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INVERSE CHARACTER FORMULA FOR VILENKIN SYSTEMS

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Abstract. The aim of this paper is to express indices of inverse characters in the Vilenkin system. This latter follows specific transformations when constructed upon different bases.

1. Introduction

A common way to define the Vilenkin system $(\chi_n)_n$ is from the so-called basis of Rademacher functions $(r_n)_n$ known in a specific class of Vilenkin groups obtained by an infinite product of the groups $\mathbb{Z}_{p_n} := \{0, 1, \ldots, p_n - 1\}$.

Let G be a Vilenkin group i.e. an infinite, totally disconnected, compact abelian group which satisfies the second axiom of countability. The topology on G is determined by a chain of open subgroups $G = G_0 \supset G_1 \supset \ldots \supset G_n \supset \ldots \supset \{0\}$, $\bigcap_{n=0}^{\infty} G_n = \{0\}$. We may assume that G_n/G_{n+1} is a cyclic group of prime order p_{n+1} for every natural number n.

For $j \in \mathbb{N}$ we denote $m_j := p_1 p_2 \ldots p_j$, $(m_0 := 1)$.

A classical example of a Vilenkin group is the product space

$$\prod_{j=1}^{\infty} \mathbb{Z}_2,$$

where $\mathbb{Z}_2 = \{0, 1\}$ is the discrete cyclic group of second order, equipped with the discrete topology and component adding (note that addition in each component is done modulo 2). Its direct generalization is the group

$$\prod_{k=0}^{\infty} \mathbb{Z}_{n_k},$$

where $\mathbb{Z}_{n_k} := \{0, 1, \ldots, n_k - 1\}$, $n_k \geq 2$, is a cyclic group of order n_k $(k \in \mathbb{N} \cup \{0\})$ equipped with the discrete topology.

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For any fixed sequence of elements $g_i \in G_i \setminus G_{i+1}, i \in \mathbb{N} \cup \{0\}$, every x from G can be written in a unique way as

$$x = \sum_{i=0}^{\infty} x_i g_i, \tag{1.1}$$

where $0 \leq x_i < p_{i+1}$.

Let μ be a Haar measure on G. It is chosen so that $\mu(G_0) = 1$, and $\mu(G_n) = m_n^{-1}$, where $m_n = p_1 p_2 \ldots p_n$.

Each $n \in \mathbb{N}$ can be uniquely represented in the following form

$$n = \sum_{j=0}^{\infty} n_j m_j, \; n_j \in \{0, 1, \ldots, p_{j+1} - 1\} \tag{1.2}$$

where only a finite number of n_j differ from zero. If $j(n) \in \mathbb{N} \cup \{0\}$ is the smallest of all $s \in \mathbb{N} \cup \{0\}$ with the following property $n_j = 0 \; (\forall j > s)$ and $n_{j(n)} \neq 0$, then

$$n = \sum_{j=0}^{j(n)} n_j m_j, \; n_j \in \{0, 1, \ldots, p_{j+1} - 1\}, \; 1 \leq n_{j(n)} < p_{j(n)+1} \tag{1.3}$$

which is equivalent to $m_{j(n)} \leq n < m_{j(n)+1}$.

G has a countable collection of characters, i.e., continuous complex valued functions χ, that satisfy the following condition

$$\chi(x + y) = \chi(x)\chi(y), \; (\forall x, y \in G).$$

That collection is denoted by Γ. The characters form an abelian group with respect to the pointwise product of functions. The topology in Γ is defined by a neighborhood basis around the unit

$$\chi_0 \in \Gamma \;\; (\chi_0(x) = 1, \forall x \in G)$$

using the collection of all sets

$$U(A, \varepsilon) := \{\chi \in \Gamma : |\chi(a) - 1| < \varepsilon, \, \forall a \in A\},$$

where A goes over the collection of all compact subsets in G and ε varies in the set of positive numbers.

It is known [5] that $(\Gamma, \cdot)$ is a discrete, countable and abelian group with torsion. Additionally, Vilenkin proved in [7] that Γ is the union of the increasing sequence of groups $\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1, \forall x \in G_n\}$ and that Γ_{n+1}/Γ_n ($\forall n \in \mathbb{N} \cup \{0\}$) is cyclic with prime order p_{n+1}.

