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SINGLE-FACILITY WEBER LOCATION PROBLEM BASED ON THE LIFT METRIC

Predrag S. Stanimirović*,†, Marija S. Ćirić, Lev A. Kazakovtsev, Idowu A. Osinuga‡

Abstract. The continuous planar single-facility min-sum Weber location problem based upon the lift metric is investigated. Essentially, the problem is reduced to the known algorithm which assumes the use of rectilinear distances in the plane. An effective algorithm is developed for its solution. A numerical example illustrating the introduced algorithm is given.

1. Introduction

Location problems represent a very important class of optimization tasks, where the coordinate of locations and distances between them are the main parameters. In the general case, the task of location problem is to define positions of some new facilities from the actual space in which some other relevant objects (points) are already placed. New facilities are centers that provide services and are called *suppliers*. The existing facilities are the service users or clients, and are called *customers*. Location problems occur frequently in real life. Many systems in the public and private sectors are characterized by facilities that provide homogeneous services at their locations to a given set of fixed points or customers. Examples of such facilities include warehouse location, positioning a computer and communication units, locating hospitals, police stations, locating fire stations in a city, or locating base stations in wireless networks.

Different classifications of the location problems are known. The classification scheme from [16] assumes five positions.

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*Corresponding author
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In the present article we pay attention to the selection of the distance function as the characterization criterion of the location problem. The distance between two points is the length of the shortest path connecting them. The metric by which the (generalized) distance between two points is measured may be different in various instances [3]. In the calculating of distance between two points, the most common distance metrics in a continuous space are those known as the class of l_p distance metrics, primarily rectangular (l_1), Euclidean (l_2) and Chebyshev (l_∞) metric. Detailed explanation of various metrics can be found in Dictionary of distances [6]. Many factors affect the process of metrics choosing. The most important factor is the nature of the problem. For example, if it is possible to move rectilinearly between two points, the distance between them is exactly given by the Euclidean (or straight-line distance) metric. On the other hand, in the cities where streets intersect under the right angle mainly, the distance between two points will be approximated using the rectangular metric (also known as the Manhattan, "city block" distance, the right-angle distance metric or taxicab distance). Measures of distances in chess are a characteristic example. The distance between squares on the chessboard for rooks is measured in Manhattan distance; kings and queens use the Chebyshev distance, and bishops use the Manhattan distance.

We emphasize the next main contribution of our paper.

The Weber location problem (also called the Fermat-Weber problem) is a basic model in the location theory which has received significant attention in the scientific literature. For a detailed review see, for example, [34]. The paper [26] investigated a reformulation of the unconstrained form of the classical Weber problem into an unconstrained minimum norm problem. The classical Weber problem is established with the Euclidean norm underlying the definition of the distance function. But, other measures, principally l_p norms, also play an important role in the theory and practice of location problems. The norms are arbitrary, in general. The most popular method to solve the Weber problem with Euclidean distances is given by a one-point iterative procedure which was first proposed by Weiszfeld [33]. The procedure is readily generalized to l_p distances (see, for example, [21], Ch. 2). Solution of the continuous Weber problem in l_1 distance is described in [8]. The three-dimensional Fermat-Weber facility location problem with Tchebychev distance is investigated in [27]. The Weber location problem with squared Euclidean distances is considered in [8]; the same problem under the assumption that the weights are selected from a given set of intervals at any point, is studied in [10].

The l_p norms have received the most attention from location analysts. But, many other types of distances have been exploited in the facility location problem. A review of metrics that are exploited in many variations of location problems is presented in [8]:

- central metrics [28],

- distance functions based on altered norms [22, 23],

- weighted one-infinity norms [32],

- mixed norms [17],
- block and round norms [29],
- mixed gauges [11],
- asymptotic distances [18],
- weighted sums of order p [2, 31].

In the present article we solve the Weber problem in the plane, under the assumption that the distance is measured by the lift metric.

The paper is organized as follows. Some basic definitions and algorithms are restated in the second section. In the third section we present an effective algorithm for the solution of the single-facility continuous planar Weber problem, assuming that distances are measured by the lift metric.

