

SHARP FUNCTION ESTIMATE FOR VECTOR-VALUED
MULTILINEAR COMMUTATOR OF FRACTIONAL AREA
INTEGRAL OPERATOR *

Weiping Kuang

Abstract. In this paper, we prove the sharp function inequality for vector-valued multilinear commutator of fractional area integral operator. By using the sharp inequality, we obtain the boundedness of the commutator from $L^p(R^n)$ to $L^q(R^n)$, where $\vec{b} = (b_1, \dots, b_m)$, $b_j \in BMO(R^n)$, $1 \leq j \leq m$.

1. Introduction

As the development of singular integral operators, their commutators have been well studied (see [1-4]). In [4][11-13], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The sharp inequalities of Littlewood-Paley commutators have been well studies(see [5-9]). The main purpose of this paper is to prove the sharp function inequality for vector-valued multilinear commutator of fractional area integral operator. By using the sharp inequality, we obtain the boundedness of the commutator from $L^p(R^n)$ to $L^q(R^n)$, where $\vec{b} = (b_1, \dots, b_m)$, $b_j \in BMO(R^n)$, $1 \leq j \leq m$.

2. Notations and Results

First let us introduce some notations (see [3][11][12]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cube Q , let $f_Q = |Q|^{-1} \int_Q f(x)dx$. The sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

Received July 22, 2012.

2010 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25

*The authors were supported in part by Scientific Research Project of Hunan Province Educational Department 11C0982 and Scientific Research Project of Huaihua University HHUY2010-01 of P.R. China.

It is well-known that (see [3])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{C \in C} \frac{1}{|Q|} \int_Q |f(y) - C| dy.$$

We say that b belongs to $BMO(R^n)$ if $b^\#$ belongs to $L^\infty(R^n)$ and define $\|b\|_{BMO} = \|b^\#\|_{L^\infty}$. It has been known that (see [14])

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck\|b\|_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y)| dy.$$

We write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$. For $0 < \delta < n$, $0 < l < \infty$, set

$$M_{l,\delta}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-\delta l/n}} \int_Q |f(y)|^l dy \right)^{1/l}.$$

For $0 < l \leq p < n/\delta$, $1/q = 1/p - \delta/n$, we have

$$\|M_{l,\delta}(f)\|_{L^q} \leq C\|f\|_{L^p}.$$

Given some functions b_j ($j = 1, \dots, m$) and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

In this paper, we will study vector-valued multilinear commutator of fractional area integral operator as follows.

Definition. Let $0 < \delta < n$. Suppose functions ψ satisfies the following properties:

- (1) $\int_{R^n} \psi(x) dx = 0$;
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$;
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+2-\delta)}$, where $2|y| < |x|$.

Set $1 < r < \infty$, b_j ($j = 1, \dots, m$) are fixed local integrable function on R^n . Let $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x-y| \leq t\}$, its characteristic function is $\chi_{\Gamma(x)}$, vector-valued multilinear commutator of fractional area integral operator is defined as follows:

$$|S_{\psi,\delta}^{\vec{b}}(f)(x)|_r = \left(\sum_{i=1}^{\infty} (S_{\psi,\delta}^{\vec{b}}(f_i)(x))^r \right)^{1/r},$$

where

$$S_{\psi,\delta}^{\vec{b}}(f)(x) = \left(\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y-z) f(z) dz$$

when $m = 1$,

$$S_{\psi,\delta}^b(f)(x) = \left(\int \int_{\Gamma(x)} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where

$$F_t^b(f)(x,y) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y-z) f(z) dz,$$

$\psi_t(x) = t^{-n+\delta} \psi(x/t)$, $t > 0$. Define

$$S_{\psi,\delta}(f)(x) = \left[\int \int_{\Gamma(x)} |F_t(f)(x)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

This is the fractional area of integral operator [12].

Now we state our theorems as following.

Theorem 1. Let $1 < r < \infty$, $0 < \delta < n$, $b_j \in BMO(R^n)$, where $j = 1, \dots, m$. Then for any $1 < l < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,

$$(|S_{\psi,\delta}^{\vec{b}}(f)|_r)^\#(\tilde{x}) \leq C \left(\|\vec{b}\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} M_l(|S_{\psi,\delta}^{\vec{b}_{\sigma^c}}(f)|_r)(\tilde{x}) \right).$$

Theorem 2. Let $1 < r < \infty$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$, $0 < \delta < n$, $b_j \in BMO(R^n)$, where $j = 1, \dots, m$. Then $|S_{\psi,\delta}^{\vec{b}}|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas, which is well known.

