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M-STEP ITERATIVE PROCESS FOR A FINITE FAMILY OF MULTIVALUED GENERALIZED NONEXPANSIVE MAPPINGS IN CAT(0) SPACES *

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Abstract. In this paper, we introduce a new m-step iterative process for a finite family of multivalued mappings satisfying the condition (E). We then prove some strong and Δ-convergence theorems in CAT(0) spaces. In this way, we extend results of [8].

1. Introduction

A metric space X is said to be a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. For more information on these spaces and on the fundamental role they play in geometry we refer the reader to Bridson and Haefliger [1].

Fixed point theory for single-valued mappings in CAT(0) spaces was first studied by W. A. Kirk (see [2] and [3]). Since then the fixed point theory for single valued and multivalued mappings in CAT(0) spaces has been rapidly developed. The study of fixed points for multivalued nonexpansive mappings by using the Hausdorff metric was initiated by Markin [4].

In 2008, Suzuki [5] introduced a condition which is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. Suzuki's condition was named by himself the condition (C).

In 2011, Garcia-Falset et al.[6] introduced two conditions on single valued mappings, called conditions (E) and (C_λ), which are weaker than nonexpansiveness and stronger than quasi nonexpansiveness.

Recently, Abkar and Eslamian [7] used a modified version of these conditions for multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying these conditions in a CAT(0) space.

The aim of this paper is to introduce a new m-step iterative process for a finite family of mulivalued mappings satisfying condition (E) and prove some strong and

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Δ-convergence theorems in a CAT(0) space. In this way, we extend the iterative process that were introduced by Abkar and Eslamian in [8].

2. Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ (or, more briefly, a geodesic) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, l]$. In particular, the mapping c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic. The space X is said to be uniquely geodesic if there is exactly one geodesic joining x and y, for each $x, y \in X$. A subset Y of X is said to be convex if Y includes every geodesic segment joining any two points of itself.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.
A geodesic metric space X is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, $d(x, y) \le d_{R^2}(\overline{x}, \overline{y})$.

We begin with the following property of a CAT(0) space.

Lemma 2.1. [10] *Let (X, d) be a CAT(0) space.*
(a) For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying above relation.
(b) For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1 - t)x \oplus ty, z) \le (1 - t)d(x, z) + td(y, z).$$

(c) ([1]) Let X be a CAT(0) space and D be a closed convex subset of X then
(1). For each $x \in X$, there exists an element $P_D(x) \in C$ such that $d(x, P_D(x)) = dist(x, D)$.
(2). $P_D(x) = P_D(x')$ for all $x' \in [x, P_D(x)]$.
(3). The mapping $x \mapsto P_D$ is nonexpansive.
The mapping P_D is called the convex projection from X into D.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It's Known (see, e.g., [11], Proposition 7) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.
A notion of Δ-convergence in CAT(0) spaces based on the fact that in Hilbert spaces a bounded sequence is weakly convergent to its unique asymptotic center has been introduced in [12].

Definition 2.1. ([12] and [13]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one writes Δ-$\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2. *([12]) Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergence subsequence.*

Lemma 2.3. *([14]) If D is a closed convex subset of a complete $CAT(0)$ space, and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ lies in D.*

Lemma 2.4. *([15]) Let X be a complete $CAT(0)$ space, and let $x \in X$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and that $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(t_n x_n \oplus (1 - t_n)y_n, x) = r$ for some $r \geq 0$. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Let D be a subset of a CAT(0) space X. We denote by $CB(D)$, $K(D)$, $P(D)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, nonempty proximal subsets of D, respectively. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(D)$ where $dist(x, B) = \inf_{y\in B} d(x, y)$.
An element $x \in X$ is said to be a fixed point of a multivalued mapping T, if $x \in Tx$. We denote by $Fix(T)$ the set of all fixed points of T.

Definition 2.2. A multivalued mapping $T : X \to CB(X)$ is said to be nonexpansive provided that

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in X.$$

Definition 2.3. A multivalued mapping $T : X \to CB(X)$ is said to be quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $H(Tx, Tp) \leq d(x, p)$ for all $x \in X, p \in Fix(T)$.

Definition 2.4. A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (E_μ) provided that

$$dist(x, Ty) \leq \mu dist(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Lemma 2.5. *Let $T : X \to CB(X)$ be a multivalued nonexpansive mapping. Then T satisfies the condition (E_1).*

Lemma 2.6. *([10]) Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X, such that $A(\{x_n\}) = \{x\}$. Suppose $\{u_n\}$ is a subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$ and let $\{d(x_n, u)\}$ converges in $\mathbb{R}$. Then $x = u$.*

3. The main results

First, we recall the following lemma.