2. Preliminaries

The *lift metric* (or the *raspberry picker metric, jungle river metric, barbed wire metric*) in the plane $\mathbb{R}^2$ is defined by

$$L(A,B) = \begin{cases} |x_1^A - x_1^B|, & x_2^A = x_2^B \\ |x_1^A| + |x_2^A - x_2^B| + |x_1^B|, & x_2^A \neq x_2^B \end{cases} \tag{2.1}$$

where $A(x_1^A, x_2^A)$ and $B(x_1^B, x_2^B)$ are given points (see, for example [6, 7]). It can be defined as the minimum Euclidean length of all admissible connecting curves between two given points, where a curve is called admissible if it consists of only segments of straight lines parallel to x-axis, and of segments of y-axis [6, 7]. Therefore, under the assumption $x_2^A \neq x_2^B$ the distance between two points A and B in the lift metric equals the sum of lengths AA', $A'B'$ and $B'B$, where A' and B' are orthogonal projections of the points A and B to the y-axis, respectively (Figure 2.1, Left). In the opposite case, $x_2^A = x_2^B$, the distance between A and B is simply the length of the segment AB (Figure 2.1, Right).

The lift metric can be used as the distance measurement in the cities with only one main street (corresponding to the y-axis) and other streets crossing it at right angles. We are observed that in the main city of Zakynthos island in Greek-Zakynthos, the streets are deployed on this way. Similar situation also occurs in tier buildings where the lift (in the role of y-axis) connects tiers.

In this case, the rectangular metric does not coincide with the lift metric. For example, in l_1 metric we have

$$l_1(A_1(0,2), B_1(3,0)) = l_1(A_2(1,2), B_2(4,0)) = 3 + 2 = 5.$$

On the other hand,

$$L(A_1(0,2), B_1(3,0)) = 3 + 2 = 5 \neq L(A_2(1,2), B_2(4,0)) = 4 + 2 + 1 = 7.$$

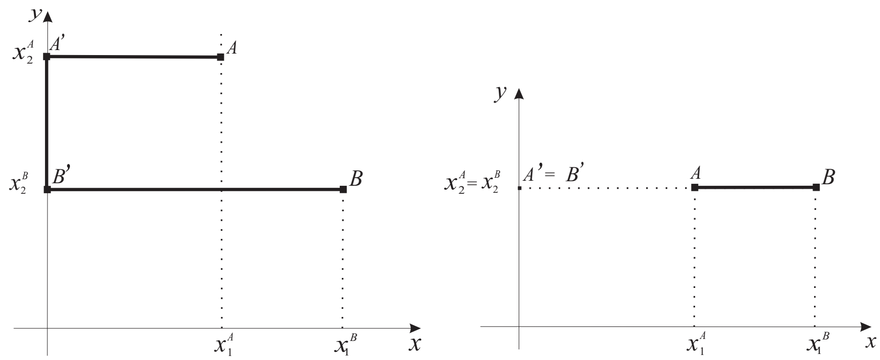

FIG. 2.1: Left) The case $x_2^A \neq x_2^B$ Right) The case $x_2^A = x_2^B$

Lift metric is defined as $\mathcal{R}^2$ metric. In the $3D$ case, it can be defined as follows. Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are some points in $\mathcal{R}^3$. We define the distance between A and B in the $3D$ lift metric in the following way:

$$L(A,B) = \begin{cases} |x_1| + |y_1| + |z_2 - z_1| + |x_2| + |y_2|, & z_1 \neq z_2 \\ |x_1 - x_2| + |y_1 - y_2|, & z_1 = z_2 \end{cases} \tag{2.2}$$

Here, we assume that the underlying metric by levels is l_1.

This metric can be used for a building with a lift (an elevator) or for location problems in the mines. Also, the location problems in the rack storages can be formulated as the problems with the lift metric if the goods transportation is performed by the loading machines which are not able to move horizontally unless the load is lowered. This is a real situation for the warehouses with highest rack storages.

The 2-dimensional continuous Weber location problem can be briefly restated as follows (see, for example [9, 34]). Let m demand centers $A_1, \dots, A_m$ be given in the plane $\mathbb{R}^2$ (locations of given customers), where $A_i(a_1^i, a_2^i)$, $i = 1, \dots, m$. It is necessarily to find a new point $X(x_1, x_2) \in \mathbb{R}^2$ which has minimal sum of weighted distances with respect to given points. Therefore, one needs to solve the unconstrained optimization problem (single-facility min-sum problem), where it is necessary to minimize the sum

$$\min_X \; f(X) = \sum_{i=1}^{m} w_i \cdot d(A_i, X). \tag{2.3}$$

The real quantity w_i is a positive weighted coefficient of the point A_i. Essentially, the weight w_i converts the distance $d(A_i, B_k)$ into a cost of serving the demand of customer A_i considerate towards kth offered facility location B_k.