The characters on G are constructed in the following way:

For any $n \in \mathbb{N} \cup \{0\}$, there exists some $\chi_{m_n} \in \Gamma_{n+1} \backslash \Gamma_n$, such that $\chi_{m_n}(g_n) = e^{\frac{2\pi i}{p_{n+1}}}$. However, the characters χ_{m_n} are arbitrary in $\Gamma_{n+1} \backslash \Gamma_n$, hence not uniquely determined.

In [7] (see also [1]), the characters of Γ are ordered as follows: put $\chi_0(x) = 1, (\forall x \in G)$, the constant function, $\Gamma_0 := \{\chi_0\}$. Suppose that we have already ordered all the elements of the subgroup

$$\Gamma_j = \{\chi_0, \chi_1, \ldots, \chi_{m_j - 1}\}.$$

In $\Gamma_{j+1} \backslash \Gamma_j$ choose an element χ of minimal order and denote it by χ_{m_j}. For every k such that $m_j \le k < m_{j+1}$, put

$$\chi_k = \chi_{m_j}^{k_j} \cdot \chi_r,$$

where

$$k = k_j \cdot m_j + r,\ 1 \le k_j < p_{j+1}\ \wedge\ 0 \le r < m_j.$$

Then all the elements of the subgroup Γ_{j+1} have been ordered and by induction all the elements of Γ. For n given by (1.2), we obviously have

$$\chi_n = \prod_{j=0}^{\infty} \chi_{m_j}^{n_j},\ 0 \le n_j < p_{j+1}.$$

Since Γ_{j+1}/Γ_j is a cyclic group of prime order p_{j+1}, we have

$$\Gamma_{j+1} \backslash \Gamma_j = \{\chi_{m_j}^s : 1 \le s \le p_{j+1} - 1\}$$

and

$$\Gamma_{j+1}/\Gamma_j = \{[\chi_{m_j}^s] : 1 \le s \le p_{j+1} - 1\},\ [\chi_{m_j}^s] := \chi_{m_j}^s \cdot \Gamma_j.$$

Therefore, every $g_0 \in G_j \setminus G_{j+1}$ satisfies $\chi_{m_j}(g_0) \neq 1\ \wedge\ \chi_{m_j}^{p_{j+1}}(g_0) = 1$. This means that $\chi_{m_j}(g_0) = e^{\frac{2\pi i k}{p_{j+1}}}$, for some $1 \le k < p_{j+1}$. Therefrom,

$$\{e^{\frac{2\pi i k x_j}{p_{j+1}}}\ :\ 1 \le x_j \le p_{j+1} - 1\}$$

is the set of all primitive p_{j+1} *-th* roots of 1 and follows $\{x_j g_0 : 1 \le x_j \le p_{j+1} - 1\} = G_j \setminus G_{j+1}$. Hence, χ_{m_j} takes, on $G_j \setminus G_{j+1}$, all values from the set

$$\{e^{\frac{2\pi i s}{p_{j+1}}}\ :\ 1 \le s \le p_{j+1} - 1\}$$

and only those values. As a consequence, $\exists g_j \in G_j \setminus G_{j+1}$ such that

$$\chi_{m_j}(g_j) = e^{\frac{2\pi i}{p_{j+1}}}. \tag{1.4}$$

Without loss of generality we can assume that g_j appearing in (1.1), are exactly those g_j with property (1.4).

Meanwhile, the author in [6] works on groups of the form $\prod_{k=0}^{\infty} \mathbb{Z}_{n_k}$. In these groups the generalized Rademacher functions are defined as

$$r_n(x) := e^{\frac{2\pi i x_n}{m_n}} \ (x \in G_m, n \in \mathbb{N}).$$

Here we mention that the Rademacher functions can not be defined in every Vilenkin group even if the element x is identified with the sequence $(x_n)_n$ for which $x = \sum\limits_{n=0}^{\infty} x_n g_n$. Because from $r_i^{p_{n+1}}(g_n) = 1$, for every $i, n \geq 0$, we get $r_i(p_{n+1} g_n) = 1$, for arbitrary $i, n \geq 0$, and it implies that $p_{n+1} g_n = 0$ for all n. This property is valid for $\prod_{k=0}^{\infty} \mathbb{Z}_{n_k}$ but not true in every Vilenkin group.