Lemma 3.1. *([8]) let D be a nonempty closed convex subset of a complete $CAT(0)$ space X. Suppose that a multivalued mapping $T : D \to K(D)$ satisfies condition (E). If $\{x_n\}$ is a sequence in D such that $\Delta\text{-}\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} dist(x_n, Tx_n) = 0$. Then $x \in Tx$.*

Lemma 3.2. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let the multivalued mapping $T : D \to K(D)$ satisfies condition (E). Suppose that $\{x_n\}$ is a bounded sequence in D such that $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in Fix(T)$. Then $W_w(\{x_n\}) \subset Fix(T)$. Here $W_w(\{x_n\}) := \cup A(\{x_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $W_w(\{x_n\})$ consists of exactly one point.*

Proof. Let $u \in W_w(\{x_n\})$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 3.1, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\{v_n\}$ is Δ-convergent in X. Let $\Delta - \lim_{n\to\infty} v_n = v$. By Lemma 3.2, $v \in Tv$, $v \in K$. By Lemma 2.12, $v = u$. Thus, $W_w(\{x_n\}) \subset Fix(T)$. It is easy to show that $W_w(\{x_n\})$ consists of exactly one point. $\square$

Lemma 3.3. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let the multivalued mapping $T : D \to K(D)$ satisfy condition (E) and be such that $\inf\{dist(x, Tx) : x \in D\} = 0$. Then T has a fixed point in D.*

Proof. let $A(\{x_n\}) = \{x\}$. By the proof of Theorem 3.4 in [8], we can deduce the argument. $\square$

Now, we introduce the following iterative process.
(I). Let X be a CAT(0) space, D be a nonempty closed convex subset of X and $T_{i,k} : D \to CB(D)$ (i=1,...,l and k=1,...,m) be $l \times m$ given multivalued mappings. Then, for $x_1 \in D$ and $a_{(i)n,k} \in [0,1]$, we consider the following iterative process:

$$\begin{aligned}
y_{n,1} &= a_{(1)n,1}z_{(1)n,1} \oplus (1-a_{(1)n,1})(a_{(2)n,1}z_{(2)n,1} \oplus (1-a_{(2)n,1})(\ldots \\
&\quad \oplus(1-a_{(l-1)n,1})(a_{(l)n,1}z_{(l)n,1} \oplus (1-a_{(l)n,1})x_n)\ldots)), \qquad n \geq 1, \\
y_{n,2} &= a_{(1)n,2}z_{(1)n,2} \oplus (1-a_{(1)n,2})(a_{(2)n,2}z_{(2)n,2} \oplus (1-a_{(2)n,2})(\ldots \\
&\quad \oplus(1-a_{(l-1)n,2})(a_{(l)n,2}z_{(l)n,2} \oplus (1-a_{(l)n,2})x_n)\ldots)), \qquad n \geq 1, \\
&\vdots \\
y_{n,m-1} &= a_{(1)n,m-1}z_{(1)n,m-1} \oplus (1-a_{(1)n,m-1})(a_{(2)n,m-1}z_{(2)n,m-1} \\
&\quad \oplus(1-a_{(2)n,m-1})(\ldots \oplus (1-a_{(l-1)n,m-1})(a_{(l)n,m-1}z_{(l)n,m-1} \\
&\quad \oplus(1-a_{(l)n,m-1})x_n)\ldots)), \qquad n \geq 1, \\
x_{n+1} &= a_{(1)n,m}z_{(1)n,m} \oplus (1-a_{(1)n,m})(a_{(2)n,m}z_{(2)n,m} \oplus (1-a_{(2)n,m})(\ldots \\
&\quad \oplus(1-a_{(l-1)n,m})(a_{(l)n,m}z_{(l)n,m} \oplus (1-a_{(l)n,m})x_n)\ldots)), \qquad n \geq 1,
\end{aligned}$$

where $z_{(i)n,1} \in T_{i,1}(x_n)$ and $z_{(i)n,k} \in T_{i,k}(y_{n,k-1})$ for $i = 1,\ldots,l$ and $k = 2,\ldots,m$. Note that if $l = m = 1$ then $\{x_n\}$ reduces to Mann's iteration, and if $m = 2$ and $l = 1$ then $\{x_n\}$ will be the Ishikawa's iteration (see [15]); if $l = 1$ and $m \in \mathbb{N}$ then $\{x_n\}$ is the iteration that which was introduced by Abkar and Eslamian (see [8]).

Theorem 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_{i,k} : D \to CB(D)$, $(i = 1,\ldots,l$ and $k = 1,\ldots,m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying condition (E) with $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k}) \neq \emptyset$ and $T_{i,k}(p) = \{p\}$ for each $p \in F$. Let $x_n \in X$ be the iterative process defined by (I) and $a_{(i)n,k} \in (0,1)$ $(i = 1,\ldots,l$ and $k = 1,\ldots,m)$. Then*
(i) $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F$.
(ii) If we assume that
$(*)$ $\lim_{n\to\infty} d(t_n y_{n,k} \oplus (1-t_n)z_{(i)n,k}, p) = \lim_{n\to\infty} d(x_n,p) = \lim_{n\to\infty} d(t_n x_n \oplus (1-t_n)z_{(i)n,k}, p)$ *for $(t_n) \subset (0,1)$ and $p \in F$, then* $\lim_{n\to\infty} dist(x_n, T_{i,k}(x_n)) = 0$, $i = 1,\ldots,l$ *and* $k = 1,\ldots,m$.

