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CONTINUITY FOR MULTILINEAR COMMUTATOR OF BOCHNER-RIESZ OPERATOR ON BESOV SPACES *

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Abstract. In this paper, we prove the the continuity for the multilinear operator associated to the Bochner-Riesz operator on the Besov spaces.

1. Introduction

As the development of the singular integral operators, their commutators and multilinear operators have been well studied(see [1-7]). From [2][7][13][15][18], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce the multilinear operator associated to the Bochner-Riesz operator and prove the continuity properties for the multilinear operator on the Besov spaces.

2. Preliminaries and Theorems

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that(see [14][15])

$$f^{\#}(x) \approx \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

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For $\beta \geqslant 0$, the Besov space $\dot{\Lambda}_\beta(R^n)$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x,\, h \,\in\, R^n \\ h \neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator(see [18]).

For $b_j \in \dot{\Lambda}_\beta(R^n)$ $(j = 1, \cdots, m)$, set

$$||\vec{b}||_{\dot{\Lambda}_\beta} = \prod_{j=1}^m ||b_j||_{\dot{\Lambda}_\beta}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leqslant j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{\dot{\Lambda}_\beta} = ||b_{\sigma(1)}||_{\dot{\Lambda}_\beta} \cdots ||b_{\sigma(j)}||_{\dot{\Lambda}_\beta}$.

Definition 1. Let $0 < p, q \leqslant \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leqslant 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,\ p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,\ p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,\ p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,\ p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,\ p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,\ p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 2. Let $1 \leqslant q < \infty$, $\alpha \in R$. The central Campanato space is defined by

$$CL_{\alpha,\ q}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{CL_{\alpha,\ q}} < \infty\},$$

where

$$||f||_{CL_{\alpha,\ q}} = \sup_{r>0} |B(0,r)|^{-\alpha} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx\right)^{1/q}.$$

Definition 3. Suppose $b_j's$ are the fixed locally integral functions on R^n and $m \in N, (j = 1, \cdots, m)$. The maximal operator $B^{\vec{b}}_{\delta,*}$ associated with the multilinear commutator generated by the Bochner-Riesz operator is defined by

$$B^{\vec{b}}_{\delta,*}(f)(x) = \sup_{t>0} |B^{\vec{b}}_{\delta,t}(f)(x)|,$$

where

$$B^{\vec{b}}_{\delta,t}(f)(x) = \int_{R^n} B^{\delta}_t(x-y)f(y)\prod_{j=1}^{m}(b_j(x)-b_j(y))dy,$$

$B^{\delta}_t(x) = t^{-n}B^{\delta}(x/t)$ and $(B^{\delta}_t(f))\hat{}(\xi) = (1-t^2|\xi|^2)^{\delta}_{+}\hat{f}(\xi)$. We also define

$$B^{\delta}_*(f)(x) = \sup_{t>0} |B^{\delta}_t(f)(x)| = \sup_{t>0} \left| \int_{R^n} B^{\delta}_t(x-y)f(y)dy \right|,$$

which is the Bochner-Riesz operator([8]).

Let H be the space $H = \{h : ||h|| = \sup_{t>0}|h(t)| < \infty\}$, then,$B^{\vec{b}}_{\delta,t}(f)(x)$ may be viewed as a mapping from R^n to H, and it is clear that

$$B^{\delta}_*(f)(x) = ||B^{\delta}_t(f)(x)||$$

and

$$B^{\vec{b}}_{\delta,*}(f)(x) = ||B^{\vec{b}}_{\delta,t}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $B^{\vec{b}}_{\delta,*}$ is just the commutator of order m. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors. Our main purpose is to study the boundedness properties for the maximal multilinear commutator on Besov spaces.

Now we state our theorems as following.

Theorem 1. Let $\delta > (n-1)/2$, $0 < \beta < 1/m$ and $b_j \in \dot{\Lambda}_{\beta}(R^n)$ for $j = 1, \cdots, m$. Then $B^{\vec{b}}_{\delta,*}$ is bounded from $L^p(R^n)$ to $\dot{\Lambda}_{m\beta-n/p}(R^n)$ for any $n/(m\beta+\delta) \le p \le n/\delta$.

