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ON PEDAL CURVES OF NORMAL SURFACES OF BIHARMONIC CURVES IN $\widetilde{\mathcal{SL}_2}(\mathbb{R})$

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Abstract. In this paper we obtain some results on the pedal curves of normal ruled surfaces of biharmonic curves in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$. Also, we find out their explicit parametric equations. Finally, we illustrate our main result.

1. Introduction

Ruled surfaces have been popular in architecture. With a focus on structural elegance, these and many other contributions are in contrast to recent free-form architecture. Applied mathematics and in particular geometry have initiated the implementation of comprehensive frameworks for modeling and mastering the complexity of today's architectural needs and shapes in an optimal sense by ruled surfaces.

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper we obtain some results on the pedal curves of normal ruled surfaces of biharmonic curves in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$. Also, we find out their explicit parametric equations. Finally, we illustrate our main result.

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2. $\widetilde{\mathcal{SL}_2}(\mathbb{R})$

We identify $\widetilde{\mathcal{SL}_2}(\mathbb{R})$ with

$$\mathbb{R}^3_+ = \left\{(x,y,z) \in \mathbb{R}^3 : z > 0\right\}$$

endowed with the metric

$$g = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $\widetilde{\mathcal{SL}_2}(\mathbb{R})$

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, \mathbf{e}_3 = z\frac{\partial}{\partial z}.$$

The characterising properties of g defined by

$$\begin{aligned} g(\mathbf{e}_1,\mathbf{e}_1) &= g(\mathbf{e}_2,\mathbf{e}_2) = g(\mathbf{e}_3,\mathbf{e}_3) = 1, \\ g(\mathbf{e}_1,\mathbf{e}_2) &= g(\mathbf{e}_2,\mathbf{e}_3) = g(\mathbf{e}_1,\mathbf{e}_3) = 0. \end{aligned}$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul's formula.

Using the Koszul's formula, we obtain

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= 0,\ \nabla_{\mathbf{e}_1}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_3,\ \ \nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_2, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 &= \frac{1}{2}\mathbf{e}_3,\ \ \nabla_{\mathbf{e}_2}\mathbf{e}_2 = \mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_1 - \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 &= -\frac{1}{2}\mathbf{e}_2,\ \ \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1,\ \ \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{2.1}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\ \ R_{ijkl} = R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3

$$R_{1212} = R_{1313} = \frac{1}{4},\ R_{2323} = -\frac{7}{4}. \tag{2.2}$$

3. Biharmonic Curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Let us consider biharmonicity of curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.2}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1,\ g(\mathbf{N}, \mathbf{N}) = 1,\ g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ *is a biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 + \tau^2 &= -\frac{1}{4} + \frac{15}{4}B_1^2, \\ \tau' &= 2N_1B_1. \end{aligned} \tag{3.4}$$

Proof. Using (3.1) and Frenet formulas (3.2), we have (3.4).

Theorem 3.2. ([9]) *Let* $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ *be a unit speed non-geodesic biharmonic curve. Then, the parametric equations of* γ *are*

$$\begin{aligned} x(s) &= \frac{1}{\aleph}\sin\varphi\sin[\aleph s + C] + \frac{1}{\aleph}\sin\varphi\cos[\aleph s + C] + \wp_2, \\ y(s) &= \frac{1}{\aleph^2 + \cos^2\varphi}\sin\varphi\wp_1 e^{\cos\varphi s}(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C]), \\ z(s) &= \wp_1 e^{\cos\varphi s}, \end{aligned} \tag{3.5}$$

where $\aleph$, C, $\wp_1$, $\wp_2$ *are constants of integration.*

4. Normal Ruled Surfaces of Biharmonic Curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

The purpose of this section is to study pedal curves of normal ruled surfaces of biharmonic curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$.

The normal surface of γ is a ruled surface

$$\Omega(s,u)=\gamma(s)+u\mathbf{N}. \tag{4.1}$$

Let Ω be a ruled surface given by equation (4.1) in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. Since the tangent plane is constant along rulings of Ω, it is clear that the pedal of Ω is a curve. Thus, for the pedal of Ω, we can write

$$\bar{\gamma}(s)=\gamma(s)+R(s)\mathbf{N}(s), \tag{4.2}$$

where $R(s)$ is the distance between the points $\gamma(s)$ and $\bar{\gamma}(s)$.

Firstly, we need following lemma.

Lemma 4.1. *Let* $\gamma: I\longrightarrow\widetilde{\mathcal{SL}_2(\mathbb{R})}$ *be a unit speed non-geodesic biharmonic curve. Then the position vector of* γ *is*

$$\begin{aligned}\gamma(s)=&[\frac{1}{\aleph}\sin\varphi\sin[\aleph s+C]+\frac{1}{\aleph}\sin\varphi\cos[\aleph s+C]+\wp_2\\&+[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C])]]\mathbf{e}_1\\&[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C])]\mathbf{e}_2+\mathbf{e}_3,\end{aligned} \tag{4.3}$$

where $\aleph, C, \wp_2$ *are constants of integration.*

From the the above lemma we have following theorem:

Theorem 4.2. *Let* Ω *be a normal surface of a unit speed non-geodesic biharmonic curve in* $\widetilde{\mathcal{SL}_2(\mathbb{R})}$ *and* $\bar{\gamma}$ *its pedal curve. Then, the equation of this pedal* *curve is given by*

$$
\begin{aligned}
\bar{\gamma}(s) \;=\; & [\frac{1}{\aleph}\sin\varphi\sin[\aleph s+C]+\frac{1}{\aleph}\sin\varphi\cos[\aleph s+C]+\wp_2 \\
& +[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C])] \\
& -\frac{R(s)\aleph}{\kappa}\sin\varphi\sin[\aleph s+C]]\mathbf{e}_1 \\
& [\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C]) \\
& +\frac{R(s)}{\kappa}(\aleph\sin\varphi\cos[\aleph s+C]-\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C] \\
& -\cos\varphi\sin\varphi\cos[\aleph s+C])]\mathbf{e}_2 \\
& (1+\frac{R(s)}{\kappa}(\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C]+\sin^2\varphi\sin^2[\aleph s+C]))\mathbf{e}_3,
\end{aligned} \tag{4.4}
$$

where $\aleph$, C, $\wp_1$, $\wp_2$ *are constants of integration and* $R(s)$ *is the distance between the points* $\gamma(s)$ *and* $\bar{\gamma}(s)$.

