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DRAGOMIR'S, BUZANO'S AND KERUPA'S INEQUALITIES IN HILBERT C^*-MODULES

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Abstract. In this paper we prove a type of Dragomir's, Buzano's and Kurepa's inequalities in Hilbert C*-modules. Some applications for discrete and integral inequalities improving the Cauchy-Schwartz result are given.

1. Introduction

In ([1]), M.L. Buzano obtained the following extension of the Cauchy-Schwartz's inequality in a real or complex Hilbert space $(H, (.,.))$

$$|(a,x)(x,b)| \leq \frac{1}{2}[||a||\,||b|| + |(a,b)|]\,||x||^2\,, \tag{1.1}$$

for any $a, b, x \in H$. It is clear that for $a = b$, the above inequality becomes the standard Cauchy-Schwartz's inequality

$$|(a,x)|^2 \leq ||a||^2\,||x||^2 : \forall a, x \in H, \tag{1.2}$$

with equality if and only if there exists a scalar $\lambda \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) such that $x = \lambda a$.

It might be useful to observe that, out of (1.1) one may get the following discrete inequality

$$\left|\sum_{i=1}^{n} p_i a_i \bar{x}_i \sum_{i=1}^{n} p_i x_i \bar{b}_i\right| \leq \frac{1}{2}[(\sum_{i=1}^{n} p_i\,|a_i|^2 \sum_{i=1}^{n} p_i\,|b_i|^2)^{\frac{1}{2}} + |\sum_{i=1}^{n} p_i a_i \bar{b}_i|] \sum_{i=1}^{n} p_i\,|x_i|^2\,, \tag{1.3}$$

where $p_i \geq 0$, $a_i, b_i, x_i \in \mathbb{C}$, $i \in \{1, ..., n\}$.

If one takes in (**??**) $b_i = \bar{a}_i$ for $i \in \{1, ..., n\}$, then one obtains

$$|\sum_{i=1}^{n} p_i a_i \bar{x}_i \sum_{i=1}^{n} p_i a_i x_i| \leq \frac{1}{2}[\sum_{i=1}^{n} p_i\,|a_i|^2 + |\sum_{i=1}^{n} p_i a_i^2|] \sum_{i=1}^{n} p_i\,|x_i|^2\,, \tag{1.4}$$

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for any $p_i \geq 0$, $a_i, x_i \in \mathbb{C}$, $i \in \{1, ..., n\}$.

Note that, if $x_i, i \in \{1, ..., n\}$ are real numbers, then out of (1.4), we may deduce the Buijn's inequality ([2])

$$|\sum_{i=1}^{n} p_i x_i z_i|^2 \leq \frac{1}{2}[\sum_{i=1}^{n} p_i \, |z_i|^2 + |\sum_{i=1}^{n} p_i z_i^2|] \sum_{i=1}^{n} p_i x_i^2, \tag{1.5}$$

where $z_i \in \mathbb{C}$, $i \in \{1, ..., n\}$. In this way, Buzano's inequality (1.1) may be regarded as a generalization of the Buijn's inequality.

In ([4]) S.S, Dragomir established the following refinement of Busano's inequality

$$|\frac{(a,x)(x,b)}{||x||^2} - \frac{(a,b)}{\alpha}| \leq \frac{||b||}{|\alpha| \, ||x||}[|\alpha - 1|^2 |(a,x)|^2 + ||x||^2 \, ||a||^2 - |(a,x)|^2] \tag{1.6}$$

where $a, b, x \in H$, $x \neq 0$ and $\alpha \in \mathbb{K} - \{0\}$.

The case of equality holds in (1.6) if and only if there exists a scalar $\lambda \in \mathbb{K}$ such that

$$\alpha \frac{\langle a, x \rangle}{||x||^2} x = a + \lambda b. \tag{1.7}$$

The goal of this paper is to show some related as well as a extension of Buzano's and Dragomir's inequality (1.1) and (1.6) to Hilbert C^*-module. We can obtain various particular inequalities in Hilbert C^*-module. In section 3, we are given a extension of Kurepa's and Dragomir's inequality to Hilbert C^*-module, the corresponding applications for discrete and integral inequalities are also provided.

2. Preliminaries in Hilbert C^*-modules

In this section we briefly recall the definitions and examples of Hilbert C^*-modules. For information about Hilbert C^*-module, we refer to ([5,6,7,9,11]). Our references for C^*-algebras are ([3,13]).