Proof. Let $p \in F$. Then using (I) and Lemma 2.1, and quasi nonexpansiveness of $T_{i,k}$, we have

$$\begin{aligned}
d(y_{n,1},p) &= d(a_{(1)n,1}z_{(1)n,1} \oplus (1-a_{(1)n,1})(a_{(2)n,1}z_{(2)n,1} \oplus (1-a_{(2)n,1})(\ldots \\
&\quad \oplus(1-a_{(l-1)n,1})(a_{(l)n,1}z_{(l)n,1} \oplus (1-a_{(l)n,1})x_n)\ldots)),p) \\
&\leq a_{(1)n,1}d(z_{(1)n,1},p) + (1-a_{(1)n,1})d(a_{(2)n,1}z_{(2)n,1} \oplus (1-a_{(2)n,1})(\ldots \\
&\qquad \oplus)(1-a_{(l-1)n,1})(a_{(l)n,1}z_{(l)n,1} \oplus (1-a_{(l)n,1})x_n)\ldots),p)
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\leq a_{(1)n,1}d(z_{(1)n,1},p)+(1-a_{(1)n,1})(a_{(2)n,1}d(z_{(2)n,1},p)+(1-a_{(2)}n,1) \\
&\quad(...+a_{(l)n,1}d(z_{(l)n,1},p)+(1-a_{(l)n,1})d(x_n,p)...)) \\
&= a_{(1)n,1}dist(z_{(1)n,1},T_{1,1}p)+(1-a_{(1)n,1})(a_{(2)n,1}dist(z_{(2)n,1},T_{2,1}p)+ \\
&\quad(1-a_{(2)n,1})(...+a_{(l)n,1}dist(z_{(l)n,1},T_{l,1}p)+(1-a_{(l)n,1})d(x_n,p)...)) \\
&\leq a_{(1)n,1}H(T_{1,1}x_n,T_{1,1}p)+(1-a_{(1)n,1})(a_{(2)n,1}H(T_{2,1}x_n,T_{2,1}p)+ \\
&\quad(1-a_{(2)n,1})(...+a_{(l)n,1}H(T_{l,1}x_n,T_{l,1}p)+(1-a_{(l)n,1})d(x_n,p)...)) \\
&\leq a_{(1)n,1}d(x_n,p)+(1-a_{(1)n,1})(a_{(2)n,1}d(x_n,p)+(1-a_{(2)}n,1)(... \\
&\quad+a_{(l)n,1}d(x_n,p)+(1-a_{(l)n,1})d(x_n,p)...)) \\
&= (a_{(1)n,1}+(1-a_{(1)}n,1)(a_{(2)n,1}+(1-a_{(2)n,1})(...a_{(l)n,1}+(1-a_{(l)n,1})...)))d(x_n,p) \\
&= d(x_n,p),
\end{aligned}
$$