For the sake of completeness, we restate well-known method which gives solution of the next optimization problem

$$\min_{x} \ f(x) = \sum_{i=1}^{m} w_i|x - a^i|, \tag{2.4}$$

$x, a^i \in \mathcal{R}$, $w_i \in \mathcal{R}^+$. Therefore, it is necessary to minimize the function f, i.e. to find x for which the value of f is minimal.

Let the condition

$$a^1 \le a^2 \le \ldots \le a^m$$

be valid. The derivation of the function (2.4) has the form

$$f'(x) = \begin{cases} -\sum_{j=1}^{m} w_j, & x < a^1 \\ \sum_{j=1}^{i} w_j - \sum_{j=i+1}^{m} w_j, & a^i < x \le a^{i+1} \\ \sum_{j=1}^{m} w_j, & x > a^m \end{cases} \tag{2.5}$$

Therefore, the function $f(x)$ is piecewise lineal and convex. As the coefficient of the direction obviously grow up at (2.5) with the growing of index i, it is easy to see that the next statement, which gives the conditions for finding the minimum of the function (2.4), x^*, holds.

Proposition 2.1. [25] *Under the solving of the task* (2.4),(2.5)*, let mark with* i^* *the index satisfying*

$$s_1 = \sum_{j=1}^{i^*-1} w_j < \frac{1}{2}\sum_{j=1}^{m} w_j \le \sum_{j=1}^{i^*} w_j = s_2. \tag{2.6}$$

If the strict inequality holds at (2.6)*, then* $x^* = a^{i^*}$*, and if holds equality at* (2.6)*, then the optimal solution is any value from the interval* $[a^{i^*}, a^{i^*+1}]$. ■

According to Proposition 2.1 we come to the well-known algorithm for solving the problem (2.4) (one can find it in [13, 25, 30]). The first step in this procedure is optional, but it can be used to accelerate the remaining two steps.

Algorithm 2.1. Solve the optimization task (2.4).

Input. Real quantities $a^1, \ldots, a^m$.

Step *I*. For each subset of identical elements $a^{i_1} = a^{i_2} = \cdots = a^{i_j}$ perform the following activities: put $w_{i_1} = w_{i_1} + w_{i_2} \cdots + w_{i_j}$, eliminate multiple elements $a^{i_2}, \ldots, a^{i_j}$ as well as corresponding weights $w_{i_2}, \ldots, w_{i_j}$ and later perform appropriate shifting of the indices of residual elements a^j and their weights w_j.

Step *II*. If Step *I* is applied, denote by q the cardinal number of different elements in the set $\{a^1, \ldots, a^m\}$; otherwise, use $q = m$. Sort coordinates a^i, $i = 1, \ldots, q$ in non-descending order. Let the sorted sequence of coordinates is

$$a^{1'} \le a^{2'} \le \cdots \le a^{q'}.$$

Rearrange the weighting coefficients $\{w_1, \dots, w_q\} \to \{w_{1'}, \dots, w_{q'}\}$ applying identical replacements on the weights. For the sake of simplicity, let us denote partial sums of the array $\{w_{1'}, \dots, w_{q'}\}$ by

$$S_w[0] = 0, \quad S_w[k] = \sum_{i=1}^{k} w_{i'}, \; 1 \le k \le q. \tag{2.7}$$

Step III. There are two possibilities, denoted by P_1 and P_2.

P_1. If the condition

$$S_w[k^* - 1] < \frac{1}{2} S_w[q] < S_2[k^*] \tag{2.8}$$

is satisfied for some $k^* \in \{1, \dots, q\}$, then we end the algorithm returning the solution $x^* = a^{k^*}$.

P_2. If the condition

$$\frac{1}{2} S_w[q] = S_w[k^*] \tag{2.9}$$

holds for some $k^* \in \{1, \dots, q\}$, then the solution is multiple, i.e. the searched coordinate x^* can to have any value from the interval $[a^{k^*}, a^{k^*+1}]$.

Note that the previous algorithm is also used for the solution of the Weber problem (2.3) in the case when the underlying distance function is defined by the l_1 metric. More precisely, two unknown coordinates in the planar Weber problem can be derived from two separate optimization problems of the general form (2.4) (see, for example, [13], Chapter 3).

