} D^{\alpha_1} \tilde{A}_1(z) \psi_t(y-z) f_1(z) dz \\ &\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, z) (x-z)^{\alpha_2}}{|x-z|^m} D^{\alpha_2} \tilde{A}_2(z) \psi_t(y-z) f_1(z) dz \\ &\quad + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x-z)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(z) D^{\alpha_2} \tilde{A}_2(z)}{|x-z|^m} \psi_t(y-z) f_1(z) dz, \end{aligned}$$

then

$$\begin{aligned} &|g_\lambda^A(f)(x) - g_\lambda^{\tilde{A}}(f_2)(x_0)| \\ &= \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^A(f)(x, y) \right\| - \left\| \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0, y) \right\| \\ &\leq \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^A(f)(x, y) - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0, y) \right\| \\ &\leq \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z)}{|x-z|^m} \psi_t(y-z) f_1(z) dz \right\| \\ &\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, z) (x-z)^{\alpha_1}}{|x-z|^m} D^{\alpha_1} \tilde{A}_1(z) \psi_t(y-z) f_1(z) dz \right\| \\ &\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, z) (x-z)^{\alpha_2}}{|x-z|^m} D^{\alpha_2} \tilde{A}_2(z) \psi_t(y-z) f_1(z) dz \right\| \\ &\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x-z)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(z) D^{\alpha_2} \tilde{A}_2(z)}{|x-z|^m} \psi_t(y-z) f_1(z) dz \right\| \end{aligned}$$

$$+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x,y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x_0,y)\right\|\\:=I_1(x)+I_2(x)+I_3(x)+I_4(x)+I_5(x),$$

thus,

$$\begin{aligned}&\frac{1}{|Q|}\int_Q\left|g_{\lambda}^{\tilde{A}}(f)(x)-g_{\lambda}^{\tilde{A}}(f_2)(x_0)\right|dx\\&\leq \frac{1}{|Q|}\int_QI_1(x)dx+\frac{C}{|Q|}\int_QI_2(x)dx+\frac{C}{|Q|}\int_QI_3(x)dx+\frac{C}{|Q|}\int_QI_4(x)dx+\frac{1}{|Q|}\int_QI_5(x)dx\\&:=I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5 , respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j; x, y) \leq C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO},$$

thus, by the (L^r, L^q) -boundedness of g_{λ} for $1 < r < n/\delta$ and $1/q = 1/r - \delta/n$, we obtain

$$\begin{aligned}I_1 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q |g_{\lambda}(f_1)(x)| dx \\&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_Q |g_{\lambda}(f_1)(x)|^q dx \right)^{1/q} \\&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) |Q|^{-1/q} \left(\int_Q |f_1(x)|^r dx \right)^{1/r} \\&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).\end{aligned}$$

For I_2 , denoting $r = pq$ for $1 < p < n/\delta$, $q > 1$, $1/q + 1/q' = 1$ and $1/s = 1/p - \delta/n$, we have, by Hölder's inequality,

$$\begin{aligned}I_2 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |g_{\lambda}(D^{\alpha_1} \tilde{A}_1 f_1)(x)| dx \\&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |g_{\lambda}(D^{\alpha_1} \tilde{A}_1 f_1)(x)|^s dx \right)^{1/s} \\&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} |Q|^{-1/s} \left(\int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) f_1(x)|^p dx \right)^{1/p} \\&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(x)|^{pq'} dx \right)^{1/pq'} \left(\frac{1}{|Q|^{1-r\delta/n}} \int_{\tilde{Q}} |f(x)|^{pq} dx \right)^{1/pq} \\&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).\end{aligned}$$

For I_3 , similar to the proof of I_2 , we get

$$I_3 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

Similarly, for I_4 , denoting $r = pq_3$ for $1 < p < n/\delta$, $q_1, q_2, q_3 > 1$, $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $1/s = 1/p - \delta/n$, we obtain

$$\begin{aligned} I_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |g_\lambda(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)| dx \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |g_\lambda(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)|^s dx \right)^{1/s} \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/s} \left(\int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x) f_1(x)|^p dx \right)^{1/p} \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(x)|^{pq_1} dx \right)^{1/pq_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_2} \tilde{A}_2(x)|^{pq_2} dx \right)^{1/pq_2} \\ &\quad \times \left(\frac{1}{|Q|^{1-r\delta/n}} \int_{\tilde{Q}} |f(x)|^{pq_3} dx \right)^{1/pq_3} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