and

$$
\begin{aligned}
d(y_{n,2},p) &= d(a_{(1)n,2}z_{(1)n,2}\oplus(1-a_{(1)n,2})(a_{(2)n,2}z_{(2)n,2}\oplus(1-a_{(2)n,2})(... \\
&\quad\oplus(1-a_{(l-1)n,2})(a_{(l)n,2}z_{(l)n,2}\oplus(1-a_{(l)n,2})x_n)...)),p) \\
&\leq a_{(1)n,2}d(z_{(1)n,2},p)+(1-a_{(1)n,2})d(a_{(2)n,2}z_{(2)n,2}\oplus(1-a_{(2)n,2})(... \\
&\quad\oplus)(1-a_{(l-1)n,2})(a_{(l)n,2}z_{(l)n,2}\oplus(1-a_{(l)n,2})x_n)...),p) \\
&\quad\vdots \\
&\leq a_{(1)n,2}d(z_{(1)n,2},p)+(1-a_{(1)n,2})(a_{(2)n,2}d(z_{(2)n,2},p)+(1-a_{(2)}n,2) \\
&\quad(...+a_{(l)n,2}d(z_{(l)n,2},p)+(1-a_{(l)n,2})d(x_n,p)...)) \\
&= a_{(1)n,2}dist(z_{(1)n,2},T_{1,2}p)+(1-a_{(1)n,2})(a_{(2)n,2}dist(z_{(2)n,2},T_{2,2}p)+ \\
&\quad(1-a_{(2)n,2})(...+a_{(l)n,2}dist(z_{(l)n,2},T_{l,2}p)+(1-a_{(l)n,2})d(x_n,p)...)) \\
&\leq a_{(1)n,2}H(T_{1,2}y_{n,1},T_{1,2}p)+(1-a_{(1)n,2})(a_{(2)n,2}H(T_{2,2}y_{n,1},T_{2,2}p)+ \\
&\quad(1-a_{(2)n,2})(...+a_{(l)n,2}H(T_{l,2}y_{n,1},T_{l,2}p)+(1-a_{(l)n,2})d(x_n,p)...)) \\
&\leq a_{(1)n,2}d(y_{n,1},p)+(1-a_{(1)n,2})(a_{(2)n,2}d(y_{n,1},p)+ \\
&\quad(1-a_{(2)n,2})(...+a_{(l)n,2}d(y_{n,1},p)+(1-a_{(l)n,2})d(x_n,p)...)) \\
&\leq a_{(1)n,2}d(x_n,p)+(1-a_{(1)n,2})(a_{(2)n,2}d(x_n,p)+(1-a_{(2)}n,2)(... \\
&\quad+a_{(l)n,2}d(x_n,p)+(1-a_{(l)n,2})d(x_n,p)...)) \\
&= (a_{(1)n,2}+(1-a_{(1)}n,2)(a_{(2)n,2}+(1-a_{(2)n,2})(...a_{(l)n,2}+(1-a_{(l)n,2})...)))d(x_n,p) \\
&= d(x_n,p).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
d(y_{n,m-1},p) &= d(a_{(1)n,m-1}z_{(1)n,m-1}\oplus(1-a_{(1)n,m-1})(a_{(2)n,m-1}z_{(2)n,m-1}\oplus(1- \\
&\quad a_{(2)n,m-1})(...\oplus(1-a_{(l-1)n,m-1})(a_{(l)n,m-1}z_{(l)n,m-1}\oplus(1-a_{(l)n,m-1})x_n)...)),p) \\
&\leq a_{(1)n,m-1}d(z_{(1)n,m-1},p)+(1-a_{(1)n,m-1})d(a_{(2)n,m-1}z_{(2)n,m-1}\oplus(1- \\
&\quad a_{(2)n,m-1})(...\oplus)(1-a_{(l-1)n,m-1})(a_{(l)n,m-1}z_{(l)n,m-1}\oplus(1-a_{(l)n,m-1})x_n)...),p) \\
&\quad\vdots \\
&\leq a_{(1)n,m-1}d(z_{(1)n,m-1},p)+(1-a_{(1)n,m-1})(a_{(2)n,m-1}d(z_{(2)n,m-1},p) \\
&\quad+(1-a_{(2)}n,m-1)(...+a_{(l)n,m-1}d(z_{(l)n,m-1},p)+(1-a_{(l)n,m-1})d(x_n,p)...)) \\
&= a_{(1)n,m-1}dist(z_{(1)n,m-1},T_{1,m-1}p)+(1-a_{(1)n,m-1})(a_{(2)n,m-1} \\
&\quad dist(z_{(2)n,m-1},T_{2,m-1}p)+(1-a_{(2)n,m-1})(...+a_{(l)n,m-1}dist(z_{(l)n,m-1},T_{l,m-1}p) \\
&\quad+(1-a_{(l)n,m-1})d(x_n,p)...)) \\
&\leq a_{(1)n,m-1}H(T_{1,m-1}y_{n,m-2},T_{1,m-1}p)+(1-a_{(1)n,m-1})(a_{(2)n,m-1} \\
&\quad H(T_{2,m-1}y_{n,m-2},T_{2,m-1}p)+(1-a_{(2)n,m-1})(...+a_{(l)n,m-1}
\end{aligned}
$$

$$\begin{aligned}
& H(T_{l,m-1}y_{n,m-2}, T_{l,m-1}p) + (1 - a_{(l)n,m-1})d(x_n, p)...)) \\
\leq\ & a_{(1)n,m-1}d(y_{n,m-2}, p) + (1 - a_{(1)n,m-1})(a_{(2)n,m-1}d(y_{n,m-2}, p)+ \\
& (1 - a_{(2)n,m-1})(... + a_{(l)n,m-1}d(y_{n,m-2}, p) + (1 - a_{(l)n,m-1})d(x_n, p)...)) \\
\leq\ & a_{(1)n,m-1}d(x_n, p) + (1 - a_{(1)n,m-1})(a_{(2)n,m-1}d(x_n, p) + (1 - a_{(2)}n, m - 1)(... \\
& +a_{(l)n,m-1}d(x_n, p) + (1 - a_{(l)n,m-1})d(x_n, p)...)) \\
=\ & (a_{(1)n,m-1} + (1 - a_{(1)}n, m - 1)(a_{(2)n,m-1} + (1 - a_{(2)n,m-1})(...a_{(l)n,m-1}+ \\
& (1 - a_{(l)n,m-1})...)))d(x_n, p) = d(x_n, p),
\end{aligned}$$