3. Continuous Weber problem and lift metric

In the sequel we solve the single-facility min-sum Weber problem (2.3) applying the lift metric (2.1). Therefore, the distance function is defined by

$$d(A_i, X) = L(A_i, X) = \begin{cases} |x_1 - a_1^i|, & x_2 = a_2^i \\ |x_1| + |x_2 - a_2^i| + |a_1^i|, & x_2 \neq a_2^i, \end{cases} \quad i = 1, \dots, m. \tag{3.1}$$

Two major steps (denoted as **Step 1** and **Step 2**) are separated in our algorithm, as in the following.

Step 1. Generate the list $\mathfrak{X}$ of permissible solutions of the problem. Its initial value is the empty set $\mathfrak{X} = \emptyset$. Two different procedures are separated during the construction of the set $\mathfrak{X}$, in accordance with the definition (3.1).

Procedure 1. Let us consider the quotient set $\mathfrak{S}$ of the set $S = \{a_2^1, \dots, a_2^m\}$

$$\mathfrak{S} = \{S_1 = [a_2^{i_1}], \dots, S_d = [a_2^{i_d}]\}, \tag{3.2}$$

where the class S_j contains elements from S whose values are $a_2^{i_j}$. For each $j = 1, \dots, d$ seek the second coordinate of the optimal point $X(x_1, x_2)$ in the form

$$x_2 \in S_j \Leftrightarrow x_2 = a_2^{i_j}. \tag{3.3}$$

So, as we know value of the coordinate x_2 ($x_2 = a_2^{i_j}$), it is necessary to determine value of the coordinate x_1. Denote by Q_j the set of indices corresponding to points whose second coordinates are contained in the set S_j. According to (3.1) and (3.3), the objective function $f(X)$ consists of two separated sums

$$f(X) = \sum_{i=1,\ i\notin Q_j}^{m} w_i \left(|x_1| + |x_2 - a_2^i| + |a_1^i|\right) + \sum_{i\in Q_j} w_i|x_1 - a_1^i| \tag{3.4}$$

Taking into account $x_2 = a_2^{i_j}$ and grouping the first term in the first sum with the second sum, we obtain

$$f(X) = \sum_{i=1,\ i\notin Q_j}^{m} w_i \left(|a_2^{i_j} - a_2^i| + |a_1^i|\right) + \sum_{i=1}^{m} w_i|x_1 - a_1^{\beta(i)}|, \tag{3.5}$$

where

$$a_1^{\beta(i)} = \begin{cases} a_1^i, & \beta(i) = i \in Q_j \\ 0, & \beta(i) \notin Q_j. \end{cases} \tag{3.6}$$

As the first sum in the expression (3.5) is constant, the problem is reduced on determining the minimum of the function

$$f_1(x_1) = \sum_{i=1}^{m} w_i|x_1 - a_1^{\beta(i)}|. \tag{3.7}$$

We apply Algorithm 2.1 in adapted form for our specific situation (3.6), (3.7). That process consists of three major steps.

Step I. For each subset of identical elements $a_1^{\beta(i_1)} = a_1^{\beta(i_2)} = \cdots = a_1^{\beta(i_j)}$ perform the following: put $w_{i_1} = w_{i_1} + w_{i_2} + \cdots + w_{i_j}$, eliminate multiple elements $a_1^{\beta(i_2)}, \ldots, a_1^{\beta(i_j)}$ as well as their weights $w_{i_2}, \ldots, w_{i_j}$ and then perform appropriate renumeration of the indices of the remainder elements $a_1^{\beta(j)}$ and their weights w_j.

Step II. If Step I is applied, denote by p the cardinal number of different elements in the set $\{a_1^{\beta(1)}, \ldots, a_1^{\beta(m)}\}$; otherwise, use $p = m$. Sort coordinates $a_1^{\beta(1)}, \ldots, a_1^{\beta(p)}$ in non-descending array. Furthermore we suppose that the ordered sequence is

$$a_1^{1'} \le a_1^{2'} \le \cdots \le a_1^{p'}.$$

Rearrange the corresponding weighting coefficients $w_1, \ldots, w_p$ analogously in the sequence

$$w_{1'}, w_{2'}, \ldots, w_{p'}.$$

For the sake of simplicity, let us denote partial sums of the array $\{w_{1'}, \ldots, w_{p'}\}$ by

$$S_w[0] = 0, \quad S_w[k] = \sum_{i=1}^{k} w_{i'},\ 1 \le k \le p.$$

Step III. There are two possible cases capable to produce permissible minimizers for f_1, denoted by C_1 and C_2.