Let $\mathbb{A}$ be a C^*-algebra (not necessarily unitary) and $\mathcal{H}$ be a complex linear space.

Definition 2.1. A pre-Hilbert $\mathbb{A}$-module is a right $\mathbb{A}$-module $\mathcal{H}$ equipped with a sesquilinear map $(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{A}$ satisfying

(i) $(x, x) \geq 0; (x, x) = 0$ if and only if $x = 0$ for all x in $\mathcal{H}$,

(ii) $(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z)$ for all x, y, z in $\mathcal{H}, \alpha, \beta$ in $\mathbb{C}$,

(iii) $(x, y) = (y, x)^*$ for all x, y in $\mathcal{H}$,

(iv) $(x, y.a) = (x, y)a$ for all x, y in $\mathcal{H}$, a in $\mathbb{A}$.

The map $(.,.)$ is called an $\mathbb{A}$-valued inner product of $\mathcal{H}$, and for $x \in \mathcal{H}$, we define $||x|| = ||(x, x)||^{\frac{1}{2}}$.

Proposition 2.1. *Let $\mathcal{H}$ be a pre-Hilbert $\mathbb{A}$-module, then*

(i) $||.||$ *is a norm on* $\mathcal{H}$,

(ii) $||x.a|| \leq ||x||\,||a||$ *for all* $x \in \mathcal{H}$, $a \in \mathbb{A}$,

(iii) $(x, y)(y, x) \leq ||y||^2\,(x, x)$ *for all* $x, y \in \mathcal{H}$,

(iv) $||(x, y)|| \leq ||x||\,||y||$ *for all* $x, y \in \mathcal{H}$.

It is clair that (iii) and (iv) are a generalization of Cauchy-Schwartz's inequality to a pre-Hilbert $\mathbb{A}$-module. The equality holds in (iv) if there exists $\lambda \in \mathbb{C}$ so that $y = \lambda x$.

Definition 2.2. The completion of a pre-Hilbert -module with respect to the norm induced by the $\mathbb{A}$-valued inner product is called a Hilbert $\mathbb{A}$-module.

Example 2.1.

1. Let $\mathbb{A}$ be a C^*-algebra. $\mathbb{A}$ is Hilbert $\mathbb{A}$-module if an $\mathbb{A}$-valued inner product is defined as $(x, y) = x^*y$ for all $x, y \in \mathbb{A}$. Any closed right ideal of $\mathbb{A}$ is sub-$\mathbb{A}$-module under the above $\mathbb{A}$-valued inner product.

2. Let $\{\mathcal{H}_i : i \in I\}$ be a finite family of Hilbert $\mathbb{A}$-modules. Then $\oplus_{i\in I}\mathcal{H}_i$ is a Hilbert $\mathbb{A}$-module with its $\mathbb{A}$-valued inner product is defined as

$$((x_i), (y_i)) = \sum_{i\in I}(x_i, y_i).$$

When $\{\mathcal{H}_i : i \in I\}$ is an infinite family of Hilbert $\mathbb{A}$-modules we define

$$\oplus_{i\in I}\mathcal{H}_i = \{(x_i)_{i\in I} : \sum_i (x_i, x_i) \text{ converges in norm in } \mathbb{A}\}.$$

Thus $\oplus_{i\in I}\mathcal{H}_i$ is a Hilbert $\mathbb{A}$-module with its $\mathbb{A}$-valued inner product is defined as

$$((x_i), (y_i)) = \sum_{i\in I}(x_i, y_i).$$

3. Let $\mathbb{A}$ be a C^*-algebra and n be a integer ≥ 1, then $\mathbb{A}^n \cong \oplus_{i=1}^n\mathbb{A}$ is a Hilbert $\mathbb{A}$-module with its $\mathbb{A}$-valued inner product is defined as

$$(a, b) = \sum_{i=1}^n a_i^* b_i \quad \text{for all } a = (a_1, ..., a_n),\ b = (b_1, ..., b_n) \in \mathbb{A}^n.$$

