

**OSTROWSKI TYPE INEQUALITIES INVOLVING THE RIGHT  
 CAPUTO FRACTIONAL DERIVATIVES BELONG TO  $L_p$**

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**Abstract.** In this paper, we have established Ostrowski type inequalities involving the right Caputo fractional derivatives belong to  $L_p$  spaces ( $1 \leq p \leq \infty$ ) via the right Caputo fractional Taylor formula with integral remainder.

### 1. Introduction

The following result is known in the literature as Ostrowski's inequality. The inequality of Ostrowski gives us an estimate for the deviation of the values of a smooth function from its mean value. In 1938, the classical integral inequality is proven by A.M. Ostrowski as the following:

**Theorem 1.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,*

$$\|f'\|_{\infty} = \sup_{t \in (a, b)} |f'(t)| < +\infty.$$

*Then the following inequality holds:*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq (b-a) \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \|f'\|_{\infty}$$

*for any  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is the best possible.*

In [1], Anastassiou established a new Ostrowski inequality which holds higher order derivatives functions.

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**Theorem 1.2.** Let  $f \in C^{n+1}([a, b])$ ,  $n \in \mathbb{N}$  and  $x \in [a, b]$  be fixed, such that  $f^{(k)}(0) = 0$ ,  $k = 1, \dots, n$ . Then it holds

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{1}{(n+2)!} \left[ \frac{(x-a)^{n+2} + (b-x)^{n+2}}{(b-a)} \right] \|f^{(n+1)}\|_{\infty}.$$

Clearly inequality (1.2) generalizes inequality (1.1) for higher order derivatives of  $f$ . During the past few years many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [14]–[18] and the references cited therein.

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see [2]–[13]. Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type. The main aim of this work is to establish Ostrowski type inequalities involving the right Caputo fractional derivatives belong to  $L_p$  spaces ( $1 \leq p \leq \infty$ ) via the right Caputo fractional Taylor formula with integral remainder. Let us begin by introducing this type of inequality.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 1.1.** Let  $f \in L_1[a, b]$ . The Riemann-Liouville integrals  $J_{a+}^{\alpha} f$  and  $J_{b-}^{\alpha} f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively where  $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ . Here is  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

**Definition 1.2.** Let  $f \in AC^m([a, b])$  ( $f^{(m-1)}$  is in  $AC([a, b])$ ),  $m \in \mathbb{N}$ ,  $m = [\alpha]$ ,  $\alpha > 0$  ( $[\cdot]$  the ceiling of the number). The right Caputo fractional derivative of order  $\alpha > 0$  is defined by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t-x)^{m-\alpha-1} f^{(m)}(t) dt, \quad x \leq b.$$

If  $\alpha = m \in \mathbb{N}$ , then

$$D_{b-}^m f(x) = (-1)^m f^{(m)}(x), \quad \forall x \in [a, b].$$

If  $x > b$ , we define  $D_{b-}^{\alpha} f(x) = 0$ .

Properties concerning this operator can be found ([10]-[13]). For some recent results connected with fractional integral inequalities see ([1]-[13]).

In order to prove our main results, we need the following theorem proved by Anastassiou in [3].

**Theorem 1.3.** *Let  $f \in AC^m([a, b])$ ,  $x \in [a, b]$ ,  $m = [\alpha]$ ,  $\alpha > 0$ . Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} D_{b-}^\alpha f(t) dt$$

*the right Caputo fractional Taylor formula with integral remainder.*

In [5], Anastassiou established general univariate right Caputo fractional Ostrowski inequalities with respect to  $\|\cdot\|_p$ ,  $1 \leq p \leq \infty$ .

**Theorem 1.4.** *Let  $f \in AC^m([a, b])$ ,  $m = [\alpha]$  and  $f^{(k)}(b) = 0$ ,  $k = 1, \dots, m-1$ . Then*

$$(1.3) \quad \begin{aligned} & \left| f(b) - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \begin{cases} \frac{(b-a)^\alpha}{\Gamma(\alpha+2)} \|D_{b-}^\alpha f\|_{\infty, [a, b]}, & \text{if } D_{b-}^\alpha f \in L_\infty([a, b]), \alpha > 0, \\ \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)} \|D_{b-}^\alpha f\|_{L_1([a, b])}, & \text{if } D_{b-}^\alpha f \in L_1([a, b]), \alpha \geq 1, \\ \frac{(b-a)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(\alpha+\frac{1}{p})} \|D_{b-}^\alpha f\|_{L_q([a, b])}, & \text{if } D_{b-}^\alpha f \in L_q([a, b]), p, q > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \alpha > 1 - \frac{1}{p}. & \end{cases} \end{aligned}$$