However, the generalized Rademacher functions introduced in [2] or [8] are not included in the previous discussion.

The principal motivations of this work are the comments made in [4] about the non-validity of Lemma 7 obtained in [6]. The assertions made in Lemma 1 and Theorem 1 in [4] are not general, but only related to the assumption made in formula (21) [4]. This latter is supposed not to affect the general case, therefore there is no reason to claim that Lemma 7 in [6] is not valid.

We prove that the counterexample provided in [4] is not an adequate tool. In fact, it is based on a very specific construction of the Vilenkin system. This latter is clearly not uniquely determined. Of course, this only means that the enumeration of characters is modified, because the dual of a given group is obviously unique.

Here we give a proof of Lemma 7 [6] obtained for the group $\prod_{k=0}^{\infty} \mathbb{Z}_{n_k}$. For each $n \in \mathbb{N}$ let $\tilde{n}$ denote the positive integer for which $\chi_n(x)\chi_{\tilde{n}}(x) = 1$, for every $x \in G$.

Lemma 1.1. *[6] If $n \in [m_j, m_{j+1})$, that is $n = \sum\limits_{i=0}^{j} n_i m_i$, $1 \leq n_j < p_{j+1}$; $0 \leq n_i < p_{i+1}$, $i = 0, 1, \ldots, j-1$, then $\tilde{n} \in [m_j, m_{j+1})$ and $\tilde{n} = (p_{j+1} - n_j)m_j + \sum\limits_{i=0, n_i \neq 0}^{j-1} (p_{i+1} - n_i)m_i = m_{j+1} + \sum\limits_{i=0, n_i \neq 0}^{j-1} m_{i+1} - n$.*

Proof. We only need to verify that $\chi_n\chi_{\tilde{n}} \equiv 1$.

We have

$$\begin{aligned}\chi_n\chi_{\tilde{n}} &= (\prod_{i=0,n_i\neq 0}^{j} \chi_{m_i}^{n_i})\chi_{m_j}^{(p_{j+1}-n_j)} \prod_{i=0,n_i\neq 0}^{j-1} \chi_{m_i}^{(p_{i+1}-n_i)} \\ &\quad (\prod_{i=0,n_i\neq 0}^{j-1} \chi_{m_i}^{n_i}\chi_{m_i}^{(p_{i+1}-n_i)})\chi_{m_j}^{n_j}\chi_{m_j}^{(p_{j+1}-n_j)} \\ &= \prod_{i=0,n_i\neq 0}^{j} \chi_{m_i}^{p_{i+1}} \equiv 1.\end{aligned}$$

□. Using the notations of [3] and those of the previous result, as $n \oplus \tilde{n} = 0$, it follows that Lemma1.1 is a direct consequence of $\chi_n\chi_m = \chi_{n\oplus m}$ (see [3, Sect. 1.5]).

Let analyze formula (21) in [4] upon which the counterexample in Proposition 1 [4] was constructed. It claims that $\chi_{m_N}^{p_{N+1}} = \chi_{m_{N-1}}$. It is clear that this assumption does not match with the definition of Rademacher functions where $\chi_{m_N}^{p_{N+1}} = \chi_0$. It is an expected fact that the authors in [4] and [6] obtain different formulae for the expression of $\tilde{k}$.

On the other hand we analyze Lemma 1 (b) in [4] in which it is stated that $\chi_{m_N}(g_{N-1}) = \exp(2\pi i\frac{1}{p_N p_{N+1}})$. But this contradicts the part of introduction where it is assumed that addition is made componentwise modulo p_n, because in this case $\chi_{m_N}^{p_N}(g_{N-1}) = \chi_{m_N}(p_N g_{N-1}) = \chi_{m_N}(0) = 1$. Then the results obtained in [4] are not compatible with the structure of $\prod_{k=0}^{\infty}\mathbb{Z}_{n_k}$.

In the following section, we investigate the general formula that gives $\tilde{k}$ in every Vilenkin group.