For I_5 , we write

$$\begin{aligned} &\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x, y) - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0, y) \\ &= \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \right] \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z)}{|x-z|^m} \psi_t(y-z) f_2(z) dz \\ &\quad + \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \int_{R^n} \left(\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z) \psi_t(y-z) f_2(z) dz \\ &\quad + \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \int_{R^n} (R_{m_1}(\tilde{A}_1; x, z) - R_{m_1}(\tilde{A}_1; x_0, z)) \frac{R_{m_2}(\tilde{A}_2; x, z)}{|x_0-z|^m} \psi_t(y-z) f_2(z) dz \\ &\quad + \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \int_{R^n} (R_{m_2}(\tilde{A}_2; x, z) - R_{m_2}(\tilde{A}_2; x_0, z)) \frac{R_{m_1}(\tilde{A}_1; x_0, z)}{|x_0-z|^m} \psi_t(y-z) f_2(z) dz \\ &\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \frac{R_{m_2}(\tilde{A}_2; x, z)(x-z)^{\alpha_1}}{|x-z|^m} \right. \\ &\quad \left. - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \frac{R_{m_2}(\tilde{A}_2; x_0, z)(x_0-z)^{\alpha_1}}{|x_0-z|^m} \right] D^{\alpha_1} \tilde{A}_1(z) \psi_t(y-z) f_2(z) dz \\ &\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \frac{R_{m_1}(\tilde{A}_1; x, z)(x-z)^{\alpha_2}}{|x-z|^m} \right. \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \frac{R_{m_1}(\tilde{A}_1; x_0, z)(x_0-z)^{\alpha_2}}{|x_0-z|^m} \Big] D^{\alpha_2} \tilde{A}_2(z) \psi_t(y-z) f_2(z) dz \\
& + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \frac{(x-z)^{\alpha_1+\alpha_2}}{|x-z|^m} - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \frac{(x_0-z)^{\alpha_1+\alpha_2}}{|x_0-z|^m} \right] \\
& \quad \times D^{\alpha_1} \tilde{A}_1(z) D^{\alpha_2} \tilde{A}_2(z) \psi_t(y-z) f_2(z) dz \\
& = I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)} + I_5^{(7)}.
\end{aligned}$$

By Lemma 1 and the following inequality (see [16]):

$$|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $z \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}
|R_m(\tilde{A}; x, z)| &\leq C|x-z|^m \sum_{|\alpha|=m} (\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,z)} - (D^\alpha A)_{\tilde{Q}}|) \\
&\leq Ck|x-z|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}.
\end{aligned}$$

Note that $|x-z| \sim |x_0-z|$ for $x \in Q$ and $z \in R^n \setminus \tilde{Q}$ and by using the inequality: $a^{1/2} - b^{1/2} \leq (a-b)^{1/2}$ for $a \geq b > 0$, we obtain, similar to the proof of Lemma 4,

$$\begin{aligned}
\|I_5^{(1)}\| &\leq C \int_{R^n} \left(\int_{R_+^{n+1}} \left[\frac{t^{n\lambda/2} |x-x_0|^{1/2} |\psi_t(y-z)| |f_2(z)| \prod_{j=1}^2 |R_{m_j}(\tilde{A}_j; x, z)|}{(t+|x-y|)^{(n\lambda+1)/2}} \right]^2 \frac{dydt}{t^{n+1}} \right)^{1/2} dz \\
&\leq C \int_{R^n} \frac{|x-x_0|^{1/2} |f_2(z)| \prod_{j=1}^2 |R_{m_j}(\tilde{A}_j; x, z)|}{|x-z|^m} \\
&\quad \times \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda+1} \frac{t^{-n} dydt}{(t+|y-z|)^{2n+2-2\delta}} \right)^{1/2} dz \\
&\leq C \int_{R^n} \frac{|x-x_0|^{1/2} |f_2(z)| \prod_{j=1}^2 |R_{m_j}(\tilde{A}_j; x, z)|}{|x-z|^m} \left(\int_0^\infty \frac{dt}{(t+|x-z|)^{2n+2-2\delta}} \right)^{1/2} dz \\
&\leq C \int_{R^n} \frac{\prod_{j=1}^2 |R_{m_j}(\tilde{A}_j; x, z)| |f_2(z)| |x-x_0|^{1/2}}{|x_0-z|^{m+n+1/2-\delta}} dz \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}} |f(z)| dz \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \sum_{k=1}^\infty k^2 2^{-k/2} \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}} \int_{2^k\tilde{Q}} |f(z)| dz \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \\
\|I_5^{(2)}\| &\leq C \int_{R^n} \frac{|x-x_0|}{|x_0-z|^{m+n+1-\delta}} \prod_{j=1}^2 |R_{m_j}(\tilde{A}_j; x, z)| |f_2(z)| dz \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|}{|x_0-z|^{n+1-\delta}} |f(z)| dz
\end{aligned}$$

$$\begin{aligned} &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 2^{-k} \frac{1}{|2^k \tilde{Q}|^{1-\delta/n}} \int_{2^k \tilde{Q}} |f(z)| dz \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