and also

$$\begin{aligned}
d(x_{n+1}, p) =\ & d(a_{(1)n,m}z_{(1)n,m} \oplus (1 - a_{(1)n,m})(a_{(2)n,m}z_{(2)n,m} \oplus (1 - a_{(2)n,m})(... \\
& \oplus(1 - a_{(l-1)n,m})(a_{(l)n,m}z_{(l)n,m} \oplus (1 - a_{(l)n,m})x_n)...)), p) \\
\leq\ & a_{(1)n,m}d(z_{(1)n,m}, p) + (1 - a_{(1)n,m})d(a_{(2)n,m}z_{(2)n,m} \oplus (1 - a_{(2)n,m})(... \\
& \oplus)(1 - a_{(l-1)n,m})(a_{(l)n,m}z_{(l)n,m} \oplus (1 - a_{(l)n,m})x_n)...), p) \\
& \vdots \\
\leq\ & a_{(1)n,m}d(z_{(1)n,m}, p) + (1 - a_{(1)n,m})(a_{(2)n,m}d(z_{(2)n,m}, p) + (1 - a_{(2)}n, m) \\
& (... + a_{(l)n,m}d(z_{(l)n,m}, p) + (1 - a_{(l)n,m})d(x_n, p)...)) \\
=\ & a_{(1)n,m}dist(z_{(1)n,m}, T_{1,m}p) + (1 - a_{(1)n,m})(a_{(2)n,m}dist(z_{(2)n,m}, T_{2,m}p)+ \\
& (1 - a_{(2)n,m})(... + a_{(l)n,m}dist(z_{(l)n,m}, T_{l,m}p) + (1 - a_{(l)n,m})d(x_n, p)...)) \\
\leq\ & a_{(1)n,m}H(T_{1,m}y_{n,m-1}, T_{1,m}p) + (1 - a_{(1)n,m})(a_{(2)n,m}H(T_{2,m}y_{n,m-1}, T_{2,m}p)+ \\
& (1 - a_{(2)n,m})(... + a_{(l)n,m}H(T_{l,m}y_{n,m-1}, T_{l,m}p) + (1 - a_{(l)n,m})d(x_n, p)...)) \\
\leq\ & a_{(1)n,m}d(y_{n,m-1}, p) + (1 - a_{(1)n,m})(a_{(2)n,m}d(y_{n,m-1}, p)+ \\
& (1 - a_{(2)n,m})(... + a_{(l)n,m}d(y_{n,m-1}, p) + (1 - a_{(l)n,m})d(x_n, p)...)) \\
\leq\ & a_{(1)n,m}d(x_n, p) + (1 - a_{(1)n,m})(a_{(2)n,m}d(x_n, p) + (1 - a_{(2)}n, m)(... \\
& +a_{(l)n,m}d(x_n, p) + (1 - a_{(l)n,m})d(x_n, p)...)) \\
=\ & (a_{(1)n,m} + (1 - a_{(1)}n, m)(a_{(2)n,m} + (1 - a_{(2)n,m})(...a_{(l)n,m} \\
& +(1 - a_{(l)n,m})...)))d(x_n, p) = d(x_n, p).
\end{aligned}$$

And $d(z_{(i)n,k}, p) = dist(z_{(i),n,k}, T_{i,k}p) \leq d(x_n, p)$. Therefore, $\{d(x_n, p)\}$ is decreasing and bounded below. This implies that $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F$. We suppose that $\lim_{n\to\infty} d(x_n, p) = c$ for some $c \geq 0$. Thus $\lim_{n\to\infty} d(x_{n+1}, p) = c$. Since $d(z_{(i)n,k}, p) \leq d(x_n, p)$ then by taking limit superior on both sides, we obtain $\limsup_n d(z_{(i)n,k}, p) \leq c$ ($i = 1, ..., l$ and $k = 1, ..., m$). Also, $\lim_n d(y_{n,k}, p) \leq \lim_n d(x_n, p) = c$. On the other hand, we have

$$\begin{aligned}
d(t_n y_{n,k} \oplus (1 - t_n)z_{(i)n,k}, p) &\leq t_n d(y_{n,k}, p) + (1 - t_n)d(z_{(i)n,k}, p) \\
&\leq t_n d(x_n, p) + (1 - t_n)d(x_n, p) = d(x_n, p)
\end{aligned}$$

so that $\lim_n d(t_n y_{n,k} \oplus (1-t_n)z_{(i)n,k}, p) \leq \lim_n d(x_n, p)$ for all $(t_n)_n \subseteq [a, b] \subset (0, 1)$. But by the assumption $(*)$, $\lim_n d(t_n y_{n,k} \oplus (1 - t_n)z_{(i)n,k}, p) = \lim_n d(x_n, p) = c$. Then by Lemma 2.5, $\lim_n d(y_{n,k}, z_{(i)n,k}) = 0$. Again, by using Lemma 2.5 and the assumption $(*)$, we get $\lim_n d(x_n, z_{(i)n,k}) = 0$. Hence, $dist(x_n, T_{i,1}x_n) \leq d(x_n, z_{(i)n,1}) \to 0$, as $n \to \infty$ and $dist(x_n, T_{i,k}y_{n,k-1}) \leq d(x_n, z_{(i)n,k}) \to 0$, as $n \to \infty$ for ($i = 1, ..., l$ and $k = 2, ..., m$). Also, we have $d(x_n, y_{n,k-1}) \leq d(x_n, z_{1n,k-1}) + d(y_{n,k-1}, z_{(1)n,k-1})$, so that $\lim_n d(x_n, y_{n,k-1}) = 0$. By condition (E), we get for some $\mu \geq 1$,