2. Main results

As previously mentioned the character $\chi_{m_N}^{p_{N+1}}$ is not necessarily of the form $\chi_{m_{N-1}}$, neither it is forcefully from $\Gamma_{N-1} \setminus \Gamma_{N-2}$. In the following result we give the form of $\tilde{n}$ in the general setting.

As the character $\chi_{m_N}^{p_{N+1}}$ belongs to Γ_{N-1}, it must be of the form

$$\chi_{m_N}^{p_{N+1}} = \prod_{j=0}^{N-1} \chi_{m_j}^{\alpha_j^N},$$

where the nonnegative integers $\alpha_j^N \leq p_{j+1}, j \leq N-1$, are uniquely identified.

Theorem 2.1. *Let G be any Vilenkin group. Then, if $n=\sum_{i=0}^{N} n_i m_i$, then $\tilde{n}=\sum_{l=0}^{N} c_l m_l$, where c_l satisfy $\sum_{i=l}^{N} b_l^i n_i+\sum_{i=l+1}^{N} F_i\alpha_l^i=F_l p_{l+1}+c_l$, for some explicit non-negative integers $(F_i)_i$, and b_l^i, $i\geq l$ satisfy the equations*

$$b_i^i=p_{i+1}-1,$$

$$b_l^i+\sum_{t=l+1}^{i} R_{t+1}\alpha_l^t=R_{l+1}p_{l+1},$$

for $l<i$, where the positive integers R_j, $j\geq 0$, are recursively uniquely determined as $0\leq b_l^i,\alpha_l^t\leq p_{l+1}-1$, for every $0\leq l\leq i,t$.

Proof. We first prove that in the case of $n=m_N$, $c_l=b_l^N$ for every $l=0,\dots,N$. As $b_l^N+\sum_{i=l+1}^{N} F_i\alpha_l^i=F_l p_{l+1}+c_l$, we only need to show that $F_i=0$ for each $i=0,\dots,N$.

From $b_N^N=F_N p_{N+1}+c_N$, it is clear that $c_N=b_N^N$ and then $F_N=0$.

Assume that $F_i=0$ for every $i=l+1,\dots,N$, for some fixed $l\leq N-1$. It follows that $b_l^N=F_l p_{l+1}+c_l$, hence $c_l=b_l^N$ and $F_l=0$.

If we prove the result for numbers of the form $n=m_i$, then we will have for any $n=\sum_{i=0}^{N} n_i m_i$ that

$$\begin{aligned}
\chi_{\tilde{n}} &= \bar{\chi}_n\\
&= \prod_{i=0}^{N}\bar{\chi}_{m_i}^{n_i}\\
&= \prod_{i=0}^{N}\chi_{\tilde{m}_i}^{n_i}\\
&= \prod_{i=0}^{N}\Big(\prod_{l=0}^{i}\chi_{m_l}^{b_l^i}\Big)^{n_i}\\
&= \prod_{l=0}^{N}\prod_{i=l}^{N}\chi_{m_l}^{b_l^i n_i}\\
&= \prod_{l=0}^{N}\chi_{m_l}^{\sum_{i=l}^{N} b_l^i n_i}\\
&= \chi_{m_N}^{b_N^N n_N}\prod_{l=0}^{N-1}\chi_{m_l}^{\sum_{i=l}^{N} b_l^i n_i}
\end{aligned}$$

$$= \chi_{m_N}^{F_N p_{N+1}} \chi_{m_N}^{c_N} \prod_{l=0}^{N-1} \chi_{m_l}^{\sum\limits_{i=l}^{N} b_l^i n_i}$$

$$= \chi_{m_N}^{c_N} \prod_{l=0}^{N-1} \chi_{m_l}^{(\sum\limits_{i=l}^{N} b_l^i n_i + F_N \alpha_l^N)}.$$

Here, the nonnegative integers F_N and $c_N \in \{0, 1, \ldots, p_{N+1} - 1\}$ are clearly uniquely determined.