## 2. Main Results

We start with the following theorem:

**Theorem 2.1.** *Let  $f, g \in AC^m([a, b])$ ,  $m = [\alpha]$ ,  $\alpha > 0$ . Assume  $f^{(k)}(b) = g^{(k)}(b) = 0$ ,  $k = 1, \dots, m-1$  and  $D_{b-}^\alpha f, D_{b-}^\alpha g \in L_\infty([a, b])$ . Then*

$$(2.1) \quad \begin{aligned} & \left| 2 \int_a^b f(x)g(x) dx - \int_a^b (f(x)g(b) + g(x)f(b)) dx \right| \\ & \leq \|D_{b-}^\alpha f\|_\infty J_{a+}^{\alpha+1} |g(b)| + \|D_{b-}^\alpha g\|_\infty J_{a+}^{\alpha+1} |f(b)|. \end{aligned}$$

*Proof.* Let  $x \in [a, b]$ . Using Theorem 1.3 and from the hypothesis of Theorem 2.1, we have the following identities

$$(2.2) \quad f(x) - f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} D_{b-}^\alpha f(t) dt$$

$$(2.3) \quad g(x) - g(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} D_{b-}^\alpha g(t) dt.$$

Multiplying both sides of (2.2) and (2.3) by  $g(x)$  and  $f(x)$  respectively and adding the resulting identities, we have

$$\begin{aligned} & 2f(x)g(x) - f(b)g(x) - f(x)g(b) \\ &= \frac{g(x)}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} D_{b-}^\alpha f(t) dt + \frac{f(x)}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} D_{b-}^\alpha g(t) dt. \end{aligned}$$

Integrating the resulting inequality with respect to  $x$  over  $[a, b]$  and using the properties of modulus, we obtain

$$\begin{aligned} (2.4) \quad & \left| 2 \int_a^b f(x)g(x) dx - \int_a^b (f(b)g(x) + f(x)g(b)) dx \right| \\ & \leq \int_a^b \frac{|g(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{\alpha-1} |D_{b-}^\alpha f(t)| dt \right) dx \\ & \quad + \int_a^b \frac{|f(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{\alpha-1} |D_{b-}^\alpha g(t)| dt \right) dx. \end{aligned}$$

Hence it holds

$$\begin{aligned} & \left| 2 \int_a^b f(x)g(x) dx - \int_a^b (f(b)g(x) + f(x)g(b)) dx \right| \\ & \leq \frac{\|D_{b-}^\alpha f\|_\infty}{\Gamma(\alpha)} \int_a^b |g(x)| \left( \int_x^b (t-x)^{\alpha-1} dt \right) dx \\ & \quad + \frac{\|D_{b-}^\alpha g\|_\infty}{\Gamma(\alpha)} \int_a^b |f(x)| \left( \int_x^b (t-x)^{\alpha-1} dt \right) dx \\ & = \frac{\|D_{b-}^\alpha f\|_\infty}{\Gamma(\alpha+1)} \int_a^b (b-x)^\alpha |g(x)| dx \\ & \quad + \frac{\|D_{b-}^\alpha g\|_\infty}{\Gamma(\alpha+1)} \int_a^b (b-x)^\alpha |f(x)| dx \\ & = \|D_{b-}^\alpha f\|_\infty J_{a+}^{\alpha+1} |g(b)| + \|D_{b-}^\alpha g\|_\infty J_{a+}^{\alpha+1} |f(b)| \end{aligned}$$

which the proof is completed.  $\square$

**Remark 2.1.** If we take  $g(x) = 1$  in Theorem 2.1, the inequality (2.1) reduces the first inequality in (1.3).

**Theorem 2.2.** Let  $f, g \in AC^m([a, b])$ ,  $m = [\alpha]$ ,  $\alpha \geq 1$ . Assume  $f^{(k)}(b) = g^{(k)}(b) = 0$ ,  $k = 1, \dots, m-1$  and  $D_{b-}^\alpha f, D_{b-}^\alpha g \in L_1([a, b])$ . Then

$$(2.5) \quad \begin{aligned} & \left| 2 \int_a^b f(x)g(x)dx - \int_a^b (f(x)g(b) + g(x)f(b)) dx \right| \\ & \leq \|D_{b-}^\alpha f\|_{L_1([a, b])} J_{a+}^\alpha |g(b)| + \|D_{b-}^\alpha g\|_{L_1([a, b])} J_{a+}^\alpha |f(b)|. \end{aligned}$$