For $I_5^{(3)}$ and $I_5^{(4)}$, by the formula (see [5]):

$$R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x, x_0)(x - z)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)| \leq C \sum_{|\beta| < m} \sum_{|\alpha|=m} |x - x_0|^{m-|\beta|} |x - z|^{|\beta|} \|D^\alpha A\|_{BMO},$$

thus, similar to the proof of Lemma 4,

$$\begin{aligned} \|I_5^{(3)}\| &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} k \frac{|x - x_0|}{|x_0 - z|^{n+1-\delta}} |f(y)| dy \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \\ \|I_5^{(4)}\| &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

Similarly, we get

$$\begin{aligned} \|I_5^{(5)}\| &\leq C \sum_{|\alpha_1|=m_1} \int_{R^n} \left\| \left[\left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} \frac{R_{m_2}(\tilde{A}_2; x, z)(x - z)^{\alpha_1}}{|x - z|^m} \right. \right. \\ &\quad \left. \left. - \left(\frac{t}{t + |x_0 - y|} \right)^{n\lambda/2} \frac{R_{m_2}(\tilde{A}_2; x_0, z)(x_0 - z)^{\alpha_1}}{|x_0 - z|^m} \right] \psi_t(y - z) \right\| |D^{\alpha_1} \tilde{A}_1(z)| |f_2(z)| dz \\ &\leq C \sum_{|\alpha|=m_2} \|D^\alpha A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k (2^{-k/2} + 2^{-k}) \\ &\quad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{r'} dy \right)^{1/r'} \left(\frac{1}{|2^k \tilde{Q}|^{1-r\delta/n}} \int_{2^k \tilde{Q}} |f(y)|^r dy \right)^{1/r} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}); \\ \|I_5^{(6)}\| &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

For $I_5^{(7)}$, taking $q_1, q_2 > 1$ such that $1/r + 1/q_1 + 1/q_2 = 1$, then

$$\begin{aligned} \|I_5^{(7)}\| &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left\| \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \frac{(x-z)^{\alpha_1+\alpha_2}}{|x-z|^m} \right. \right. \\ &\quad \left. \left. - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \frac{(x_0-z)^{\alpha_1+\alpha_2}}{|x_0-z|^m} \right] \psi_t(y-z) \right\| |D^{\alpha_1} \tilde{A}_1(z)| |D^{\alpha_2} \tilde{A}_2(z)| |f_2(z)| dz \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} k(2^{-k/2} + 2^{-k}) \left(\frac{1}{|2^k \tilde{Q}|^{1-p\delta/n}} \int_{2^k \tilde{Q}} |f(y)|^r dy \right)^{1/r} \\ &\quad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{q_1} dy \right)^{1/q_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_2} \tilde{A}_2(y)|^{q_2} dy \right)^{1/q_2} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

Thus

$$\|I_5\| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

We choose $1 < r < p$ in (1), then (2) follows from Lemma 2. For (3), taking $1 < r < \min(p, (n-\sigma)/p\delta)$ in (1) and by Lemma 3, we obtain

$$\begin{aligned} \|g_\lambda^A(f)\|_{L^{q,\sigma}} &\leq C \|M(g_\lambda^A(f))\|_{L^{q,\sigma}} \leq C \|(g_\lambda^A(f))^\#\|_{L^{q,\sigma}} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|M_{\delta,r}(f)\|_{L^{q,\sigma}} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|(M_{r\delta/n}(|f|^r))^{1/r}\|_{L^{q,\sigma}} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|M_{r\delta/n}(|f|^r)\|_{L^{q/r,\sigma}}^{1/r} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \||f|^r\|_{L^{p/r,\sigma}}^{1/r} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|f\|_{L^{p,\sigma}}. \end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. There only remains to prove (1). Let $Q, \tilde{Q}, \tilde{A}_j(x), f_1$ and f_2 be the same as the proof of Theorem 1. We write

$$F_t^A(f)(x, y) = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z) dz$$

$$\begin{aligned}
& + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f_1(z) dz \\
& - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, z)(x - z)^{\alpha_1} D^{\alpha_1} \tilde{A}_1(z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f_1(z) dz \\
& - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, z)(x - z)^{\alpha_2} D^{\alpha_2} \tilde{A}_2(z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f_1(z) dz \\
& + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x - z)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(z) D^{\alpha_2} \tilde{A}_2(z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f_1(z) dz,
\end{aligned}$$