Suppose that for some $k \in \{1, 2, \ldots, N-1\}$ we have

$$\chi_{\tilde{n}} = \prod_{l=N-k+1}^{N} \chi_{m_l}^{c_l} \prod_{l=0}^{N-k} \chi_{m_l}^{(\sum\limits_{i=l}^{N} b_l^i n_i + \sum\limits_{i=N-k+1}^{N} F_i \alpha_l^i)}.$$

This yields

$$\chi_{\tilde{n}} = \prod_{l=N-k+1}^{N} \chi_{m_l}^{c_l} \chi_{m_{N-k}}^{(\sum\limits_{i=N-k}^{N} b_{N-k}^i n_i + \sum\limits_{i=N-k+1}^{N} F_i \alpha_{N-k}^i)} \prod_{l=0}^{N-k-1} \chi_{m_l}^{(\sum\limits_{i=l}^{N} b_l^i n_i + \sum\limits_{i=N-k+1}^{N} F_i \alpha_l^i)}.$$

Let F_{N-k} and c_{N-k} be the unique nonnegative integers satisfying

$$\sum_{i=N-k}^{N} b_{N-k}^i n_i + \sum_{i=N-k+1}^{N} F_i \alpha_{N-k}^i = F_{N-k} p_{N-k+1} + c_{N-k},$$

with $c_{N-k} \in \{0, 1, \ldots, p_{N-k+1} - 1\}$. We have

$$\begin{aligned}
\chi_{\tilde{n}} &= \prod_{l=N-k+1}^{N} \chi_{m_l}^{c_l} \chi_{m_{N-k}}^{F_{N-k} p_{N-k+1}} \chi_{m_{N-k}}^{c_{N-k}} \prod_{l=0}^{N-k-1} \chi_{m_l}^{(\sum\limits_{i=l}^{N} b_l^i n_i + \sum\limits_{i=N-k+1}^{N} F_i \alpha_l^i)} \\
&= \prod_{l=N-k}^{N} \chi_{m_l}^{c_l} \prod_{l=0}^{N-k-1} \chi_{m_l}^{(\sum\limits_{i=l}^{N} b_l^i n_i + \sum\limits_{i=N-k+1}^{N} F_i \alpha_l^i + F_{N-k} \alpha_l^{N-k})} \\
&= \prod_{l=N-k}^{N} \chi_{m_l}^{c_l} \prod_{l=0}^{N-k-1} \chi_{m_l}^{(\sum\limits_{i=l}^{N} b_l^i n_i + \sum\limits_{i=N-k}^{N} F_i \alpha_l^i)}.
\end{aligned}$$

Then, for $k = N-1$, $\chi_{\tilde{n}}$ has the form

$$\begin{aligned}
\chi_{\tilde{n}} &= \prod_{l=1}^{N} \chi_{m_l}^{c_l} \chi_{m_0}^{(\sum\limits_{i=0}^{N} b_0^i n_i + \sum\limits_{i=1}^{N} F_i \alpha_0^i)} \\
&= \prod_{l=0}^{N} \chi_{m_l}^{c_l}.
\end{aligned}$$

Let prove the result for $n = m_N$ for some fixed N.

As a first step we show the relation

$$(2.1)\quad \chi_{m_N}(g_s) = \exp(2\pi i(\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,N-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{N}}{p_{i_1+1}\cdots p_{i_m+1}p_{N+1}} + \frac{k_N}{p_{N+1}})),$$

for $s < N$, where the integer $k_N \in \{0,\ldots,p_{N+1}-1\}$ is not uniquely determined but fixed, and $\varepsilon_{i_1}^s = 1$ if $i_1 \neq s$ and $\varepsilon_{i_1}^s = 0$ when $i_1 = s$.

The relation (2.1) can be obtained recursively. We first calculate $\chi_{m_{s+1}}(g_s)$. We have

$$\begin{aligned} \chi_{m_{s+1}}^{p_{s+2}}(g_s) &= \chi_{m_s}^{\alpha_s^{s+1}}(g_s) \\ &= \exp(2\pi i\frac{\alpha_s^{s+1}}{p_{s+1}}), \end{aligned}$$

because $\chi_{m_j}(g_s) = 1$ if $j \leq s-1$. This is true if and only if $\chi_{m_{s+1}}(g_s) = \exp(2\pi i(\frac{\alpha_s^{s+1}}{p_{s+1}p_{s+2}} + \frac{k_{s+1}}{p_{s+2}}))$, for some integer $k_{s+1} \in \{0,\ldots,p_{s+2}-1\}$.