4. Let $\mathbb{A}$ be a C^*-algebra. Denote by $l^2(\mathbb{A})$ the linear space of all sequences $z = (z_n)_{n\geq 1}$ of elements $\mathbb{A}$ so that

$$\sum_{n=1}^{+\infty} ||z_n||^2 \langle +\infty.$$

Then $l^2(\mathbb{A})$ is a Hilbert $\mathbb{A}$-module with its $\mathbb{A}$-valued inner product is defined as

$$(z, z') = \sum_{n=1}^{+\infty} z_n^* z_n' \quad \text{for all } z = (z_n)_{n\geq 1}, z' = (z_n')_{n\geq 1} \in l^2(\mathbb{A}).$$

5. Let $\mathbb{A}$ be a unitary C^*-algebra with unit e, and $(\rho_n)_{n\geq 1}$ be a sequence of positives reals numbers so that
$$\sum_{n=1}^{+\infty} \rho_n = 1.$$
Denote by $l^2_\rho(\mathbb{A})$ the linear space of all sequences $z = (z_n)_{n\geq 1}$ of elements $\mathbb{A}$ so that
$$\sum_{n=1}^{+\infty} \rho_n ||z_n||^2 \langle +\infty.$$
Then $l^2_\rho(\mathbb{A})$ is a Hilbert $\mathbb{A}$-module with its $\mathbb{A}$-valued inner product is defined as
$$(z, z') = \sum_{n=1}^{+\infty} \rho_n z_n^* z'_n \quad \text{for all } z = (z_n)_{n\geq 1}, z' = (z'_n)_{n\geq 1} \in l^2_\rho(\mathbb{A}).$$

6. Let $\mathbb{A}$ be a unitary commutative C^*-algebra with unit e, (S, Σ, μ) be a positive measure space and φ be a $\mathbb{A}$-valued function $S \longrightarrow \mathbb{A}$ so that $\varphi(t)$ is hermitian positive in $\mathbb{A}$ for which $t \in S$, and
$$\int_S ||\varphi(t)||^2 d\mu(t) \langle +\infty \quad \text{and} \quad \int_S \varphi(t) d\mu(t) = e.$$
Denote by $\mathbb{L}^2_\varphi(S, \Sigma, \mu, \mathbb{A})$ the linear space of all $\mathbb{A}$-valued functions $f : S \longrightarrow \mathbb{A}$ such that
$$\int_S ||\varphi(t)|| ||f(t)||^2 d\mu(t) \langle +\infty.$$
Then $\mathbb{L}^2_\varphi(S, \Sigma, \mu, \mathbb{A})$ is a Hilbert $\mathbb{A}$-module with its $\mathbb{A}$-valued inner product is defined as
$$(f, g)_\varphi = \int_S \varphi(t) f(t)^* g(t) d\mu(t) \quad \text{for all} \quad f, g \in \mathbb{L}^2_\varphi(S, \Sigma, \mu, \mathbb{A}).$$

3. Buzano's and Dragomir's inequality in Hilbert C^*-modules

We let $\mathbb{A}$ be a unitary C^*-algebra with unit e and $\mathcal{H}$ be a Hilbert C^*-module over $\mathbb{A}$. The following results may be stated. It is a generalization of Dragomir's result (1.6).

Theorem 3.1. *For all $x, y, z \in \mathcal{H}$ so that (x, x) is invertible in $\mathbb{A}$ and for each invertible $a \in \mathbb{A}$, one has the inequality*

$$\begin{aligned} &||(y, x)(x, x)^{-1}(x, z) - a^{-1}(y, z)|| \\ \leq\ & \frac{||z||}{||a||} ||(a - e)(y, x)(x, x)^{-1}(x, y)(a^* - e) + (y, y) - (y, x)(x, x)^{-1}(x, y)||^{\frac{1}{2}} . \end{aligned} \tag{3.1}$$

The case of equality holds in (**??**) *if there exists $\lambda \in \mathbb{C}$ such that*

$$x.(x, x)^{-1}(y, x)a^* = y + \lambda z.a^*. \tag{3.2}$$

Proof. Using Cauchy-Schwartz's inequality (Proposition 2(iii)), we have that

$$\begin{aligned} &(x.(x,x)^{-1}(x,y) - y.(a^{-1})^*, z)(z, x.(x,x)^{-1}(x,y) - y.(a^{-1})^*) \\ \leq\ & ||z||^2 (x.(x,x)^{-1}(x,y) - y.(a^{-1})^*, x.(x,x)^{-1}(x,y) - y.(a^{-1})^*), \end{aligned} \tag{3.3}$$