then

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q \left| \mu_\lambda^A(f)(x) - \mu_\lambda^{\tilde{A}}(f_2)(x_0) \right| dx \\
& \leq \frac{1}{|Q|} \int_Q \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} F_t^A(f)(x, y) - \left(\frac{t}{t + |x_0 - y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0, y) \right\| dx \\
& \leq \frac{1}{|Q|} \int_Q \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f_1(z) dz \right\| dx \\
& \quad \frac{C}{|Q|} \int_Q \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, z)(x - z)^{\alpha_1} D^{\alpha_1} \tilde{A}_1(z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f_1(z) dz \right\| dx \\
& \quad + \frac{C}{|Q|} \int_Q \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, z)(x - z)^{\alpha_2} D^{\alpha_2} \tilde{A}_2(z)}{|x - z|^m} \frac{\Omega(y - z)}{|y - z|^{n-1-\delta}} f_1(z) dz \right\| dx \\
& \quad v + \frac{C}{|Q|} \int_Q \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \frac{(x - z)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(z) D^{\alpha_2} \tilde{A}_2(z) \Omega(y - z)}{|x - z|^m |y - z|^{n-1-\delta}} f_1(z) dz \right\| dx \\
& \quad + \frac{1}{|Q|} \int_Q \left\| \left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x, y) - \left(\frac{t}{t + |x_0 - y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0, y) \right\| dx \\
& := J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Similar to the proof of Theorem 1, we get

$$\begin{aligned}
J_1 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q |\mu_\lambda(f_1)(x)| dx \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_Q |\mu_\lambda(f_1)(x)|^q dx \right)^{1/q} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta, r}(f)(\tilde{x}). \\
J_2 & \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |\mu_\lambda(D^{\alpha_1} \tilde{A}_1 f_1)(x)| dx \\
& \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |\mu_\lambda(D^{\alpha_1} \tilde{A}_1 f_1)(x)|^s dx \right)^{1/s}
\end{aligned}$$

$$\begin{aligned}
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \\
J_3 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}). \\
J_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |\mu_\lambda(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)| dx \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |\mu_\lambda(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)|^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,p}(f)(\tilde{x}).
\end{aligned}$$

For J_5 , we write

$$\begin{aligned}
&\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x, y) - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0, y) \\
&= \int_{R^n} \left(\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \right) \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z) dz \\
&\quad + \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \int_{R^n} \left(\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z) \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z) dz \\
&\quad + \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \int_{R^n} (R_{m_1}(\tilde{A}_1; x, z) - R_{m_1}(\tilde{A}_1; x_0, z)) \frac{R_{m_2}(\tilde{A}_2; x, z)}{|x_0-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z) dz \\
&\quad + \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \int_{R^n} (R_{m_2}(\tilde{A}_2; x, z) - R_{m_2}(\tilde{A}_2; x_0, z)) \frac{R_{m_1}(\tilde{A}_1; x_0, z)}{|x_0-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z) dz \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \frac{R_{m_2}(\tilde{A}_2; x, z)(x-z)^{\alpha_1}}{|x-z|^m} \right. \\
&\quad \left. - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \frac{R_{m_2}(\tilde{A}_2; x_0, z)(x_0-z)^{\alpha_1}}{|x_0-z|^m} \right] \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} D^{\alpha_1} \tilde{A}_1(z) f_2(z) dz \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \frac{R_{m_1}(\tilde{A}_1; x, z)(x-z)^{\alpha_2}}{|x-z|^m} \right. \\
&\quad \left. - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \frac{R_{m_1}(\tilde{A}_1; x_0, z)(x_0-z)^{\alpha_2}}{|x_0-z|^m} \right] \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} D^{\alpha_2} \tilde{A}_2(z) f_2(z) dz \\
&\quad + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} \frac{(x-z)^{\alpha_1+\alpha_2}}{|x-z|^m} - \left(\frac{t}{t+|x_0-y|} \right)^{n\lambda/2} \frac{(x_0-z)^{\alpha_1+\alpha_2}}{|x_0-z|^m} \right] \\
&\quad \times \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} D^{\alpha_1} \tilde{A}_1(z) D^{\alpha_2} \tilde{A}_2(z) f_2(z) dz,
\end{aligned}$$

then, similar to the proof of Lemma 4 and Theorem 1, we get

$$\|J_5\| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

This finishes the proof.

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