Suppose now that (2.1) is valid for $\chi_{m_j}(g_s)$, for every $s+1 \leq j \leq N$ and show it for $N+1$. We have

$$\begin{aligned} \chi_{m_{N+1}}^{p_{N+2}}(g_s) &= \prod_{j=s}^{N}\chi_{m_j}^{\alpha_j^{N+1}}(g_s) \\ &= \exp(2\pi i(\frac{\alpha_s^{N+1}}{p_{s+1}} \\ &+ \sum_{j=s+1}^{N}\alpha_j^{N+1}(\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}}{p_{i_1+1}\cdots p_{i_m+1}p_{j+1}} + \frac{k_j}{p_{j+1}}))). \end{aligned}$$

It follows that

$$\begin{aligned} \chi_{m_{N+1}}(g_s) &= \exp(2\pi i(\frac{\alpha_s^{N+1}}{p_{s+1}p_{N+2}} \\ &+\sum_{j=s+1}^{N}\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}\alpha_j^{N+1}}{p_{i_1+1}\cdots p_{i_m+1}p_{j+1}p_{N+2}} + \frac{k_j\alpha_j^{N+1}}{p_{j+1}p_{N+2}} + \frac{k_{N+1}}{p_{N+2}})) \\ &= \exp(2\pi i(\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,N\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{N+1}}{p_{i_1+1}\cdots p_{i_m+1}p_{N+2}} + \frac{k_{N+1}}{p_{N+2}})). \end{aligned}$$

The relation (2.1) is proved.

Following [4] in the proof of Theorem 1, the equations verified by b_s^N can also be proved recursively. We first show that $b_N^N = p_{N+1} - 1$. We have

$$\begin{aligned}
\exp(2\pi i \frac{p_{N+1}-1}{p_{N+1}}) &= \bar{\chi}_{m_N}(g_N) \\
&= \chi_{\tilde{m}_N}(g_N) \\
&= \prod_{l=0}^{N} \chi_{m_l}^{c_l}(g_N) \\
&= \prod_{l=0}^{N} \chi_{m_l}^{b_l^N}(g_N) \\
&= \chi_{m_N}^{b_N^N}(g_N) \\
&= \exp(2\pi i \frac{b_N^N}{p_{N+1}}).
\end{aligned}$$

Then $b_N^N = p_{N+1} - 1$ follows immediately from $b_N^N \in \{0, 1, \ldots, p_{N+1} - 1\}$.

In order to calculate b_{N-1}^N, we write

$$\begin{aligned}
\bar{\chi}_{m_N}(g_{N-1}) &= \chi_{\tilde{m}_N}(g_{N-1}) \\
&= \prod_{l=0}^{N} \chi_{m_l}^{b_l^N}(g_{N-1}) \\
&= \chi_{m_{N-1}}^{b_{N-1}^N}(g_{N-1})\chi_{m_N}^{b_N^N}(g_{N-1}) \\
&= \exp(2\pi i \frac{b_{N-1}^N}{p_N}) \exp(2\pi i (p_{N+1} - 1)(\frac{k_N}{p_{N+1}} + \frac{\alpha_{N-1}^N}{p_N p_{N+1}})) \\
&= \exp(2\pi i (\frac{b_{N-1}^N + \alpha_{N-1}^N}{p_N} - (\frac{k_N}{p_{N+1}} + \frac{\alpha_{N-1}^N}{p_N p_{N+1}}))) \\
&= \bar{\chi}_{m_N}(g_{N-1}) \exp(2\pi i \frac{b_{N-1}^N + \alpha_{N-1}^N}{p_N}),
\end{aligned}$$

from which we get $b_{N-1}^N + \alpha_{N-1}^N = p_N$. Hence, $R_N = 1$, where we have taken that $R_{N+1} = 1$.

Suppose there exists $s \le N-2$ such that for all $j = s+1, \ldots, N-1$, we have $b_j^N + \sum\limits_{i=j+1}^{N} R_{i+1}\alpha_j^i = R_{j+1}p_{j+1}$. Then, we prove that $b_s^N + \sum\limits_{i=s+1}^{N} R_{i+1}\alpha_s^i \equiv 0 \pmod{p_{s+1}}$.