Theorem 2. Let $\delta > (n-1)/2$, $0 < \beta < 1/m$, $1 < q_1 < n/m\beta$, $1/q_2 = 1/q_1 - m\beta/n$, $-n/q_2 - 1 < \alpha \le -n/q_2$ and $b_j \in \dot{\Lambda}_{\beta}(R^n)$ for $j = 1, \cdots, m$. Then $B^{\vec{b}}_{\delta,*}$ is bounded from $\dot{K}^{\alpha,\infty}_{q_1}(R^n)$ to $CL_{-\alpha/n-1/q_2,q_2}(R^n)$.

Remark. Theorem 2 also hold for the nonhomogeneous Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1.(see [9]) Let $\delta > (n-1)/2$ and $1 < p < \infty$. Then B^{δ}_* is bounded on $L^p(R^n)$.

Lemma 2.(see [18]) For $0<\beta<1, 1\leqslant p\leqslant\infty$, we have

$$||b||_{\dot\wedge_\beta}\approx\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|b(x)-b_Q|dx\approx\sup_Q\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q|b(x)-b_Q|^pdx\right)^{1/p}$$
$$\approx\sup_Q\inf_c\frac{1}{|Q|^{1+\beta/n}}\int_Q|b(x)-c|dx\approx\sup_Q\inf_c\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q|b(x)-c|^pdx\right)^{1/p}.$$

Lemma 3.(see [17]) For $\alpha<0, 0<q<\infty$, we have

$$||f||_{\dot K_q^{\alpha,\infty}}\approx\sup_{\mu\in Z}2^{\mu\alpha}||f\chi_{B_\mu}||_{L^q}.$$

Lemma 4. Let $0<\eta<n$, $1<p<n/\eta$. Suppose $b\in\dot\wedge_\beta(R^n)$, then

$$|b_{2^{k+1}B}-b_B|\leq C||b||_{\dot\wedge_\beta}k|2^{k+1}B|^{\beta/n}\text{ for }k\geq1\ .$$

Proof.

$$\begin{aligned}|b_{2^{k+1}B}-b_B| &\leq \sum_{j=0}^{k}|b_{2^{j+1}B}-b_{2^jB}|\\ &\leq \sum_{j=0}^{k}\frac{1}{|2^jB|}\int_{2^jB}|b(y)-b_{2^{j+1}B}|dy\\ &\leq C\sum_{j=0}^{k}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|^pdy\right)^{1/p}\\ &\leq C||b||_{\dot\wedge_\beta}\sum_{j=0}^{k}|2^{j+1}B|^{\beta/n}\\ &\leq C||b||_{\dot\wedge_\beta}(k+1)|2^{k+1}B|^{\beta/n}\\ &\leq C||b||_{\dot\wedge_\beta}k|2^{k+1}B|^{\beta/n}.\end{aligned}$$

Lemma 5.(see [6]) Let $0<\eta<1$, $1<r<n/\eta$, $1/r-1/s=\eta/n$ and $b_j\in\dot\wedge_\beta(R^n)$ for $j=1,\cdots,m$. Then $B^{\vec b}_{\delta,*}$ is bounded from $L^r(R^n)$ to $L^s(R^n)$.

Proof of Theorem 1. It is only to prove that there exists a constant C_0 such that

$$\frac{1}{|B|^{1+m\beta/n-1/p}}\int_B|B^{\vec b}_{\delta,*}(f)(x)-C_0|dx\leq C||f||_{L^p}.$$

Fix a ball B, $B=B(x_0,r)$, we decompose f into $f=f_1+f_2$ with $f_1=f\chi_B,\ f_2=f\chi_{(R^n\backslash B)}$.