Lemma 1. Let $1 < r < \infty$, $b_j \in BMO(R^n)$, $j = 1, \dots, k$, $k \in N$. Then

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leq C \prod_{j=1}^k \|b_j\|_{BMO}$$

and

$$\left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \leq C \prod_{j=1}^k \|b_j\|_{BMO}.$$

Proof of Lemma 1. Let $1 < p_j < \infty$, $j = 1, \dots, k$ satisfies $1/p_1 + \dots + 1/p_k = 1$, by the Hölder's inequality,

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leq \prod_{j=1}^k \left(\frac{1}{|Q|} \int_Q |b_j(y) - (b_j)_Q|^{p_j} dy \right)^{1/p_j} \leq C \prod_{j=1}^k \|b_j\|_{BMO},$$

and

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \leq \prod_{j=1}^k \left(\frac{1}{|Q|} \int_Q |b_j(y) - (b_j)_Q|^{p_j r} dy \right)^{1/p_j r} \\ & \leq C \prod_{j=1}^k \|b_j\|_{BMO}. \end{aligned}$$

Lemma 2. [12] Let $1 < r < \infty$, $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$. then $|S_{\psi, \delta}|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\begin{aligned} & \frac{1}{|Q|} \int_Q ||S_{\psi, \delta}^{\vec{b}}(f)(x)|_r - C_0| dx \\ & \leq C \left(\|\vec{b}\|_{BMO} M_{l, \delta}(|f|_r)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} M_l(|S_{\psi, \delta}^{\vec{b}_{\sigma^c}}(f)|_r)(\tilde{x}) \right). \end{aligned}$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$, write, $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_{2Q}$, $h_i = f_i \chi_{(2Q)^c}$.

We first consider the **Case** $m = 1$.

Write,

$$F_t^{b_1}(f_i)(x, y) = (b_1(x) - (b_1)_{2Q}) F_t(f_i)(y) - F_t((b_1 - (b_1)_{2Q}) g_i)(y) - F_t((b_1 - (b_1)_{2Q}) h_i)(y).$$

By Minkowski's inequality, we get

$$\begin{aligned} & \frac{1}{|Q|} \int_Q ||S_{\psi, \delta}^{b_1}(f)(x)|_r - |S_{\psi, \delta}((b_1)_{2Q} - b_1)h|(x_0)_r| dx \\ & \leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left\| (b_1(x) - (b_1)_{2Q}) \chi_{\Gamma(x)} F_t(f_i)(y) \right\|^r \right)^{1/r} dx \\ & \quad + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left\| \chi_{\Gamma(x)} F_t((b_1 - (b_1)_{2Q}) g_i)(y) \right\|^r \right)^{1/r} dx \\ & \quad + \frac{1}{|Q|} \int_Q |||\chi_{\Gamma(x)} F_t((b_1 - (b_1)_{2Q}) h)(y) - \chi_{\Gamma(x_0)} F_t((b_1 - (b_1)_{2Q}) h)(y)|||_r dx \\ & = I + II + III. \end{aligned}$$

For I , by the Hölder's inequality with exponent, $1/l + 1/l' = 1$, we get

$$\begin{aligned} I &= \frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}| |S_{\psi, \delta}(f)(x)|_r dx \\ &\leq C \left(\frac{1}{|2Q|} \int_{2Q} |b_1(x) - (b_1)_{2Q}|^{l'} dx \right)^{1/l'} \left(\frac{1}{|Q|} \int_Q |S_{\psi, \delta}(f)(x)|_r^l dx \right)^{1/l} \\ &\leq C \|b_1\|_{BMO} M_l(|S_{\psi, \delta}(f)|_r)(\tilde{x}). \end{aligned}$$