C_1. If the inequalities

$$S_w[k'-1] < \frac{1}{2}S_w[p] < S_w[k'], \tag{3.8}$$

are satisfied for some $k' \in \{1,\dots,p\}$, then the searched coordinate is $x_1 = a_1^{k'}$. Later, we use $X(x_1,x_2)$ as the possible optimal point: $\mathfrak{X} = \mathfrak{X} \cup \{(a_1^{k'}, a_2^{i_j})\}$.

C_2. If the condition

$$\frac{1}{2}S_w[p] = S_w[k'] \tag{3.9}$$

is satisfied for some $k' \in \{1,\dots,p\}$, then the solution is multiple, i.e. the searched coordinate x_1 can to have any value from the interval $[a_1^{k'}, a_1^{k'+1}]$. In our implementation we use the value $x_1 = (a_1^{k'} + a_1^{k'+1})/2$. Thus, we found the additional possible solution of the starting problem (2.3), which implies $\mathfrak{X} = \mathfrak{X} \cup \{((a_1^{k'} + a_1^{k'+1})/2, a_2^{i_1})\}$.

Procedure 2. Compute x_2 under the assumption

$$x_2 \notin S \Leftrightarrow x_2 \neq a_2^i \ \text{ for each } i \in \{1,\dots,m\} \tag{3.10}$$

(under the assumptions opposite with respect to (3.3)), the function $f(X)$ is reduced to

$$f(X) = \sum_{i=1}^{m} w_i \left(|x_1| + |x_2 - a_2^i| + |a_1^i|\right). \tag{3.11}$$

It is necessary to minimize that function. Since the third term $w_i|a_1^i|$ in the function $f(X)$ defined in (3.11) is constant, one needs to minimize the next two objectives:

$$\min_{x_1} \ f_1(x_1) = \sum_{i=1}^{m} w_i|x_1| \tag{3.12}$$

$$\min_{x_2} \ f_2(x_2) = \sum_{i=1}^{m} w_i|x_2 - a_2^i|. \tag{3.13}$$

Thus, solving the problem (3.11) with two variables was reduced to solving two independent tasks of unconstrained optimization (3.12) and (3.13) with one variable (x_1 and x_2, respectively).

Solution of the optimization problem (3.12) is evidently $x_1 = 0$. In order to find optimal value for x_2 it suffices to apply Algorithm 2.1 assuming that the input sequence is $a^1 = a_1^1, \dots, a^m = a_2^m$ and taking into account conditions (3.10).

Step I. For each subset of identical elements $a_2^{i_1} = a_2^{i_2} = \cdots = a_2^{i_j}$ perform the following activities: put $w_{i_1} = w_{i_1} + w_{i_2} + \cdots + w_{i_j}$, eliminate multiple elements $a_2^{i_2}, \dots, a_2^{i_j}$ as well as corresponding weights $w_{i_2}, \dots, w_{i_j}$ and later perform appropriate shifting of the indices of residual elements a_2^j and their weights w_j.

Step II. If Step I is applied, denote by q the cardinal number of different elements in the set $\{a_2^1, \ldots, a_2^m\}$; otherwise, use $q = m$. Sort coordinates a_2^i, $i = 1, \ldots, q$ in non-descending order. Let the sorted sequence of coordinates be

$$a_2^{1'} \le a_2^{2'} \le \cdots \le a_2^{q'}.$$

Rearrange the weighting coefficients $\{w_1, \ldots, w_q\} \to \{w_{1'}, \ldots, w_{q'}\}$ applying identical replacements on the weights. Subsequently, generate the partial sums $S_w[i],\ i = 0, \ldots, q$ of the array $\{w_{1'}, \ldots, w_{q'}\}$ as in (2.7).

Step III. There are two possibilities, denoted by P_1 and P_2.

P_1. If the condition (2.8) is satisfied for some $k^* \in \{1, \ldots, q\}$, then the algorithm is finished without any solution. Indeed, the formal solution $x_2 = a_2^{k^*}$ is eliminated according to assumption (3.10), actual for this case.

P_2. If the condition (2.9) holds for some $k^* \in \{1, \ldots, q\}$, then the solution is multiple, i.e. the searched coordinate x_2 can to have any value from the interval $(a_2^{k^*}, a_2^{k^*+1})$. We use the midpoint value $x_2 = (a_2^{k^*} + a_2^{k^*+1})/2$, so that the possible optimal point is $X(0, x_2)$. Place the point X at the end of the list $\mathfrak{X}$ by $\mathfrak{X} = \mathfrak{X} \cup \{(0, (a_2^{k^*} + a_2^{k^*+1})/2)\}$.