When $m=1$, for $C_0=B^\delta_*(((b_1)_B-b_1)f_2)(x_0)$, we have

$$B^{b_1}_{\delta,t}(f)(x)=(b_1(x)-(b_1)_B)B^\delta_t(f)(x)-B^\delta_t((b_1-(b_1)_B)f_1)(x)-B^\delta_t((b_1-(b_1)_B)f_2)(x).$$

Then

$$\begin{aligned}
& |B^{b_1}_{\delta,*}(f)(x) - B^\delta_*(((b_1)_B - b_1)f_2)(x_0)| \\
= & \left| ||B^{b_1}_{\delta,t}(f)(x)|| - ||B^\delta_t(((b_1)_B - b_1)f_2)(x_0)|| \right| \\
\le & ||B^{b_1}_{\delta,t}(f)(x) - B^\delta_t(((b_1)_B - b_1)f_2)(x_0)|| \\
\le & ||(b_1(x) - (b_1)_B)B^\delta_t(f)(x)|| + ||B^\delta_t((b_1 - (b_1)_B)f_1)(x)|| \\
& + ||B^\delta_t((b_1 - (b_1)_B)f_2)(x) - B^\delta_t((b_1 - (b_1)_B)f_2)(x_0)|| \\
= & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by the L^p-boundedness of B^δ_* with $1 < p < \infty$, we obtain, using Hölder's inequality with $1/p' + 1/p = 1$,

$$\begin{aligned}
& \frac{1}{|B|^{1+\beta/n-1/p}} \int_B |A(x)|dx \\
= & \frac{1}{|B|^{1+\beta/n-1/p}} \int_B |(b_1(x) - (b_1)_B)B^\delta_*(f)(x)|dx \\
\le & C\frac{1}{|B|^{1+\beta/n-1/p}} \left(\int_B |(b_1(x) - (b_1)_B)|^{p'} dx \right)^{1/p'} \left(\int_B |B^\delta_*(f)(x)|^p dx \right)^{1/p} \\
\le & C\frac{|B|^{\beta/n+1/p'}}{|B|^{1+\beta/n-1/p}} \frac{1}{|B|^{\beta/n}} \left(\frac{1}{|B|} \int_B |(b_1(x) - (b_1)_B)|^{p'} dx \right)^{1/p'} \left(\int_B |(f)(x)|^p dx \right)^{1/p} \\
\le & C\frac{|B|^{\beta/n+1/p'}}{|B|^{1+\beta/n-1/p}} ||b_1||_{\dot\Lambda_\beta} ||f||_{L^p} \\
\le & C||b_1||_{\dot\Lambda_\beta} ||f||_{L^p}.
\end{aligned}$$

For $B(x)$, taking $1 < r < p < \infty$ and $p = rt$, by Hölder's inequality, we have

$$\begin{aligned}
& \frac{1}{|B|^{1+\beta/n-1/p}} \int_B |B(x)|dx \\
= & \frac{1}{|B|^{1+\beta/n-1/p}} \int_B |B^\delta_*((b_1)_B - (b_1)f_1)(x)|dx \\
\le & C\frac{1}{|B|^{\beta/n-1/p}} \left(\frac{1}{|B|} \int_{R^n} |B^\delta_*((b_1(x) - (b_1)_B)f)(x)\chi_B(x)|^r dx \right)^{1/r} \\
\le & C\frac{1}{|B|^{\beta/n-1/p+1/r}} \left(\int_B |(b_1(x) - (b_1)_B)f(x)|^r dx \right)^{1/r} \\
\le & C\frac{1}{|B|^{\beta/n-1/p+1/r}} \left(\int_B |b_1(x) - (b_1)_B|^{rt'} dx \right)^{1/rt'} \left(\int_B |f(x)|^{rt} dx \right)^{1/rt} \\
\le & C\frac{|B|^{\beta/n+1/rt'}}{|B|^{\beta/n-1/p+1/r}} \frac{1}{|B|^{\beta/n}} \left(\frac{1}{|B|} \int_B |b_1(x) - (b_1)_B|^{rt'} dx \right)^{1/rt'} \left(\int_B |f(x)|^{rt} dx \right)^{1/rt} \\
\le & C\frac{|B|^{\beta/n+1/rt'}}{|B|^{\beta/n-1/p+1/r}} ||b_1||_{\dot\Lambda_\beta} ||f||_{L^p}
\end{aligned}$$

$$\leq \quad C||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

For $C(x)$, we have, for $x \in Q$,

$$\begin{aligned} C(x) &= ||B_t^\delta((b_1-(b_1)_{2Q})f_2)(x) - B_t^\delta((b_1-(b_1)_{2Q})f_2)(x_0)|| \\ &= \sup_{t>0}\left|\int_{(2Q)^c}(b_1(y)-(b_1)_{2Q})f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy\right|. \end{aligned}$$