*Proof.* From the inequality (2.4) of Theorem 2.1, we have again

$$\begin{aligned} & \left| 2 \int_a^b f(x)g(x)dx - \int_a^b (f(b)g(x) + f(x)g(b)) dx \right| \\ & \leq \int_a^b \frac{|g(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{\alpha-1} |D_{b-}^\alpha f(t)| dt \right) dx \\ & \quad + \int_a^b \frac{|f(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{\alpha-1} |D_{b-}^\alpha g(t)| dt \right) dx \\ & \leq \int_a^b \frac{|g(x)|}{\Gamma(\alpha)} (b-x)^{\alpha-1} \left( \int_x^b |D_{b-}^\alpha f(t)| dt \right) dx \\ & \quad + \int_a^b \frac{|f(x)|}{\Gamma(\alpha)} (b-x)^{\alpha-1} \left( \int_x^b |D_{b-}^\alpha g(t)| dt \right) dx \\ & = \|D_{b-}^\alpha f\|_{L_1([a, b])} \int_a^b \frac{|g(x)|}{\Gamma(\alpha)} (b-x)^{\alpha-1} dx \\ & \quad + \|D_{b-}^\alpha g\|_{L_1([a, b])} \int_a^b \frac{|f(x)|}{\Gamma(\alpha)} (b-x)^{\alpha-1} dx \\ & = \|D_{b-}^\alpha f\|_{L_1([a, b])} J_{a+}^\alpha |g(b)| + \|D_{b-}^\alpha g\|_{L_1([a, b])} J_{a+}^\alpha |f(b)| \end{aligned}$$

which completes the proof.  $\square$

**Remark 2.2.** If we take  $g(x) = 1$  in Theorem 2.2, the inequality (2.5) reduces the second inequality in (1.3).

**Theorem 2.3.** Let  $f, g \in AC^m([a, b])$ ,  $m = [\alpha]$ ,  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\alpha > 1 - \frac{1}{p}$ . Assume  $f^{(k)}(b) = g^{(k)}(b) = 0$ ,  $k = 1, \dots, m-1$  and  $D_{b-}^\alpha f, D_{b-}^\alpha g \in L_q([a, b])$ . Then

$$(2.6) \quad \begin{aligned} & \left| 2 \int_a^b f(x)g(x)dx - \int_a^b (f(x)g(b) + g(x)f(b)) dx \right| \\ & \leq \Gamma\left(\alpha + \frac{1}{p}\right) \left( \|D_{b-}^\alpha f\|_{L_q([a, b])} J_{a+}^{\alpha + \frac{1}{p}} |g(b)| + \|D_{b-}^\alpha g\|_{L_q([a, b])} J_{a+}^{\alpha + \frac{1}{p}} |f(b)| \right). \end{aligned}$$

*Proof.* From the inequality (2.4) of Theorem 2.1, we have again

$$\begin{aligned} & \left| 2 \int_a^b f(x)g(x)dx - \int_a^b (f(b)g(x) + f(x)g(b)) dx \right| \\ & \leq \int_a^b \frac{|g(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{\alpha-1} |D_{b-}^\alpha f(t)| dt \right) dx \\ & \quad + \int_a^b \frac{|f(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{\alpha-1} |D_{b-}^\alpha g(t)| dt \right) dx. \end{aligned}$$

Using the Hölder's integral inequality, we obtain

$$\begin{aligned} & \left| 2 \int_a^b f(x)g(x)dx - \int_a^b (f(b)g(x) + f(x)g(b)) dx \right| \\ & \leq \int_a^b \frac{|g(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{(\alpha-1)p} dt \right)^{\frac{1}{p}} \left( \int_x^b |D_{b-}^\alpha f(t)|^q dt \right)^{\frac{1}{q}} dx \\ & \quad + \int_a^b \frac{|f(x)|}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{(\alpha-1)p} dt \right)^{\frac{1}{p}} \left( \int_x^b |D_{b-}^\alpha g(t)|^q dt \right)^{\frac{1}{q}} dx \\ & = \frac{\|D_{b-}^\alpha f\|_{L_q([a,b])}}{\Gamma(\alpha)} \int_a^b |g(x)| \left( \int_x^b (t-x)^{(\alpha-1)p} dt \right)^{\frac{1}{p}} dx \\ & \quad + \frac{\|D_{b-}^\alpha g\|_{L_q([a,b])}}{\Gamma(\alpha)} \int_a^b |f(x)| \left( \int_x^b (t-x)^{(\alpha-1)p} dt \right)^{\frac{1}{p}} . \end{aligned}$$

By simple computation

$$\left( \int_x^b (t-x)^{(\alpha-1)p} dt \right)^{\frac{1}{p}} = \frac{(b-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}.$$

This last equality is substituted the above, then we have the conclusion.  $\square$

**Remark 2.3.** If we take  $g(x) = 1$  in Theorem 2.3, the inequality (2.6) reduces the third inequality in (1.3).

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