Step 2. Thus, we got one or more permissible solutions of the starting problem (2.3). For all obtained values X from $\mathfrak{X}$ we determine the values of the function $f(X)$ defined in (2.3), with $\mathcal{D} = L$. Solution of the Weber problem will be the point $X^*(x_1^*, x_2^*)$ for which the function $f(X)$ has a minimal value. Actually in this step we are solving generated discrete location problem, where the set $\mathfrak{X}$ contains in advance defined feasible locations of the supplier.

Let $X_1, \ldots, X_r$ be r locations on which it is possible to set a new desired object (supplier). *The sum of weighted distances* from the permissible location X_k, $k \in \{1, \ldots, r\}$ of the supplier to the customers is equal to

$$W_k = \sum_{i=1}^{m} w_i \cdot L(A_i, X_k). \tag{3.14}$$

The task is to determine the location B_{k^*} for which the sum of weighted distances is minimal, i.e.

$$W_{k^*} = \min\ \{W_k |\ 1 \le k \le r\}.$$

In accordance with the previous considerations, we state the following general algorithm.

Algorithm 3.1. Solution of the single-facility min-sum Weber problem in the lift metric.

Input. List $lp = \{(a_1^1, a_1^2), \ldots, (a_m^1, a_m^2)\}$ and the list of corresponding weights $lt = \{w_1, \ldots, w_m\}$.

Step 1: Form the quotient set of $S = \{a_2^1, \ldots, a_2^m\}$ in the form $\mathfrak{S} = \{S_1, \ldots, S_d\}$, where each equivalence class S_j contains identical elements from S with the value $a_2^{i_j}$.

Step 2: Generate the list $\mathfrak{X}$ applying the procedure included into the possibilities C_1 and C_2 (included in **Procedure 1.**) to all distinctive values $a_2^{i_j}$ of the set S, i.e. using $x_2 = a_2^{i_j}$, $j = 1, \ldots, d$.

Step 3: Extend the list $\mathfrak{X}$ applying the method defined in the case P_2 (included in **Procedure 2.**).

Step 4: Solve the discrete location problem using given locations lp, discrete set $\mathfrak{X}$ of possible solutions and the weights lt.

For the $3D$ case with l_1 underlying metric (2.2), the algorithm proposed can be easily adapted.

Let's formulate the basic equations for the $3D$ case. If the set of the demand points is $\{A_1 = (a_1^1, a_2^1, a_3^1), \cdots, A_m = (a_1^m, a_2^m, a_3^m)\}$, the equation (3.2) must be formulated as

$$S = \{S_1 = [(a_3^{i_1}], \ldots, S_d = [a_3^{i_d}]\}. \tag{3.15}$$

The Steps I, II and III of the Procedure 1 must be passed twice. For the 1st pass, we implement the original form of the equation (3.6). For the 2nd pass,

$$a_2^{\beta(i)} = \begin{cases} a_2^i, & \beta(i) = i \in Q_j \\ 0, & \beta(i) \notin Q_j. \end{cases} \tag{3.16}$$

Here, Q_j is the set of indices corresponding to points whose 3rd coordinates are contained in the set S_j.

In the Step II of the Procedure 1, the 1st pass does not differ from the $2D$ case. At the 2nd pass, denote by p_2 the cardinal number of different elements in the set $\{a_2^{\beta(1)}, \ldots, a_2^{\beta(m)}\}$ and sort coordinates $a_2^{\beta(1)}, \ldots, a_2^{\beta(p)}$ in non-descending array. Rearrange the corresponding weighting coefficients $w_1, \ldots, w_p$ analogously in the sequence

$$w_{1''}, w_{2''}, \ldots, w_{p_2''}.$$

For the sake of simplicity, let us denote partial sums of the array $\{w_{1''}, \ldots, w_{p_2'}\}$ by

$$S_w[0] = 0, \quad S_w[k] = \sum_{i=1}^{k} w_{i''}, \ 1 \le k \le p_2.$$

The 2nd pass of Step III is performed as follows. There are two possible cases, C'_1 and C'_2.

C'_1. If the inequalities

$$S_w[k'-1] < \frac{1}{2} S_w[p] < S_w[k''],$$

are satisfied for some $k'' \in \{1, \ldots, p_2\}$, then the searched coordinate is $x_2 = a_2^{k''}$. Later, we use $X(x_1, x_2, x_3)$ as the possible optimal point: $\mathfrak{X} = \mathfrak{X} \cup \{(a_1^{k'}, a_2^{k''}, a_3^{i_j})\}$.