We consider the following two cases:

Case 1. $0 < t \leqslant d$. In this case, notice that(see [8])

$$|B_1^\delta(x)| \leqslant C(1+|x|)^{-(\delta+(n+1)/2)}.$$

Using the Minkowski's inequality, we obtain

$$\begin{aligned} &\left|\int_{(B)^c}(b_1(y)-(b_1)_B)f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy\right| \\ \leq\ & Ct^{-n}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_B||f(y)|(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy \\ \leq\ & C(t/d)^{\delta-(n-1)/2}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_1(y)-(b_1)_B||f(y)|dy\right) \\ \leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\frac{1}{|2^{k+1}B|}\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_B|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{2^{k+1}B}|f(y)|^p dy\right)^{1/p} \\ \leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\frac{1}{|2^{k+1}B|}\left[\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_{2^{k+1}B}|^{p'}dy\right)^{1/p'}+|(b_1)_{2^{k+1}B}-(b_1)_B||2^{k+1}B|^{1/p'}\right] \\ & \times\left(\int_{2^{k+1}B}|f(y)|^p dy\right)^{1/p} \\ \leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\frac{1}{|2^{k+1}B|}\left[|2^{k+1}B|^{\beta/n+1/p'}\frac{1}{|2^{k+1}B|^{\beta/n}}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_1(y)-(b_1)_{2^{k+1}B}|^{p'}dy\right)^{1/p'}\right. \\ & \left.+|(b_1)_{2^{k+1}B}-(b_1)_B||2^{k+1}B|^{1/p'}\right]\left(\int_{2^{k+1}B}|f(y)|^p dy\right)^{1/p} \\ \leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\frac{1}{|2^{k+1}B|}\left[|2^{k+1}B|^{\beta/n+1/p'}||b_1||_{\dot{\Lambda}_\beta}+k|2^{k+1}B|^{\beta/n+1/p'}||b_1||_{\dot{\Lambda}_\beta}\right]||f||_{L^p} \\ \leq\ & C\sum_{k=1}^\infty k2^{k((n-1)/2-\delta)}\frac{|2^{k+1}B|^{\beta/n+1/p'}}{|2^{k+1}B|}||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p} \\ \leq\ & C|B|^{\beta/n-1/p}||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}. \end{aligned}$$

Case 2. $t > d$. In this case, we choose δ_0 such that $(n-1)/2 < \delta_0 < \min(\delta, (n+1)/2)$, notice that (see [8])

$$|(\partial/\partial x)B_1^\delta(x)| \leq C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
&\left|\int_{(B)^c}(b_1(y)-(b_1)_B)f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy\right| \\
\leq\ & Ct^{-n}\int_{(B)^c}|b_1(y)-(b_1)_B||f(y)||B^\delta((x-y)/t)-B^\delta((x_0-y)/t)|dy \\
\leq\ & Ct^{-n-1}\int_{(B)^c}|b_1(y)-(b_1)_B||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta+(n+1)/2)}dy \\
\leq\ & Ct^{-n-1}\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_B||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\
\leq\ & C(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta_0)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}QB}|b_1(y)-(b_1)_B||f(y)|dy\right) \\
\leq\ & C|B|^{\beta/n-1/p}||b_1||_{\dot\wedge_\beta}||f||_{L^p}.
\end{aligned}$$

Thus

$$\frac{1}{|B|^{1+\beta/n-1/p}}\int_B|C(x)|dx\leq C||b_1||_{\dot\wedge_\beta}||f||_{L^p}\frac{1}{|B|^{1+\beta/n-1/p}}\int_B|B|^{\beta/n-1/p}dx\leq C||b_1||_{\dot\wedge_\beta}||f||_{L^p}.$$

This completes the case $m=1$.