Indeed,

$$
\begin{aligned}
\bar{\chi}_{m_N}(g_s) &= \prod_{j=s}^{N} \chi_{m_j}^{b_j^N}(g_s) \\
&= \chi_{m_s}^{b_s^N}(g_s)\chi_{m_N}^{b_N^N}(g_s) \prod_{j=s+1}^{N-1} \chi_{m_j}^{b_j^N}(g_s) \\
&= \exp(2\pi i(\frac{b_s^N}{p_{s+1}} + b_N^N(\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,N-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{N}}{p_{i_1+1}\cdots p_{i_m+1}p_{N+1}} + \frac{k_N}{p_{N+1}}) \\
&+ \sum_{j=s+1}^{N-1} b_j^N(\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}}{p_{i_1+1}\cdots p_{i_m+1}p_{j+1}} + \frac{k_j}{p_{j+1}}))) \\
&= \exp(2\pi i(\frac{b_s^N}{p_{s+1}} + \sum_{j=s+1}^{N-1} R_{j+1} \sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}}{p_{i_1+1}\cdots p_{i_m+1}} \\
&- \sum_{j=s+1}^{N-1} (\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} \sum_{i=j+1}^{N} R_{i+1} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}\alpha_j^i}{p_{i_1+1}\cdots p_{i_m+1}p_{j+1}} \\
&+ R_{i+1}\frac{k_j\alpha_j^i}{p_{j+1}}) + \sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,N-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{N}}{p_{i_1+1}\cdots p_{i_m+1}} \\
&- \sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,N-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{N}}{p_{i_1+1}\cdots p_{i_m+1}p_{N+1}} - \frac{k_N}{p_{N+1}})).
\end{aligned}
$$

From

$$
\begin{aligned}
&\sum_{j=s+1}^{N-1} (\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} \sum_{i=j+1}^{N} R_{i+1} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}\alpha_j^i}{p_{i_1+1}\cdots p_{i_m+1}p_{j+1}} + R_{i+1}\frac{k_j\alpha_j^i}{p_{j+1}}) \\
&\quad = \sum_{i=s+2}^{N} \sum_{j=s+1}^{i-1} (\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} R_{i+1} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}\alpha_j^i}{p_{i_1+1}\cdots p_{i_m+1}p_{j+1}}) + R_{i+1}\frac{k_j\alpha_j^i}{p_{j+1}},
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=s+1}^{i-1} &(\sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,j-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{j}\alpha_j^i}{p_{i_1+1}\cdots p_{i_m+1}p_{j+1}} + \frac{k_j\alpha_j^i}{p_{j+1}}) \\
&= \sum_{i_1,i_2,\ldots,i_m\in\{s,s+1,\ldots,i-1\}} \frac{k_{i_1}^{\varepsilon_{i_1}^s}\alpha_{i_1}^{i_2}\alpha_{i_2}^{i_3}\ldots\alpha_{i_m}^{i}}{p_{i_1+1}\cdots p_{i_m+1}} - \frac{\alpha_s^i}{p_{s+1}},
\end{aligned}
$$

$\bar{\chi}_{m_N}(g_s)$ becomes

$$\bar{\chi}_{m_N}(g_s) = \bar{\chi}_{m_N}(g_s) \exp 2\pi i(\frac{b_s^N}{p_{s+1}} + \sum_{i=s+1}^{N} R_{i+1}\frac{\alpha_s^i}{p_{s+1}}).$$

This is only true when $b_s^N + \sum_{i=s+1}^{N} R_{i+1}\alpha_s^i \equiv 0 \pmod{p_{s+1}}$, or when $b_s^N + \sum_{i=s+1}^{N} R_{i+1}\alpha_s^i = R_{s+1}p_{s+1}$, where R_{s+1} is determined in a unique way.

This ends the proof. □

Lemma 7 in [6] and Theorem 1 in [4] are direct consequences of Theorem 2.1. This can be seen in the following corollaries, applicable in two different specific situations.