and since

$$\begin{aligned} & (x.(x,x)^{-1}(x,y) - y.(a^{-1})^*, x.(x,x)^{-1}(x,y) - y.(a^{-1})^*) \\ =\ & (y,x)(x,x)^{-1}(x,x)(x,x)^{-1}(x,y) - (y,x)(x,x)^{-1}(x,y)(a^{-1})^* - a^{-1}(y,x)(x,x)^{-1}(x,y) \\ & +a^{-1}(y,y)(a^{-1})^* \\ =\ & a^{-1}[a(y,x)(x,x)^{-1}(x,y)a^* - a(y,x)(x,x)^{-1}(x,y) - (y,x)(x,x)^{-1}(x,y)a^* + (y,y)](a^{-1})^* \\ =\ & a^{-1}[(a-e)(y,x)(x,x)^{-1}(x,y)(a^*-e) + (y,y) - (y,x)(x,x)^{-1}(x,y)](a^{-1})^*, \end{aligned}$$

and

$$\begin{aligned} & (x.(x,x)^{-1}(x,y) - y.(a^{-1})^*, z)(z, x.(x,x)^{-1}(x,y) - y.(a^{-1})^*) \\ =\ & [(y,x)(x,x)^{-1}(x;z) - a^{-1}(y,z)][(y,x)(x,x)^{-1}(x;z) - a^{-1}(y,z)]^*. \end{aligned}$$

Using (3.3) we get that

$$\begin{aligned} & [(y,x)(x,x)^{-1}(x;z) - a^{-1}(y,z)][(y,x)(x,x)^{-1}(x;z) - a^{-1}(y,z)]^* \\ \leq\ & ||z||^2 a^{-1}[(a-e)(y,x)(x,x)^{-1}(x,y)(a^*-e) + (y,y) - (y,x)(x,x)^{-1}(x,y)](a^{-1})^*. \end{aligned}$$

Passing to the norm in $\mathbb{A}$, we have that

$$\begin{aligned} & ||(y,x)(x,x)^{-1}(x;z) - a^{-1}(y,z)||^2 \\ \leq\ & \frac{||z||^2}{||a||^2} ||(a-e)(y,x)(x,x)^{-1}(x,y)(a^*-e) + (y,y) - (y,x)(x,x)^{-1}(x,y)||. \end{aligned}$$

This proves (3.1).

The case of equality holds in (3.3) if there exists $\lambda \in \mathbb{C}$ so that $x.(x,x)^{-1}(x,y) - y.(a^{-1})^* = \lambda z$. Or equivalently

$$x.(x,x)^{-1}(x,y)a^* = y + \lambda z.a^*.$$

□

If $\mathbb{A}$ is a commutative C^*-algebra, we obtain the following result.

Corollary 3.1. *For all $x, y, z \in \mathcal{H}$ so that (x,x) is invertible in $\mathbb{A}$ and for each invertible $a \in \mathbb{A}$, one has the inequality*

$$\begin{aligned} & ||\frac{(y,x)(x,z)}{(x,x)} - \frac{(y,z)}{a}|| \\ \leq\ & \frac{||z||}{||a||||x||} \|(a-e)(a^*-e)(y,x)(x,y) + (x,x)(y,y) - (y,x)(x,y)\|^{\frac{1}{2}}, \end{aligned} \tag{3.4}$$

where $\frac{1}{a} = a^{-1}$ *is the inverse of* a *in* $\mathbb{A}$*. The case of equality holds in* (3.4) *if there exists* $\lambda \in \mathbb{C}$ *such that*

$$x.a\frac{(y,x)}{(x,x)} = y + \lambda z. \tag{3.5}$$

The following result also holds.

Proposition 3.1. *For all* $x, y, z \in \mathcal{H}$ *such that* (x,x) *is invertible in* $\mathbb{A}$ *and for each invertible* $a \in \mathbb{A}$*, one has the double inequality*

$$\begin{aligned} &||a^{-1}(y,z)|| - \frac{||z||}{||a||}||(a-e)(y,x)(x,x)^{-1}(x,y)(a^*-e) + (y,y) - \\ &\qquad (y,x)(x,x)^{-1}(x,y)||^{\frac{1}{2}} \\ &\leq ||(y,x)(x,x)^{-1}(x,z)|| \leq \\ &||a^{-1}(y,z)|| + \frac{||z||}{||a||}||(a-e)(y,x)(x,x)^{-1}(x,y)(a^*-e) + (y,y) - \\ &\qquad (y,x)(x,x)^{-1}(x,y)||^{\frac{1}{2}}. \end{aligned} \tag{3.6}$$

Proof. Using the continuity of C^*-norm in $\mathbb{A}$, we get that

$$\big|||(y,x)(x,x)^{-1}(x,z)|| - ||a^{-1}(y,z)||\big| \leq ||(y,x)(x,x)^{-1}(x,z) - a^{-1}(y,z)||.$$

Using (3.1) we deduce the double inequality (3.6). □

The following result is generalization of Buzano's inequality (1.1) to Hilbert C^*-module.