Now, we consider the **Case** $m\geq 2$. we have, for $b=(b_1,\cdots,b_m)$,

$$\begin{aligned}
&B_{\delta,t}^{\vec{b}}(f)(x)=\int_{R^n}\prod_{j=1}^m[(b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q})]B_t^\delta(x-y)f(y)dy \\
=\ & \sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}((b(y)-(b)_{2Q})_{\sigma^c}B_t^\delta(x-y)f(y)dy \\
=\ & \prod_{j=1}^m(b_j(x)-(b_j)_{2Q})B_t^\delta(f)(x)+(-1)^mB_t^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f)(x) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}B_t^\delta(x-y)f(y)dy \\
=\ & \prod_{j=1}^m(b_j(x)-(b_j)_{2Q})B_t^\delta(f)(x)+(-1)^mB_t^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f)(x) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma B_{\delta,t}^{\vec{b}_{\sigma^c}}(f)(x),
\end{aligned}$$

thus, set $C_0=B_*^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_2)(x_0)$,

$$|B_{\delta,*}^{\vec{b}}(f)(x)-B_*^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_2)(x_0)|$$

$$\begin{aligned}
&= \left| ||B^{\vec{b}}_{\delta,t}(f)(x)|| - ||B^{\delta}_{t}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_2)(x_0)|| \right| \\
&\le ||B^{\vec{b}}_{\delta,t}(f)(x) - B^{\delta}_{t}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_2)(x_0)|| \\
&\le ||\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})B^{\delta}_{t}(f)(x)|| \\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2Q})_\sigma B^{\vec{b}_{\sigma^c}}_{\delta,t}(f)(x)|| \\
&+ ||B^{\delta}_{t}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_1)(x)|| \\
&+ ||B^{\delta}_{t}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_2)(x) - B^{\delta}_{t}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_2)(x_0)|| \\
&= I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}$$

For $I_1(x)$, by the L^p-boundedness of B^{δ}_{*} with $1<p<\infty$, we obtain, using Hölder's inequality with $1/s_1+\cdots+1/s_m+1/p=1$,

$$\begin{aligned}
&\frac{1}{|B|^{1+m\beta/n-1/p}}\int_B |I_1(x)|dx \\
&= \frac{1}{|B|^{1+m\beta/n-1/p}}\int_B |(b_1(x)-(b_1)_B)\cdots(b_m(x)-(b_m)_B)B^{\delta}_{*}(f)(x)|dx \\
&\le C\frac{1}{|B|^{1+m\beta/n-1/p}}\prod_{j=1}^{m}\left(\int_B |(b_j(x)-(b_j)_B)|^{s_j}dx\right)^{1/s_j}\left(\int_B |B^{\delta}_{*}(f)(x)|^p dx\right)^{1/p} \\
&\le C\frac{|B|^{m\beta/n+1/s_1+\cdots s_m}}{|B|^{1+m\beta/n-1/p}}\prod_{j=1}^{m}\frac{1}{|B|^{m\beta/n}}\left(\frac{1}{|B|}\int_B |(b_1(x)-(b_1)_B)|^{s_j}dx\right)^{1/s_j}\left(\int_B |(f)(x)|^p dx\right)^{1/p} \\
&\le C\frac{|B|^{m\beta/n+1/s_1+\cdots s_m}}{|B|^{1+m\beta/n-1/p}}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p} \\
&\le C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}$$

For $I_2(x)$, by the L^p-boundedness of B^{δ}_{*} with $1<r<p<\infty$ and $p=rt$, we obtain, using Hölder's inequality with $1/r'+1/r=1$ and $1/t'+1/t=1$,