Corollary 2.1. *Let G be a Vilenkin group, using the notations above, let $(\chi_n)_n$ be a Vilekin system such that $\alpha_j^N = 0$, for every $N \geq 1$, $j = 0, \dots, N-1$. Then, $c_l = p_{l+1} - n_l$ if $n_l \neq 0$ and $c_l = 0$ if $n_l = 0$.*

Proof. From the definitions of $(c_l)_l$ and b_N^l, we have for $l = N$ $c_N \equiv b_N^N n_N$ $(\text{mod} p_{N+1}) \equiv (p_{N+1} - 1)n_N$ $(\text{mod} p_{N+1})$. Then $c_N = p_{N+1} - n_N$ as $n_N \neq 0$. Applying

$$\sum_{i=l}^{N} b_l^i n_i + \sum_{i=l+1}^{N} F_i\alpha_l^i = F_l p_{l+1} + c_l,$$

for $l \leq N-1$, with $b_l^i = 0$ when $l \leq i-1$ and $b_i^i = p_{i+1} - 1$, we get $c_l + F_l p_{l+1} = (p_{l+1} - 1)n_l$, then $c_l \equiv p_{l+1} - n_l$ $(\text{mod} p_{l+1})$. Hence, $c_l = p_{l+1} - n_l$ if $n_l \neq 0$ and $c_l = 0$ if $n_l = 0$. □

Corollary 2.2. *Let G be a Vilenkin group, using the notations above, let $(\chi_n)_n$ be a Vilekin system such that $\alpha_j^N = 0$, for every $N \geq 1$, $j = 0, \dots, N-2$ and $\alpha_{N-1}^N = 1$. Then, $c_N = p_{N+1} - n_N$ and $c_l = p_{l+1} - n_l - 1$, for every $l \leq N-1$.*

Proof. We first prove this assertion for $n = m_N$ for some fixed N. This means that we need to prove that $b_l^N = p_{l+1} - 1$ for $l \leq N-1$.

From the definition of b_l^N and by $R_N = R_{N+1} = 1$, we deduce from $b_{N-1}^N + R_{N+1} = R_N p_N$, that $b_{N-1}^N = p_N - 1$.

If $R_{l+2} = 1$ for some $l \leq N-2$, then $b_l^N + R_{l+2} = R_{l+1}p_{l+1}$ implies that $R_{l+1} = 1$ and $b_l^N = p_{l+1} - 1$. By induction it follows that $R_l = 1$ for each $l = 1, \dots, N+1$, and $b_l^N = p_{l+1} - 1$, for $l = 0, \dots, N$.

This ends the proof for $n = m_N$.

From $R_l = 1$, $l = 1, \ldots, N+1$, it is easily seen that $b_l^i = p_{l+1}-1$, for $l = 0, \ldots, i-1$, $i \leq N-1$. Introducing these values in the expression of c_l for $n = \sum\limits_{i=0}^{N} n_i m_i$, we obtain

$$(p_{l+1}-1)\sum_{i=l}^{N} n_i + F_{l+1} = F_l p_{l+1} + c_l,$$

for $l \leq N-1$, and

$$(p_{N+1}-1)n_N = F_N p_{N+1} + c_N.$$

This can be written as

$$p_{l+1}\Big(\sum_{i=l}^{N} n_i - F_l\Big) + F_{l+1} = \sum_{i=l}^{N} n_i + c_l, \tag{2.2}$$

and

$$(n_N - F_N)p_{N+1} = n_N + c_N.$$

As $n_N \neq 0$, the second expression is only valid when $n_N = F_N + 1$, and then $c_N = p_{N+1} - n_N$.

Now if we suppose that $F_{l+1} = \sum_{i=l+1}^{N} n_i - 1$ for some $l \leq N-1$, then (2.2) becomes

$$p_{l+1}\Big(\sum_{i=l}^{N} n_i - F_l\Big) = 1 + n_l + c_l.$$

Similarly, this is only true when $F_l = \sum_{i=l}^{N} n_i - 1$, and then $c_l = p_{l+1} - n_l - 1$. By induction, we deduce that $F_l = \sum_{i=l}^{N} n_i - 1$, for every $l \leq N-1$, hence $c_l = p_{l+1} - n_l - 1$ for every $l \leq N-1$. $\square$

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