Corollary 3.2. *For all* $x, y, z \in \mathcal{H}$ *so that* (x,x) *is invertible in* $\mathbb{A}$*, we have the inequality*

$$\big|\big|(y,x)(x,x)^{-1}(x,z)\big|\big| \leq \frac{1}{2}[||y||\,||z|| + ||(y,z)||]. \tag{3.7}$$

Proof. In (3.6) we put $a = 2e$, then we get

$$||(y,x)(x,x)^{-1}(x,z)|| \leq \frac{1}{2}\,||y||\,||z|| + \frac{1}{2}\,||(y,z)||\,,$$

this proves (3.7). □

It is obvious that, out of (3.1) and (3.4), we can obtain various particular inequalities. A class of these which is

Corollary 3.3. *For all $x, y, z \in \mathcal{H}$ so that (x, x) is invertible in $\mathbb{A}$ and for each $\eta \in \mathbb{C}$ with $|\eta| = 1$ and $\mathrm{Re}(\eta) \neq -1$, we have the following inequality*

$$||(y,x)(x,x)^{-1}(x,z) - \frac{1}{1+\eta}(y,z)|| \leq \frac{1}{\sqrt{2}\sqrt{1+\mathrm{Re}(\eta)}}||y||||z||. \tag{3.8}$$

In particular for $\eta = 1$, we have

$$||(y,x)(x,x)^{-1}(x,z) - \frac{1}{2}(y,z)|| \leq \frac{1}{2}||y||||z||. \tag{3.9}$$

Proof. In Theorem 1, on choosing $a = (1+\eta)e$, we get that

$$||(y,x)(x,x)^{-1}(x,z) - \frac{1}{1+\eta}(y,z)|| \leq \frac{1}{|1+\eta|}||z||||(y,y)||^{\frac{1}{2}},$$

then

$$||(y,x)(x,x)^{-1}(x,z) - \frac{1}{1+\eta}(y,z)|| \leq \frac{1}{\sqrt{2}\sqrt{1+\mathrm{Re}(\eta)}}||y||||z||.$$

Finally, if $\eta = 1$ we have the inequality (3.9). $\square$

If $\mathbb{A}$ is a commutative C^*-algebra, we have the following result

Corollary 3.4. *For all $x, y, z \in \mathcal{H}$ so that (x, x) is invertible in $\mathbb{A}$ and for all unitary element $b \in \mathbb{A}$ so that $\mathrm{Re}(b) \neq -e$, we have the following inequality*

$$||\frac{(y,x)(x,z)}{(x,x)} - \frac{(y,z)}{e+b}|| \leq \frac{||y||||z||}{\sqrt{2}\sqrt{||e+\mathrm{Re}(b)||}}. \tag{3.10}$$

In particular, for $b = e$, we have the inequality

$$||\frac{(y,x)(x,z)}{(x,x)} - \frac{(y,z)}{2}|| \leq \frac{1}{2}||y||||z||. \tag{3.11}$$

Proof. Using Corollary 1, on choosing $a = e + b$, we get that

$$\begin{aligned}
& ||\frac{(y,x)(x,z)}{(x,x)} - \frac{(y,z)}{e+b}|| \leq \frac{||z||}{||e+b||||x||} \times ||bb^*(y,x)(x,y) + (x,x)(y,y) - (y,x)(x,y)||^{\frac{1}{2}} \\
\Longleftrightarrow \quad & ||\frac{(y,x)(x,z)}{(x,x)} - \frac{(y,z)}{e+b}|| \leq \frac{||z||}{||e+b||||x||}||(x,x)(y,y)||^{\frac{1}{2}} \\
\Longleftrightarrow \quad & ||\frac{(y,x)(x,z)}{(x,x)} - \frac{(y,z)}{e+b}|| \leq \frac{||z||||y||}{||e+b||}.
\end{aligned}$$