$$\begin{aligned}
dist(x_n, T_{i,k}x_n) &\leq d(x_n, y_{n,k-1}) + dist(y_{n,k-1}, T_{i,k}x_n) \\
&\leq d(x_n, y_{n,k-1}) + \mu dist(y_{n,k-1}, T_{i,k}y_{n,k-1}) + d(x_n, y_{n,k-1}) \\
&\leq d(x_n, y_{n,k-1}) + \mu dist(x_n, T_{i,k}y_{n,k-1}) + \mu d(x_n, y_{n,k-1}) + d(x_n, y_{n,k-1}) \\
&\leq (\mu + 2)d(x_n, y_{n,k-1}) + \mu dist(x_n, T_{i,k}y_{n,k-1});
\end{aligned}$$

So, for $i = 1, ..., l$ and $k = 1, ..., m$, $\lim_{n\to\infty} dist(x_n, T_{i,k}x_n) = 0$. $\square$

Now, we turn to some strong and Δ convergence theorems:

Theorem 3.2. *Suppose that $X, D, T_{i,k} : D \to K(D)$ $(i = 1, ..., l,\ k = 1, ..., m)$ are as in Theorem 3.4 with $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k}) \neq \emptyset$ and $T_{i,k}(p) = \{p\}$ for each $p \in F$. Let $x_n \in X$ be the iterative process defined by (I) and $a_{(i)n,k} \in (0,1)$ $(i = 1, ..., l$ and $k = 1, ..., m)$. Let the assumption $(*)$ of Theorem 3.4(ii) hold. Then $\{x_n\}$ is Δ-convergent to a common fixed point of $\{T_{i,k}\}_1^{l,m}$.*

Proof. By Theorem 3.4, we have $\lim_{n\to\infty} dist(x_n, T_{i,k}x_n) = 0$. Let

$$W_w(\{x_n\}) := \cup A(\{u_n\})$$

where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. By Lemma 3.2, $W_w(\{x_n\}) \subset F$ and $W_w(\{x_n\})$ consists of exactly one point, $\{x\}$. This means that $x \in F$ is the unique asymptotic center of each subsequence of $\{x_n\}$. Thus $\Delta\text{-}\lim_{n\to\infty} x_n = x$. $\square$

Theorem 3.3. *Let D be a nonempty compact convex subset of a complete CAT(0) space X. Let $T_{i,k} : D \to CB(D)$ $(i = 1, ..., l$ and $k = 1, ..., m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E) with $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k}) \neq \emptyset$ and $T_{i,k}(p) = \{p\}$ for each $p \in F$. Let $x_n \in X$ be the iterative process defined by (I) and $a_{(i)n,k} \in [a, b] \subset (0, 1)$ $(i = 1, ..., l$ and $k = 1, ..., m)$. Let the condition $(*)$ in Theorem 3.4(ii) hold. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_{i,k}\}_1^{l,m}$.*

Proof. By Theorem 3.4, for $i = 1, ..., l$ and $k = 1, ..., m$, we have

$$\lim_{n\to\infty} dist(x_n, T_{i,k}x_n) = 0.$$

Since D is compact, there exists a subsequence $\{x_{n_t}\}$ of $\{x_n\}$ such that $\lim_{t\to\infty} x_{n_t} = w$ for $w \in D$. By condition (E), we have for some $\mu \geq 1$,

$$\begin{aligned} dist(w, T_{i,k}w) &\leq d(w, x_{n_t}) + dist(x_{n_t}, T_{i,k}w) \\ &\leq \mu dist(x_{n_t}, T_{i,k}x_{n_t}) + 2d(w, x_{n_t}) \to 0,\ as\ t \to \infty. \end{aligned}$$

Thus $w \in F$. Since $\{x_{n_t}\}$ converges strongly to w and $\lim_n d(x_n, w)$ exists (by Theorem 3.4), we conclude that $\{x_n\}$ converges strongly to w. $\square$

Theorem 3.4. *Suppose that $X, D, T_{i,k}$ $(i = 1, ..., l,\ k = 1, ..., m)$ are as in Theorem 3.4 with $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k}) \neq \emptyset$ and $T_{i,k}(p) = \{p\}$ for each $p \in F$. Let $x_n \in X$ be the iterative process defined by (I) and $a_{(i)n,k} \in [a, b] \subset (0, 1)$ $(i = 1, ..., l$ and $k = 1, ..., m)$. Let the condition $(*)$ of Theorem 3.4(ii) hold. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_{i,k}\}_1^{l,m}$ if and only if $\liminf_n dist(x_n, F) = 0$.*