$$\begin{aligned}
&\frac{1}{|B|^{1+m\beta/n-1/p}}\int_B |I_2(x)|dx \\
&= \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|B|^{1+m\beta/n-1/p}}\int_B |(\vec{b}(x)-\vec{b}_B)_\sigma B^{\delta}_{*}((\vec{b}-\vec{b}_B)_{\sigma^c}f)(x)|dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|B|^{1+m\beta/n-1/p}}\left(\int_B|(\vec{b}(x)-\vec{b}_B)_\sigma|^{r'}dx\right)^{1/r'}\left(\int_B B_*^\delta((\vec{b}-\vec{b}_B)_{\sigma^c}f)(x)|^r dx\right)^{1/r}\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|B|^{|\sigma|\beta/n+1/r'}}{|B|^{1+m\beta/n-1/p}}\frac{1}{|B|^{|\sigma|\beta/n}}\left(\frac{1}{|B|}\int_B|(\vec{b}(x)-\vec{b}_B)_\sigma|^{r'}dx\right)^{1/r'}\\
&\quad\times\left(\int_B|(\vec{b}-\vec{b}_B)_{\sigma^c}f(x)|^r d\mu(x)\right)^{1/r}\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|B|^{|\sigma|\beta/n+1/r'}}{|B|^{1+m\beta/n-1/p}}\frac{1}{|B|^{|\sigma|\beta/n}}\left(\frac{1}{|B|}\int_B|(\vec{b}(x)-\vec{b}_B)_\sigma|^{r'}dx\right)^{1/r'}\\
&\quad\times\left(\int_B|(\vec{b}(x)-\vec{b}_B)_{\sigma^c}|^{rt'}dx\right)^{1/rt'}\left(\int_B|f(x)|^{rt}dx\right)^{1/rt}\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|B|^{|\sigma|\beta/n+1/r'}}{|B|^{1+m\beta/n-1/p}}\frac{1}{|B|^{|\sigma|\beta/n}}\left(\frac{1}{|B|}\int_B|(\vec{b}(x)-\vec{b}_B)_\sigma|^{r'}dx\right)^{1/r'}\\
&\quad\times|B|^{|\sigma^c|\beta/n+1/rt'}\frac{1}{|B|^{|\sigma^c|\beta/n}}\left(\frac{1}{|B|}\int_B|(\vec{b}(x)-\vec{b}_B)_{\sigma^c}|^{rt'}dx\right)^{1/rt'}||f||_{L^p}\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{\mu(B)^{m\beta/n+1/r'+1/rt'}}{|B|^{1+m\beta/n-1/p}}||\vec{b}_\sigma||_{\dot{\Lambda}_\beta}||\vec{b}_{\sigma^c}||_{\dot{\Lambda}_\beta}||f||_{L^p}\\
&\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}
$$

For $I_3(x)$, taking $1<r<p<\infty$, $p=rt$ and $1/t_1+\cdots+1/t_m+1/t=1$, by Hölder's inequality, we have

$$
\begin{aligned}
&\frac{1}{|B|^{1+m\beta/n-1/p}}\int_B|I_3(x)|dx\\
&=\frac{1}{|B|^{1+m\beta/n-1/p}}\int_B|B_*^\delta((b_1-(b_1)_B)\cdots(b_m-(b_m)_B)f_1)(x)|dx\\
&\leq\frac{1}{|B|^{m\beta/n-1/p}}\left(\frac{1}{|B|}\int_{R^n}|B_*^\delta((b_1-(b_1)_B)\cdots(b_m-(b_m)_B)f)(x)\chi_B(x)|^r dx\right)^{1/r}\\
&\leq\frac{1}{|B|^{m\beta/n-1/p+1/r}}\left(\int_B|((b_1-(b_1)_B)\cdots(b_m-(b_m)_B))f(x)|^r dx\right)^{1/r}\\
&\leq\frac{1}{|B|^{m\beta/n-1/p+1/r}}\prod_{j=1}^m\left(\int_B|b_j(x)-(b_j)_B|^{rt_j}dx\right)^{1/rt_j}\left(\int_B|f(x)|^{rt}dx\right)^{1/rt}\\
&\leq\frac{|B|^{m\beta/n+1/rt_1+\cdots 1/rt_m}}{|B|^{m\beta/n-1/p+1/r}}\prod_{j=1}^m\frac{1}{|B|^{m\beta/n}}\left(\frac{1}{|B|}\int_B|b_j(x)-(b_j)_B|^{rt_j}d\mu(x)\right)^{1/rt_j}||f||_{L^p}\\
&\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}
$$

For $I_4(x)$, similar to $C(x)$, set $1/r_1+\cdots 1/r_m+1/p=1$, we have

$$\begin{aligned} I_4(x) &= ||B_t^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_2)(x)-B_t^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_2)(x_0)|| \\ &= \sup_{t>0}|\int_{(2Q)^c}\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy|. \end{aligned}$$