Proof. From Lemma 4.1, we get

$$
\mathbf{T}=\sin\varphi\cos[\aleph s+C]\,\mathbf{e}_1+\sin\varphi\sin[\aleph s+C]\,\mathbf{e}_2+\cos\varphi\mathbf{e}_3. \tag{4.5}
$$

Using first equation of the system (3.2) and (2.3), we have

$$
\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} \;=\; & -\aleph\sin\varphi\sin[\aleph\mathbf{s}+\mathbf{C}]\,\mathbf{e}_1 \\
& +(\aleph\sin\varphi\cos[\aleph s+C]-\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C] \\
& -\cos\varphi\sin\varphi\cos[\aleph s+C])\mathbf{e}_2 \\
& +(\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C]+\sin^2\varphi\sin^2[\aleph s+C])\mathbf{e}_3.
\end{aligned} \tag{4.6}
$$

By the use of Frenet formulas and above equation, we get

$$
\begin{aligned}
\mathbf{N} \;=\; & -\frac{\aleph}{\kappa}\sin\varphi\sin[\aleph s+C]\,\mathbf{e}_1 \\
& +\frac{1}{\kappa}(\aleph\sin\varphi\cos[\aleph s+C]-\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C] \\
& -\cos\varphi\sin\varphi\cos[\aleph s+C])\mathbf{e}_2 \\
& +\frac{1}{\kappa}(\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C]+\sin^2\varphi\sin^2[\aleph s+C])\mathbf{e}_3.
\end{aligned} \tag{4.7}
$$

Combining (4.6) and (4.2), we obtain (4.4). This concludes the proof of theorem.

From the above theorem and (3.4) we can see that:

Corollary 4.3. *Let Ω be a normal surface of a unit speed non-geodesic biharmonic curve in $\widetilde{\mathcal{SL}_2}(\mathbb{R})$ and $\bar{\gamma}$ its pedal curve. Then, using torsion of biharmonic curve*

$$\begin{aligned}\bar{\gamma}(s) = {} & [\frac{1}{\aleph}\sin\varphi\sin[\aleph s + C] + \frac{1}{\aleph}\sin\varphi\cos[\aleph s + C] + \wp_2 \\ & +[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C])] \\ & -(-\tau^2 - \frac{1}{4} + \frac{15}{4}B_1^2)^{-\frac{1}{2}}R(s)\aleph\sin\varphi\sin[\aleph s + C]]\mathbf{e}_1 \\ & [\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C]) \\ & +(-\tau^2 - \frac{1}{4} + \frac{15}{4}B_1^2)^{-\frac{1}{2}}R(s)(\aleph\sin\varphi\cos[\aleph s + C] - \sin^2\varphi\cos[\aleph s + C]\sin[\aleph s + C] \\ & -\cos\varphi\sin\varphi\cos[\aleph s + C])]\mathbf{e}_2 + (1 + (-\tau^2 - \frac{1}{4} + \frac{15}{4}B_1^2)^{-\frac{1}{2}}R(s)(\sin^2\varphi\cos[\aleph s + C] \\ & \sin[\aleph s + C] + \sin^2\varphi\sin^2[\aleph s + C]))\mathbf{e}_3,\end{aligned}$$

where $\aleph$, C, $\wp_1$, $\wp_2$ are constants of integration and $R(s)$ is the distance between the points $\gamma(s)$ and $\bar{\gamma}(s)$.

5. Applications

The obtained parametric equations for Eq. (3.4) is illustrated in Fig. 5.1:

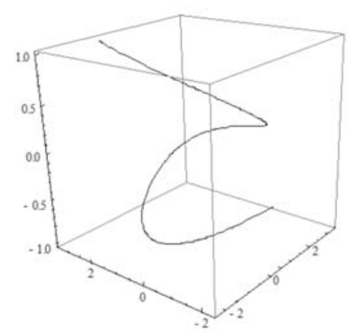

Fig. 5.1: Illustration of parametric equations

Similarly, the obtained equations for pedal curve is illustrated for constant $R(s)$ in Fig. 5.2:

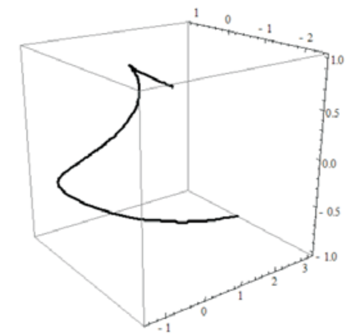

Fig. 5.2: Illustration of equations for pedal curve

If we use Mathematica both unit speed non-geodesic biharmonic curve and its pedal curve, we have

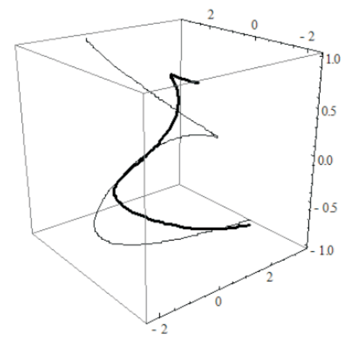

Fig. 5.3: Illustration in Mathematica

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