Proof. By using Theorem 3.4, and an argument similar to that in the proof of Theorem 3.8 in ([8]) we can finish the job. □

Inspired by the condition (A') for two mappings that was introduced in ([16]), we'd like to consider it for a finite family of multivalued mappings:
A finite family of multivalued mappings $T_{i,k} : D \to CB(D)$ is said to satisfy condition (A'') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r > 0$ such that for some $i = 1, ..., l$ and $k = 1, ..., m$

$$dist(x, T_{i,k}x) \geq f(dist(x, F)),$$

for all $x \in D$ where $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k})$.

Theorem 3.5. *Suppose that $X, D, T_{i,k}$ $(i = 1, ..., l,\ k = 1, ..., m)$ are as in Theorem 3.4 with $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k}) \neq \emptyset$ and $T_{i,k}(p) = \{p\}$ for each $p \in F$. Let $x_n \in X$ be the iterative process defined by (I) and $a_{(i)n,k} \in [a, b] \subset (0, 1)$ $(i = 1, ..., l$ and $k = 1, ..., m)$. Let the condition $(*)$ in Theorem 3.4(ii) hold and let $T_{i,k}$ satisfy the condition (A''). Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_{i,k}\}_1^{l,m}$.*

Proof. By Theorem 3.4, we have $\lim_{n\to\infty} dist(x_n, T_{i,k}x_n) = 0$ for each i, k. So by condition (A''), we get $\lim_n dist(x_n, F) = 0$. Now, the conclusion follows from Theorem 3.7. □

Corollary 3.1. *Suppose that X, D are as in Theorem 3.4, and let $T_{i,k} : D \to CB(D)$ $(i = 1, ..., l$ and $k = 1, ..., m)$ be a finite family of nonexpansive multivalued mappings with $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k}) \neq \emptyset$ and $T_{i,k}(p) = \{p\}$ for each $p \in F$. Let $x_n \in X$ be the iterative process defined by (I) and $a_{(i)n,k} \in [a, b] \subset (0, 1)$ $(i = 1, ..., l$ and $k = 1, ..., m)$. Let the condition $(*)$ in Theorem 3.4(ii) hold and let $T_{i,k}$ satisfy the condition (A''). Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_{i,k}\}_1^{l,m}$.*

Corollary 3.2. *Suppose that X, D are as in Theorem 3.4, and let $T_{i,k} : D \to D$ $(i = 1, ..., l$ and $k = 1, ..., m)$ be a finite family of single valued mappings satisfying the condition (E) with $F = \bigcap_{i,k=1}^{l,m} Fix(T_{i,k}) \neq \emptyset$. Let $x_n \in X$ be the iterative process defined by (I) and $a_{(i)n,k} \in (0, 1)$ $(i = 1, ..., l$ and $k = 1, ..., m)$. Let the condition $(*)$ in Theorem 3.4(ii) hold and let $T_{i,k}$ satisfy the condition (A''). Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_{i,k}\}_1^{l,m}$.*

Now, we introduce the improved iteration (I) for a finite family of multivalued nonself mappings:
(I'). Let $T_{i,k} : D \to CB(X)$ (i=1,...,l and k=1,...,m) be $l \times m$ given multivalued nonself mappings. Then, for $x_1 \in D$ and $a_{(i)n,k} \in [0, 1]$, we consider the following iterative process:

$$y_{n,1} \quad = \quad P_D(a_{(1)n,1}z_{(1)n,1} \oplus (1 - a_{(1)n,1})(a_{(2)n,1}z_{(2)n,1} \oplus (1 - a_{(2)n,1})(...$$

$$\begin{array}{rcll} & & \oplus(1-a_{(l-1)n,1})(a_{(l)n,1}z_{(l)n,1}\oplus(1-a_{(l)n,1})x_n)...))), & n\geq 1,\\ y_{n,2} &=& P_D(a_{(1)n,2}z_{(1)n,2}\oplus(1-a_{(1)n,2})(a_{(2)n,2}z_{(2)n,2}\oplus(1-a_{(2)n,2})(... & \\ & & \oplus(1-a_{(l-1)n,2})(a_{(l)n,2}z_{(l)n,2}\oplus(1-a_{(l)n,2})x_n)...))), & n\geq 1,\\ & & \vdots & \\ y_{n,m-1} &=& P_D(a_{(1)n,m-1}z_{(1)n,m-1}\oplus(1-a_{(1)n,m-1})(a_{(2)n,m-1}z_{(2)n,m-1} & \\ & & \oplus(1-a_{(2)n,m-1})(...\oplus(1-a_{(l-1)n,m-1})(a_{(l)n,m-1}z_{(l)n,m-1} & \\ & & \oplus(1-a_{(l)n,m-1})x_n)...))), & n\geq 1,\\ x_{n+1} &=& P_D(a_{(1)n,m}z_{(1)n,m}\oplus(1-a_{(1)n,m})(a_{(2)n,m}z_{(2)n,m}\oplus(1-a_{(2)n,m})(... & \\ & & \oplus(1-a_{(l-1)n,m})(a_{(l)n,m}z_{(l)n,m}\oplus(1-a_{(l)n,m})x_n)...))), & n\geq 1,\end{array}$$