For II , choose $1 < l < p < q < n/\delta$, $1/q = 1/p - \delta/n$, $l = ps$, by Lemma 2 and Hölder's inequality, we get

$$\begin{aligned} II &\leq \left(\frac{1}{|Q|} \int_{R^n} |S_{\psi, \delta}((b_1 - (b_1)_{2Q})f_1)(x)|_r^q dx \right)^{1/q} \\ &\leq \frac{C}{|Q|^q} \left(\int_{2Q} |(b_1(x) - (b_1)_{2Q})f(x)|_r^p dx \right)^{1/p} \\ &\leq C |Q|^{(-1/q)+(1/ps')+(1-\delta ps/n)/ps} \left(\frac{1}{|2Q|} \int_{2Q} |b_1(x) - (b_1)_{2Q}|^{ps'} dx \right)^{1/ps'} \\ &\quad \times \left(\frac{1}{|2Q|^{1-\delta ps/n}} \int_{2Q} |f(x)|_r^{ps} dx \right)^{1/ps} \\ &= C |Q|^{(-1/q)+(1/ps')+(1/ps-\delta/n)} \left(\frac{1}{|2Q|} \int_{2Q} |b_1(x) - (b_1)_{2Q}|^{ps'} dx \right)^{1/ps'} \\ &\quad \times \left(\frac{1}{|2Q|^{1-\delta l/n}} \int_{2Q} |f(x)|_r^l dx \right)^{1/l} \\ &\leq C \|b_1\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}). \end{aligned}$$

For III , by Minkowski's inequality, we obtain

$$\begin{aligned} III(x) &\leq \left[\int \int_{R_+^{n+1}} \left(\int_{(2Q)^c} |\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}| |b_1(z) - (b_1)_{2Q}| |\psi_t(y-z)| |f(z)|_r \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\ &\leq C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r \left| \int \int_{|x-y|\leq t} \frac{t^{1-n} dydt}{(t+|y-z|)^{2n+2-2\delta}} \right. \\ &\quad \left. - \int \int_{|x_0-y|\leq t} \frac{t^{1-n} dydt}{(t+|y-z|)^{2n+2-2\delta}} \right|^{1/2} dz \\ &\leq \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r \left(\int \int_{|y|\leq t, |x+y-z|\leq t} \left| \frac{1}{(t+|x+y-z|)^{2n+2-2\delta}} \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}} \left| \frac{dydt}{t^{n-1}} \right|^{1/2} dz \\
\leq & \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r \\
& \times \left(\int \int_{|y|\leq t, |x+y-z|\leq t} \frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}} dydt \right)^{1/2} dz,
\end{aligned}$$

note that when $|y| \leq t$, $2t + |x+y-z| \geq 2t + |x-z| - |y| \geq t + |x-z|$, we can easily obtain

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+3-2\delta}} = C|x-z|^{-2n-1+2\delta},$$

then, for $x \in Q$,

$$\begin{aligned}
& III(x) \\
\leq & C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r \left(\int \int_{|y|\leq t} \frac{2^{2n+3-2\delta} |x_0-x| t^{1-n} dydt}{(2t+2|x+y-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
\leq & C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r |x-x_0|^{1/2} \left(\int \int_{|y|\leq t} \frac{t^{1-n} dydt}{(2t+|x+y-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
\leq & C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r |x-x_0|^{1/2} \left(\int \int_{|y|\leq t} \frac{t^{1-n} dydt}{(t+|x-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
\leq & C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r |x-x_0|^{1/2} \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
\leq & C \int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}| |f(z)|_r \frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}} dz \\
\leq & C \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^kQ} |x_0-x|^{1/2} |x_0-z|^{-(n+1/2-\delta)} |b_1(z) - (b_1)_{2Q}| |f(z)|_r dz \\
\leq & C \sum_{k=1}^\infty 2^{-k/2} \left(\frac{1}{|2^{k+1}Q|^{1-\delta l/n}} \int_{2^{k+1}Q} |f(z)|_r^l dz \right)^{1/l} \\
& \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(z) - (b_1)_{2Q}|^{r'} dz \right)^{1/r'} \\
\leq & C \sum_{k=1}^\infty k 2^{-k/2} \|b_1\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}) \\
\leq & C \|b_1\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}).
\end{aligned}$$

Thus

$$III \leq \frac{1}{|Q|} \int_Q III(x) dx \leq C \|b_1\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}).$$