We consider the following two cases:
Case 1. $0<t\leqslant d$. In this case, notice that

$$|B_1^\delta(x)|\leqslant C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned} & |\int_{(B)^c}\prod_{j=1}^m(b_j(y)-(b_j)_B)f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy| \\ \leq\ & Ct^{-n}\sum_{k=1}^\infty\int_{2^{k+1}B\backslash 2^kB}|\prod_{j=1}^m(b_j(y)-(b_j)_B)||f(y)|(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy \\ \leq\ & C(t/d)^{\delta-(n-1)/2}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|\prod_{j=1}^m(b_j(y)-(b_j)_B)||f(y)|dy\right) \\ \leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\frac{1}{|2^{k+1}B|}\prod_{j=1}^m\left(\int_{2^{k+1}B}|b_j(y)-(b_j)_B|^{r_j}dy\right)^{1/r_j}\left(\int_{2^{k+1}B}|f(y)^p dy\right)^{1/p} \\ \leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\frac{|2^{k+1}B|^{m\beta/n+1/r_1+\cdots 1/r_m}}{|2^{k+1}B|}\prod_{j=1}^m\frac{1}{|2^{k+1}B|^{m\beta/n}}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_j(y)-(b_j)_B|^{r_j}dy\right)^{1/r_j} \\ & \times||f||_{L^P} \\ \leq\ & C\sum_{k=1}^\infty k^m 2^{k((n-1)/2-\delta)}\frac{|2^{k+1}B|^{m\beta/n+1-1/p}}{|2^{k+1}B|}||\vec{b}||_{\dot\wedge_\beta}||f||_{L^p} \\ \leq\ & C|B|^{m\beta/n-1/p}||\vec{b}||_{\dot\wedge_\beta}||f||_{L^p}. \end{aligned}$$

Case 2. $t>d$. In this case, we choose δ_0 such that $(n-1)/2<\delta_0<\min(\delta,(n+1)/2)$, notice that

$$|(\partial/\partial x)B_1^\delta(x)|\leq C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned} & |\int_{(B)^c}[\prod_{j=1}^m(b_j(y)-(b_j)_B)]f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy| \\ \leq\ & Ct^{-n}\int_{(B)^c}|\prod_{j=1}^m(b_j(y)-(b_j)_B)||f(y)||B^\delta((x-y)/t)-B^\delta((x_0-y)/t)|dy \end{aligned}$$

$$
\begin{aligned}
&\leq Ct^{-n-1}\int_{(B)^c}|\prod_{j=1}^m(b_j(y)-(b_j)_B)||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta+(n+1)/2)}dy \\
&\leq Ct^{-n-1}\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}|\prod_{j=1}^m(b_j(y)-(b_j)_B)||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\
&\leq C(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta_0)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|\prod_{j=1}^m(b_j(y)-(b_j)_B)||f(y)|dy\right) \\
&\leq C|B|^{m\beta/n-1/p}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}
$$

Thus

$$\frac{1}{|B|^{1+m\beta/n-1/p}}\int_B|I_4(x)|dx\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Fix a ball $B=B(0,l)$, there exists $\epsilon_0\in\mathbf{Z}$ such that $2^{\epsilon_0-1}\leq l<2^{\epsilon_0}$. We choose x_0 such that $2l<|x_0|<3l$. It is only to prove that

$$2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|B^{\vec{b}}_{\delta,*}(f)(x)-B^{\vec{b}}_{\delta,*}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\leq C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

We write, for $f_1=f\chi_{4B_{\epsilon_0}}$ and $f_2=f\chi_{R^n\setminus 4B_{\epsilon_0}}$, then

$$|B^{\vec{b}}_{\delta,*}(f)(x)-B^{\vec{b}}_{\delta,*}(f_2)(x_0)|\leq|B^{\vec{b}}_{\delta,*}(f_1)(x)|+|B^{\vec{b}}_{\delta,*}(f_2)(x)-B^{\vec{b}}_{\delta,*}(f_2)(x_0)|,$$

thus

$$
\begin{aligned}
&2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|B^{\vec{b}}_{\delta,*}(f)(x)-B^{\vec{b}}_{\delta,*}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2} \\
\leq\ &2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|B^{\vec{b}}_{\delta,*}(f_1)(x)|^{q_2}dx\right)^{1/q_2} \\
&+2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|B^{\vec{b}}_{\delta,*}(f_2)(x)-B^{\vec{b}}_{\delta,*}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2} \\
=\ &J_1+J_2.
\end{aligned}
$$