The inequality (3.10) result by using the fact that

$$||e+b||^2 = ||(e+b)(e+b^*)|| = ||2e + 2mathrmRe(b)|| = 2\,||e+\mathrm{Re}(b)||\,.$$

Finally, if $b = e$ we have the inequality (3.11). $\square$

Remark 3.1. Using the continuity of C^*-norm in $\mathbb{A}$, we get from (**??**) that

$$||(y,x)(x,x)^{-1}(x,z)|| \leq \frac{||z||||y|| + ||(y,z)||}{\sqrt{2}\sqrt{|1+\mathrm{Re}(\eta)|}}, \tag{3.12}$$

for each $\eta \in \mathbb{C}$ such that $|\eta| = 1$ and $\mathrm{Re}(\eta) \neq -1$.

Let n be a integer ≥ 1, $(\rho_i)_{1\leq i\leq n}$ be a finite family of positives reals numbers and $\mathcal{H} = \mathbb{A}^n \cong \oplus_{i=1}^n \mathbb{A}$ the Hilbert $\mathbb{A}$-module with the $\mathbb{A}$-valued inner product

$$(a,b) = \sum_{i=1}^n \rho_i a_i^* b_i \ \text{ for all } a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{A}^n.$$

It might be useful to observe that, out of (3.7), we obtain the following discrete inequality

$$||\sum_{i=1}^n \rho_i a_i^* x_i (\sum_{i=1}^n \rho_i x_i^* x_i)^{-1} \sum_{i=1}^n \rho_i x_i^* b_i|| \leq \frac{1}{2}[(\sum_{i=1}^n \rho_i ||a_i||^2 \sum_{i=1}^n \rho_i ||b_i||^2)^{\frac{1}{2}} + ||\sum_{i=1}^n \rho_i a_i b_i^*||], \tag{3.13}$$

where $a_i, b_i, x_i \in \mathbb{A}$, $i \in \{1, ..., n\}$ and $\sum\limits_{i=1}^n \rho_i x_i^* x_i$ is invertible in $\mathbb{A}$.

If one takes in (3.13) $b_i = a_i^*$ for $i \in \{1, ..., n\}$, then we obtain

$$||\sum_{i=1}^n \rho_i a_i x_i (\sum_{i=1}^n \rho_i x_i^* x_i)^{-1} \sum_{i=1}^n \rho_i x_i^* a_i|| \leq \sum_{i=1}^n \rho_i ||a_i||^2. \tag{3.14}$$

Note that, if x_i is hermitian for $i \in \{1, ..., n\}$ then, out of (3.13), we may deduce the following generalization of Bruijn's inequality (1.5) (see [2]).

$$||\sum_{i=1}^n \rho_i a_i x_i (\sum_{i=1}^n \rho_i x_i^2)^{-1} \sum_{i=1}^n \rho_i x_i a_i|| \leq \frac{1}{2}[(\sum_{i=1}^n \rho_i ||a_i||^2) + ||\sum_{i=1}^n \rho_i a_i^2||]. \tag{3.15}$$

We closed this section by noting the following result.

Corollary 3.5. *Let $\mathcal{H}$ be a Hilbert C^*-module over a unitary C^*-algebra $\mathbb{A}$. For all $x, y, z \in \mathcal{H}$ such that (x,x) is invertible in $\mathbb{A}$, we have the following double inequality*

$$\begin{aligned} ||(y,x)(x,x)^{-1}(x,z)|| &\leq ||(y,x)(x,x)^{-1}(x,z) - \frac{1}{2}(y,z)|| \ + \frac{1}{2}||(y,z)|| \\ &\leq \frac{1}{2}[||y||\,|z|\,| + ||(y,z)||]. \end{aligned} \tag{3.16}$$

Proof. Using (3.9) we get for all $x, y, z \in \mathcal{H}$ that

$$||(y,x)(x,x)^{-1}(x,z) - \frac{1}{2}(y,z)|| \leq \frac{1}{2}||y||||z||,$$

the continuity of C^*-norm in $\mathbb{A}$ implies that

$$\left| ||(y,x)(x,x)^{-1}(x,z)|| - \frac{1}{2}||(y,z)|| \right| \leq ||(y,x)(x,x)^{-1}(x,z) - \frac{1}{2}(y,z)||,$$

then

$$||(y,x)(x,x)^{-1}(x,z)|| \leq ||(y,x)(x,x)^{-1}(x,z) - \frac{1}{2}(x,x)(y,z)|| + \frac{1}{2}||(y,z)|| \leq \frac{1}{2}[||y||\,|z|\,| + ||(y,z)||].$$