Now, we consider the **Case** $m \geq 2$, we have known that, for $b = (b_1, \dots, b_m)$,

$$\begin{aligned}
F_t^{\vec{b}}(f_i)(x, y) &= \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y - z) f_i(z) dz \\
&= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_{2Q}) - (b_j(z) - (b_j)_{2Q})] \psi_t(y - z) f_i(z) dz \\
&= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(z) - (b)_{2Q})_{\sigma^c} \psi_t(y - z) f_i(z) dz \\
&= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f_i)(y) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_i)(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(z) - b(x))_{\sigma^c} \psi_t(y - z) f_i(z) dz \\
&= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f_i)(y) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_i)(y) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} c_{m,j} (b(x) - (b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f_i)(x, y),
\end{aligned}$$

thus, by Minkowski's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q ||S_{\psi, \delta}^{\vec{b}}(f)(x)|_r - |S_{\psi, \delta}^{\vec{b}}(((b_1)_{2Q} - b_1) \cdots ((b_m)_{2Q} - b_m)) h)(x_0)|_r| dx \\
&\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \| (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) \chi_{\Gamma(x)} F_t(f_i)(y) \|^r \right)^{1/r} dx \\
&\quad + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \| (b(x) - (b)_{2Q})_\sigma \chi_{\Gamma(x)} F_t^{\vec{b}_{\sigma^c}}(f_i)(x, y) \|^r \right)^{1/r} dx \\
&\quad + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \| \chi_{\Gamma(x)} F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) g_i)(y) \|^r \right)^{1/r} dx \\
&\quad + \frac{1}{|Q|} \int_Q \left\| \chi_{\Gamma(x)} F_t \left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) h \right) (y) - \chi_{\Gamma(x_0)} F_t \left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) h \right) (y) \right\|_r dx \\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned}$$

For K_1 , by Hölder's inequality, choose $1 < p_j < \infty$, $j = 1, \dots, m$, such that $1/p_1 + \cdots + 1/p_m + 1/l = 1$, we get

$$K_1 = \frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}| \cdots |b_m(x) - (b_m)_{2Q}| |S_{\psi, \delta}(f)(x)|_r dx$$

$$\begin{aligned}
&\leq \left(\frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}|^{p_1} \right)^{1/p_1} \cdots \left(\frac{1}{|Q|} \int_Q |b_m(x) - (b_m)_{2Q}|^{p_m} dx \right)^{1/p_m} \\
&\quad \times \left(\frac{1}{|Q|} \int_Q |S_{\psi,\delta}(f)(x)|_r^l dx \right)^{1/l} \\
&\leq C \|\vec{b}\|_{BMO} M_l(|S_{\psi,\delta}(f)|_r)(\tilde{x}).
\end{aligned}$$

For K_2 , by Lemma 1 and Minkowski's inequality, we obtain

$$\begin{aligned}
K_2 &\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma| |S_{\psi,\delta}^{\vec{b}_\sigma}(f)(x)|_r dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|2Q|} \int_{2Q} |(b(x) - (b)_{2Q})_\sigma|^{l'} dx \right)^{1/l'} \left(\frac{1}{|Q|} \int_Q |S_{\psi,\delta}^{\vec{b}_\sigma}(f)(x)|_r^l dx \right)^{1/l} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} M_l(|S_{\psi,\delta}^{\vec{b}_\sigma}(f)|_r)(\tilde{x}).
\end{aligned}$$

For K_3 , choose $1 < l < p < q < n/\delta$, $1/q = 1/p - \delta/n$, $l = ps$, by Lemma 1 and Hölder's inequality, we obtain

$$\begin{aligned}
K_3 &\leq \left(\frac{1}{|Q|} \int_{R^n} |S_{\psi,\delta}((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f(x) \chi_{2Q})(x)|_r^q dx \right)^{1/q} \\
&\leq \frac{C}{|Q|^{1/q}} \left(\int_{R^n} |(b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q})|^p |f(x)|_r^p \chi_{2Q}(x) dx \right)^{1/p} \\
&\leq C \|\vec{b}\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}).
\end{aligned}$$