where $z_{(i)n,1}\in T_{i,1}(x_n)$ and $z_{(i)n,k}\in T_{i,k}(y_{n,k-1})$ for $i=1,...,l$ and $k=2,...,m$ and P_D is the convex projection from X into D in Lemma 2.1(iii). Since P_D is nonexpansive, we have the following remark.

Remark 3.1. All of the previous results hold true for $\{x_n\}$ given by (I').

Now, we consider the following iteration process:
(II). Let $T_{i,k}:D\to CB(X)$ (i=1,...,l and k=1,...,m) be $l\times m$ given multivalued mappings and $P_{T_{i,k}}=\{y\in T_{i,k}(x):\ d(x,y)=dist(x,T_{i,k}(x))\}$. Then, for $x_1\in D$ and $a_{(i)n,k}\in[0,1]$, we consider the following iterative process:

$$\begin{array}{rcll} y_{n,1} &=& a_{(1)n,1}z_{(1)n,1}\oplus(1-a_{(1)n,1})(a_{(2)n,1}z_{(2)n,1}\oplus(1-a_{(2)n,1})(... & \\ & & \oplus(1-a_{(l-1)n,1})(a_{(l)n,1}z_{(l)n,1}\oplus(1-a_{(l)n,1})x_n)...)), & n\geq 1,\\ y_{n,2} &=& a_{(1)n,2}z_{(1)n,2}\oplus(1-a_{(1)n,2})(a_{(2)n,2}z_{(2)n,2}\oplus(1-a_{(2)n,2})(... & \\ & & \oplus(1-a_{(l-1)n,2})(a_{(l)n,2}z_{(l)n,2}\oplus(1-a_{(l)n,2})x_n)...)), & n\geq 1,\\ & & \vdots & \\ y_{n,m-1} &=& a_{(1)n,m-1}z_{(1)n,m-1}\oplus(1-a_{(1)n,m-1})(a_{(2)n,m-1}z_{(2)n,m-1} & \\ & & \oplus(1-a_{(2)n,m-1})(...\oplus(1-a_{(l-1)n,m-1})(a_{(l)n,m-1}z_{(l)n,m-1} & \\ & & \oplus(1-a_{(l)n,m-1})x_n)...)), & n\geq 1,\\ x_{n+1} &=& a_{(1)n,m}z_{(1)n,m}\oplus(1-a_{(1)n,m})(a_{(2)n,m}z_{(2)n,m}\oplus(1-a_{(2)n,m})(... & \\ & & \oplus(1-a_{(l-1)n,m})(a_{(l)n,m}z_{(l)n,m}\oplus(1-a_{(l)n,m})x_n)...)), & n\geq 1,\end{array}$$

where $z_{(i)n,1}\in P_{T_{i,1}}(x_n)$ and $z_{(i)n,k}\in P_{T_{i,k}}(y_{n,k-1})$ for $i=1,...,l$ and $k=2,...,m$. Assume that for $(t_n)\subset(0,1)$ and $p\in F$ we have

$$\lim_{n\to\infty}d(t_ny_{n,k}\oplus(1-t_n)z_{(i)n,k},p)=\lim_{n\to\infty}d(x_n,p)=\lim_{n\to\infty}d(t_nx_n\oplus(1-t_n)z_{(i)n,k},p)\ (**).$$

Theorem 3.6. *Suppose that $X,D,T_{i,k}:D\to P(D)$ $(i=1,...,l,\ k=1,...,m)$ are as in Theorem 3.4 with $F=\bigcap_{i,k=1}^{l,m}Fix(T_{i,k})\neq\emptyset$. Let $x_n\in X$ be the iterative process defined by (II) and $a_{(i)n,k}\in(0,1)$ $(i=1,...,l$ and $k=1,...,m)$. Let $T_{i,k}$ satisfy the condition (A''), and for $(t_n)\subset(0,1)$ and $p\in F$ we have*

$$\begin{aligned}&\lim_{n\to\infty}d(t_ny_{n,k}\oplus(1-t_n)z_{(i)n,k},p)\\ &=\lim_{n\to\infty}d(x_n,p)=\lim_{n\to\infty}d(t_nx_n\oplus(1-t_n)z_{(i)n,k},p).\end{aligned}$$

Then $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_{i,k}\}_1^{l,m}$.

Proof. The proof essentially goes in the same lines as in the proof of Theorem 3.4, and in the proof of Theorem 3.12 ([8]). We omit the details. □

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