For J_1, by the (L^{q_1},L^{q_2})-boundedness of $B^{\vec{b}}_{\delta,*}$(see Lemma 5) and Lemma 3, we get

$$
\begin{aligned}
J_1 &\leq C2^{\epsilon_0(\alpha+n/q_2)}2^{-\epsilon_0 n/q_2}\left(\int_{R^n}|f_1(x)|^{q_1}dx\right)^{1/q_1} \\
&\leq C2^{\epsilon_0\alpha}||f\chi_{B_{\epsilon_0}}||_{L^{q_1}} \\
&\leq C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.
\end{aligned}
$$

For J_2, similar to the estimates of Theorem 1, set $1/v_1+\cdots+1/v_m+1/q_1=1$, we obtain, by Hölder's inequality and recall that $-1/q_2<\alpha$ and $1/q_2=1/q_1-m\beta/n$,

We consider the following two cases:
Case 1. $0 < t \leqslant d$. In this case, notice that

$$|B_1^\delta(x)| \leqslant C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
&|B_{\delta,*}^{\vec{b}}(f_2)(x) - B_{\delta,*}^{\vec{b}}(f_2)(x_0)| \\
\leq\ & Ct^{-n}\sum_{k=1}^{\infty}\int_{C_{\epsilon_0+k}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)\right||f(y)|(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy \\
\leq\ & C(t/d)^{\delta-(n-1)/2}\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\left(\frac{1}{|B_{\epsilon_0+k}|}\int_{B_{\epsilon_0+k}}|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)||f(y)|dy\right) \\
\leq\ & C\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\frac{1}{|B_{\epsilon_0+k}|}\int_{B_{\epsilon_0+k}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)\right||f(y)|dy \\
\leq\ & C\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\frac{1}{|B_{\epsilon_0+k}|}\prod_{j=1}^{m}\left(\int_{B_{\epsilon_0+k}}|b_j(y)-(b_j)_B|^{v_j}dy\right)^{1/v_j}\left(\int_{B_{\epsilon_0+k}}|f(y)^{q_1}dy\right)^{1/q_1} \\
\leq\ & C\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\frac{|B_{\epsilon_0+k}|^{m\beta/n+1/v_1+\cdots 1/v_m}}{|B_{\epsilon_0+k}|} \\
& \prod_{j=1}^{m}\frac{1}{|B_{\epsilon_0+k}|^{m\beta/n}}\left(\frac{1}{|B_{\epsilon_0+k}|}\int_{B_{\epsilon_0+k}}|b_j(y)-(b_j)_B|^{v_j}dy\right)^{1/v_j}\times||f\chi_{\epsilon_0+k}||_{L^{q_1}} \\
\leq\ & C\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}|B_{\epsilon_0+k}|^{m\beta/n-1/q_1}||\vec{b}||_{\dot{\Lambda}_\beta}||f\chi_{\epsilon_0+k}||_{L^{q_1}} \\
\leq\ & C\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}2^{(\epsilon_0+k)[(m\beta-n/q_1)-\alpha]}||\vec{b}||_{\dot{\Lambda}_\beta}2^{(\epsilon_0+k)\alpha}||f\chi_{\epsilon_0+k}||_{L^{q_1}} \\
\leq\ & C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}_{q_1}^{\alpha,\infty}}.
\end{aligned}$$

Case 2. $t > d$. In this case, we choose δ_0 such that $(n-1)/2 < \delta_0 < \min(\delta, (n+1)/2)$, notice that

$$|(\partial/\partial x)B_1^\delta(x)| \leq C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
&|B_{\delta,*}^{\vec{b}}(f_2)(x) - B_{\delta,*}^{\vec{b}}(f_2)(x_0)| \\
\leq\ & Ct^{-n-1}\sum_{k=1}^{\infty}\int_{C_{\epsilon_0+k}}|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy
\end{aligned}$$

$$\leq \quad C(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta_0)}\left(\frac{1}{|B_{\epsilon_0+k}|}\int_{B_{\epsilon_0+k}}|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)||f(y)|dy\right)$$
$$\leq \quad C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

thus

$$J_2 \leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

This completes the proof of Theorem 2.

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