□

Remark 3.2. In (3.4) on choosing $a = e$, we get that

$$||(y,x)(x,x)^{-1}(x,z) - (y,z)|| \leq ||z||||(y,y) - (y,x)(x,x)^{-1}(x,y)||^{\frac{1}{2}} \tag{3.17}$$

where $x, y, z \in \mathcal{H}$ and (x,x) is invertible in $\mathbb{A}$.

4. Kurepa's and Dragomir's inequality in Hilbert C^*-module

In 1960, N.G. Bruijn ([2]), obtained the following refinement of the Cauchy-Bunyakovsky-Schwartz inequality

$$|\sum_{i=1}^{n} a_i z_i|^2 \leq \frac{1}{2}\sum_{i=1}^{n} a_i^2 [\sum_{i=1}^{n} |z_i|^2 + |\sum_{i=1}^{n} z_i|^2] \tag{4.1}$$

provided that a_i are real numbers while z_i are complex for each $i \in \{1, ..., n\}$.

In an effort to extend this result to Hilbert space, S. Kurepa ([9]) obtained the following results

Theorem 4.1. *Let $(H; (.,.))$ be a real Hilbert space and $(H_{\mathbb{C}}; (.,.)_{\mathbb{C}})$ its complexification. Then for any $a \in H$ and $z \in H_{\mathbb{C}}$, one has the following refinement of Cauchy-Schwartz's inequality*

$$|(a,z)_{\mathbb{C}}|^2 \leq \frac{1}{2}||a||^2 \ [||z||^2_{\mathbb{C}} + ||(z;\bar{z})] \ \leq ||a||^2 \ ||z||^2_{\mathbb{C}}, \tag{4.2}$$

where $\bar{z}$ denote the conjugate of $z \in H_{\mathbb{C}}$.

As consequences of these results, S. Kerupa noted the following integral, respectively, discrete inequality.

Corollary 4.1. *Let $(S, \sum, \mu)$ be a positive measure space and let $a, z \in L_2(S, \sum, \mu)$, the Hilbert space of complex-valued 2-μ-integrable functions defined on S. If a is a real function and z is a complex function, then*

$$\left|\int_S a(t)z(t)d\mu(t)\right|^2 \leq \frac{1}{2}\int_S a^2(t)d\mu(t) \ (\int_S |z(t)|^2 d\mu(t) + \left|\int_S z^2(t)d\mu(t)\right|). \tag{4.3}$$

Corollary 4.2. *If $a_1, ..., a_n$ are real numbers, $z_1, ..., z_n$ are complex numbers and $A = (\alpha_{i,j})$ is a positive definite real matrix of order $n \times n$, then*

$$|\sum_{i,j=1}^{n} \alpha_{i,j} a_i z_j|^2 \leq \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{i,j} a_i a_j (\sum_{i,j=1}^{n} \alpha_{i,j} z_i \bar{z}_j + |\sum_{i,j=1}^{n} \alpha_{i,j} z_i \bar{z}_j|). \tag{4.4}$$

In [4] S.S. Dragomir proved the following refinement of Kurepa's results

Theorem 4.2. *Let $(H; (.,.))$ be a real Hilbert space and $(H_{\mathbb{C}}; (.,.)_{\mathbb{C}})$ its complexification. Then for any $v \in H$ and $\omega \in H_{\mathbb{C}}$, one has the following inequality*

$$\begin{aligned} |(\omega, v)_{\mathbb{C}}|^2 &\leq |(\omega, v)_{\mathbb{C}}^2 - \frac{1}{2}(\omega, \bar{\omega})_{\mathbb{C}} ||v||^2| + \frac{1}{2}|(\omega, \bar{\omega})_{\mathbb{C}}| ||v||^2 \\ &\leq \frac{1}{2}||v||^2 \quad (||\omega||_{\mathbb{C}}^2 + ||(\omega, \bar{\omega})_{\mathbb{C}}||). \end{aligned} \tag{4.5}$$

In an effort to extend the Dragomir's and Kurepa's results to Hilbert C^*-module, I considered the following setting

Let $\mathbb{A}$ be a real unitary C^*-algebra and $\mathbb{A}_{\mathbb{C}}$ its complexification, it is clear that $\mathbb{A}_{\mathbb{C}}$ be a unitary C^*-algebra. A element $a \in \mathbb{A}$ is sided positive if its positive in $\mathbb{A}_{\mathbb{C}}$.