For K_4 , similar to the proof of III in Case $m = 1$, we obtain

$$K_4(x) \leq C \int_{(2Q)^c} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_{2Q}) \right| |f(z)|_r dz,$$

taking $1 < p_j < \infty$, $j = 1, \dots, m$ such that $1/p_1 + \cdots + 1/p_m + 1/r = 1$, then, for $x \in Q$,

$$\begin{aligned}
K_4(x) &\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}Q} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_{2Q}) \right| |f(z)|_r dz \\
&\leq C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^{k+1}Q|^{1-\delta l/n}} \int_{2^{k+1}Q} |f(z)|_r^l dz \right)^{1/l} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \left| \prod_{j=1}^m (b_j(z) - (b_j)_{2Q}) \right|^{l'} dz \right)^{1/l'}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=1}^{\infty} k^m 2^{-k/2} \prod_{j=1}^m \|b_j\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}) \\
&\leq C \sum_{k=1}^{\infty} k^m 2^{-k/2} \|\vec{b}\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}) \\
&\leq C \|\vec{b}\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}),
\end{aligned}$$

thus

$$K_4 \leq \frac{1}{|Q|} \int_Q K_4(x) dx \leq C \|\vec{b}\|_{BMO} M_{l,\delta}(|f|_r)(\tilde{x}).$$

This completes the proof of the theorem.

Proof of Theorem 2. We first consider the **Case** $m = 1$. Choose $1 < l < p$, we get

$$\begin{aligned}
\|S_{\psi,\delta}^{b_1}(f)|_r\|_{L^q} &\leq \|M(|S_{\psi,\delta}^{b_1}(f)|_r)\|_{L^q} \leq C \|(|S_{\psi,\delta}^{b_1}(f)|_r)^\# \|_{L^q} \\
&\leq C \|M_l(|S_{\psi,\delta}(f)|_r)\|_{L^q} + C \|M_{l,\delta}(|f|_r)\|_{L^q} \\
&\leq C \|S_{\psi,\delta}(f)|_r\|_{L^q} + C \|M_{l,\delta}(|f|_r)\|_{L^q} \\
&\leq C \|f|_r\|_{L^p}.
\end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof.

R E F E R E N C E S

1. J. ALVAREZ, R. J. BABGY, D. S. KURTZ AND C. PÉREZ: *Weighted estimates for commutators of linear operators*. Studia Math., 104(1993), 195-209.
2. R. COIFMAN, R. ROCHBERG AND G. WEISS: *Factorization theorems for Hardy spaces in several variables*. Ann. of Math., 103(1976), 611-635.
3. J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA: *Weighted norm inequalities and related topics*. North-Holland Math., 116, Amsterdam, 1985.
4. G. E. HU AND D. C. YANG: *A variant sharp estimate for multilinear singular integral operators*. Studia Math., 141(2000), 25-42.
5. L. Z. LIU: *Weighted weak type estimates for commutators of Littlewood-Paley operator*. Japanese J. of Math., 29(1)(2003), 1-13.
6. L. Z. LIU: *A sharp endpoint estimate for multilinear Littlewood-Paley operator*. Georgian Math. J., 11(2004), 361-370.
7. L. Z. LIU: *Sharp endpoint inequality for multilinear Littlewood-Paley operator*. Kodai Math. J., 27(2004), 134-143.
8. L. Z. LIU: *A sharp estimate for multilinear Marcinkiewicz integral operator*. The Asian J. of Math., 9(2)(2005), 177-184.
9. L. Z. LIU AND S.Z. LU: *Weighted weak type inequalities for maximal commutators of Bochner-Riesz operator*. Hokkaido Math. J., 32(1)(2003), 85-99.

10. C. PÉREZ: *Endpoint estimate for commutators of singular integral operators.* J. Func. Anal., 128(1995), 163-185.
11. C. PÉREZ AND G. PRADOLINI: *Sharp weighted endpoint estimates for commutators of singular integral operators.* Michigan Math. J., 49(2001), 23-37.
12. C. PÉREZ AND R. TRUJILLO-GONZALEZ: *Sharp Weighted estimates for multilinear commutators.* J. London Math. Soc., 65(2002), 672-692.
13. E. M. STEIN: *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals.* Princeton Univ. Press, Princeton NJ, 1993.
14. A. TORCHINSKY : *Real variable methods in harmonic analysis.* Pure and Applied Math., 123, Academic Press, New York, 1986.

Weiping Kuang
Department of Mathematics
Huaihua university
Huaihua 418008, Hunan, P.R.of China
kuangweipinppp@163.com