Table 3.1: Algorithm data, part 1

coordinates ($a_1^{i'}$)	0	0	4	6
weights ($w_{i'}$)	1	3	4	2
k	1	2	3	4
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	1	4	8	10

C'_2. If the condition

$$\frac{1}{2}S_w[p] = S_w[k'']$$

is satisfied for some $k'' \in \{1, \ldots, p_2\}$, then the solution is multiple, i.e. the searched coordinate x_2 can have any value from the interval $[a_2^{k''}, a_2^{k''+1}]$. In our implementation, we use the value $x_2 = (a_2^{k''} + a_2^{k''+1})/2$. $\mathfrak{X} = \mathfrak{X} \cup \{((a_1^{k'}, a_2^{k''} + a_2^{k''+1})/2, a_3^{i_1})\}$.

So, in the $3D$ case, the *Step II* and *Step III*, solve 2 problems (2.4).

Example 3.1. Solve Weber problem using the specified algorithm with the next data:

$$A_1(4,4),\ w_1 = 4, \quad A_2(3,1),\ w_2 = 1, \quad A_3(6,4),\ w_3 = 2, \quad A_4(6,2),\ w_4 = 3.$$

We have $S = \{4, 1, 4, 2\}$. The quotient set of S is defined as $S_1 = [4], S_2 = [1], S_3 = [2]$. Therefore, it is necessary to consider three possibilities for the cases C_1 and C_2.

1. Let be $x_2 = 4$. Then the function $f(X)$ has the following form:

$$\begin{aligned} f(x) &= w_2(|x_1| + |x_2 - a_2^2| + |a_1^2|) + w_4(|x_1| + |x_2 - a_2^4| + |a_1^4|) \\ &+ w_1|x_1 - a_1^1| + w_3|x_1 - a_1^3|. \end{aligned}$$

According to the constant value $x_2 = 4$ of the coordinate x_2, the function f just depend on x_1, so we can consider the next function

$$\begin{aligned} f_1(x_1) &= w_1|x_1 - a_1^1| + w_2|x_1 - 0| + w_3|x_1 - a_1^3| + w_4|x_1 - 0| \\ &= \sum_{i=1}^{4} w_i|x_1 - a_1^{\beta(i)}|, \end{aligned}$$

where $a_1^{\beta(1)} = a_1^1 = 4$, $a_1^{\beta(3)} = a_1^3 = 6$, $a_1^{\beta(2)} = a_1^{\beta(4)} = 0$.

Let us sort the coordinates $a_1^{\beta(i)} \to a_1^{i'}$ and rearrange corresponding weights $w_i \to w_{i'}$, using the same replacements (Table 3.1).

We firstly assume that Step I is omitted. According to $\frac{1}{2}\sum_{i=1}^{4} w_{i'} = 5$, the condition

$$S_w[k'-1] < \frac{1}{2}S_w[4] < S_w[k']$$

is satisfied for $k' = 3$, so $x_1 = a_1^{3'} = 4$. Therefore, one possible solution is $X_1(4,4)$.

In the case when Step I is applied, data from Table 3.1 reduce to Table 3.2.

Then conditions (3.8) are satisfied for $k = 2$, so that the same possible solution is generated.

Table 3.2: Algorithm data, part 2

coordinates ($a_1^{i'}$)	0	4	6
weights ($w_{i'}$)	4	4	2
k	1	2	3
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	4	8	10

Table 3.3: Algorithm data, part 3

coordinates ($a_1^{i'}$)	0	0	0	3
weights ($w_{i'}$)	4	2	3	1
k	1	2	3	4
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	4	6	9	10

The list of permissible solutions is now equal to $\mathfrak{X} = \{X_1\}$.

2. In this case it is assumed $x_2 = 1$. Now the function $f_1(x_1)$ looks like:

$$f_1(x_1) = w_1|x_1 - 0| + w_2|x_1 - 3| + w_3|x_1 - 0| + w_4|x_1 - 0|.$$

On the similar procedure as in the case **1.** we get the Table 3.3.

Inequalities (3.8) are valid for $k' = 2$, so $x_1 = a_1^{2'} = 0$, i.e. we got the second possible solution $X_2(0, 1)$.

In the case when Step I is applied, Table 3.3 transforms to Table 3.4.

Condition (3.8) holds for $k = 1$, so that X_2 is again the second eventual solution.

We have $\mathfrak{X} = \{X_1, X_2\}$.

3. Let us now start from the assumption $x_2 = 2$.

$$f_1(x_1) = w_1|x_1 - 0| + w_2|x_1 - 0| + w_3|x_1 - 0| + w_4|x_1 - 2|.$$

The relational table is Table 3.5.

Since the equality of the form (3.9) are satisfied for $k' = 2$, the solution x_1 is from the interval $[a_1^{2'}, a_1^{3'}] = [0, 0]$, i.e. $x_1 = 0$, which implies $X_3(0, 2)$.

Table 3.4: Algorithm data, part 4

coordinates ($a_1^{i'}$)	0	3
weights ($w_{i'}$)	9	1
k	1	2
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	9	10

Table 3.5: Algorithm data, part 5

coordinates ($a_1^{i'}$)	0	0	0	2
weights ($w_{i'}$)	4	1	2	3
k	1	2	3	4
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	4	5	7	10

Table 3.6: Algorithm data, part 6

coordinates ($a_1^{i'}$)	0	2
weights ($w_{i'}$)	5	3
k	1	2
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	7	10

The list $\mathfrak{X}$ is expanded: $\mathfrak{X} = \{X_1, X_2, X_3\}$.

Let us observe that Step I transforms Table 3.5 into the next Table 3.6.

Now, condition (3.8) is satisfied for $k = 1$, so that $X_3(0, 2)$ is possible optimal point.

4. Let $x_2 \neq 1, 2, 4$. In this case, we take $x_1 = 0$ and then seek for the minimum of the function

$$f_2(x_2) = \sum_{i=1}^{4} w_i |x_2 - a_2^i|$$

Let us sort the coordinates $a_2^i \to a_2^{i'}$ and perform analogous rearrangement $w_i \to w_{i'}$. The results are shown in Table 3.7.

Inequalities of the form (2.8) hold for $k^* = 3$. We stop algorithm. This case has no solution, since the assumption $x_2 \neq 4$ is made.

Let us mention that Step I gives the next Table 3.8.

Therefore, the conclusion is the same as from Table 3.7.

At the end, in order to solve Step 2 of Algorithm 3.1, we must compute and compare the values of the function f at each point X_i, $i = 1, 2, 3$. We get

$$f(X_1) = 50, \ f(X_2) = 55, \ f(X_3) = 62.$$

Table 3.7: Algorithm data, part 7

coordinates ($a_2^{i'}$)	1	2	4	4
weights ($w_{i'}$)	1	3	4	2
k	1	2	3	4
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	1	4	8	10

Table 3.8: Algorithm data, part 8

coordinates $(a_2^{i'})$	1	2	4
weights $(w_{i'})$	1	3	6
k	1	2	3
$S_w[k] = \sum_{i=1}^{k} w_{i'}$	1	4	10

Therefore, the solution of the Weber problem will be the point in which the function f has a minimal value, i.e. $X^* = X_1 = A_1 = (4,4)$.

4. Conclusion and future work

Our paper is the first attempt to solve the discrete and the single-facility minsum continuous location problem with the lift metric as the measure of distances.

A couple of variants and extensions of continuous location problems have been investigated in literature. Let us mention the most important among them. More complex problems include the placement of multiple facilities. Problems with barriers are the subject in [5, 15, 19, 20]. The location of undesirable (obnoxious) facilities requires maximization of minimum distances (see, e.g., [1, 12, 14, 24]. Location models with both desirable and undesirable facilities have been analyzed in [4]. It seems interesting to investigate these extensions in the sense of the lift metric or in the more general nonconvex case, where the shortest length of arc is used as distance instead of a particular metrics.

Also, the $3D$ lift metric with the underlying Euclidean metric l_2 or the rectilinear l_1 metric will be the subject of our future research.

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Predrag S. Stanimirović
University of Niš
Department of Computer Science
Faculty of Science and Mathematics
Višegradska 33
1800 Niš, Serbia
pecko@pmf.ni.ac.rs

Marija Ćirić
University of Niš
Department of Computer Science
Faculty of Science and Mathematics
Višegradska 33
1800 Niš, Serbia
marijamath@yahoo.com

Lev A. Kazakovtsev
Siberian State Aerospace University
Department of Information Technologies
prosp.Krasnoyarskiy Rabochiy, 31
660014 Krasnoyarsk, Russian Federation
levk@ieee.org

Idowu A. Osinuga
Federal University of Agriculture
College of Natural Sciences
Department of Mathematics
PMB 2240, Abeokuta, Nigeria
osinuga08gmail.com