Definition 4.1. Let $\mathbb{A}$ be a real unitary C^*-algebra, a real pre-Hilbert $\mathbb{A}$-module is a real vector space $\mathcal{H}$ which is an algebraic right $\mathbb{A}$-module equipped with a bilinear map $(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{A}$ satisfying
(i) $(x, x) \geq 0; (x, x) = 0$ if and only if $x = 0$ for all x in $\mathcal{H}$.
(ii) $(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z)$ for all x, y, z in $\mathcal{H}, \alpha, \beta$ in $\mathbb{R}$,
(iii) $(x, y) = (y, x)^*$ for all x, y in $\mathcal{H}$,
(iv) $(x, y.a) = (x, y)a$ for all x, y in $\mathcal{H}$, a in $\mathbb{A}$.

The map $(.,.)$ is called an $\mathbb{A}$-valued inner product of $\mathcal{H}$, and for all $x \in \mathcal{H}$, we define a norm in $\mathcal{H}$ by $||x|| = ||(x, x)||^{\frac{1}{2}}$
The completion of a real pre-Hilbert $\mathbb{A}$-module with respect to the norm induced by the $\mathbb{A}$-valued inner product is called a real Hilbert $\mathbb{A}$-module.

Let $\mathcal{H}$ be a real Hilbert $\mathbb{A}$ -module with the $\mathbb{A}$-valued inner product $(.,.)$ and the norm $||.||$. The complexification $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \oplus i\mathcal{H}$ of $\mathcal{H}$ endowed with the following operations

$$\begin{aligned} (x + iy) + (x' + iy') &\doteqdot (x + x') + i(y + y') : x, x', y, y' \in \mathcal{H} \\ (\alpha + i\beta).(x + iy) &\doteqdot (\alpha x - \beta y) + i(\beta x + \alpha y) : x, y \in \mathcal{H}; \alpha, \beta \in \mathbb{C} \\ (x + iy).(a + ib) &\doteqdot (x.a - y.b) + i(x.b + y.a) : x, y \in \mathcal{H}; a, b \in \mathbb{A} \end{aligned}$$

is complex vector space and right $\mathbb{A}_{\mathbb{C}}$-module. On $\mathcal{H}_{\mathbb{C}}$ one can consider the $\mathbb{A}_{\mathbb{C}}$-valued inner product defined by

$$(z, z')_{\mathbb{C}} \doteqdot (x, x') + (y, y') + i[(y, x') - (x, y')]$$

where $z = x + iy$ and $z' = x' + iy' \in \mathcal{H}_{\mathbb{C}}$, then $\mathcal{H}_{\mathbb{C}}$ is Hilbert $\mathbb{A}_{\mathbb{C}}$-module. We define the conjugate of a vector $z = x + iy \in \mathcal{H}_{\mathbb{C}}$ by $\bar{z} = x - iy$.

The next results is a generalization of Dragomir's results (Theorem 3) to Hilbert C^*-module

Theorem 4.3. *For each $v \in \mathcal{H}$ so that (v, v) is invertible in $\mathbb{A}$ and for each $w \in \mathcal{H}_{\mathbb{C}}$, one has the double inequality*

$$(4.6) \qquad \begin{aligned} ||(w,v)_{\mathbb{C}}(v,v)^{-1}(v,w)^*_{\mathbb{C}}|| &\leq \\ ||(w,v)_{\mathbb{C}}(v,v)^{-1}(w,v)_{\mathbb{C}} - \frac{1}{2}(w,\bar{w})_{\mathbb{C}}|| + \frac{1}{2}||(w,\bar{w})_{\mathbb{C}}|| &\leq \frac{1}{2}[||w||^2 + ||(w,\bar{w})_{\mathbb{C}}||] \end{aligned}$$

Proof. By applying the corollary 12, for $\mathcal{H}_{\mathbb{C}}$ and $x = v$, $y = w$ and $z = \bar{w}$, then we have

$$(4.7) \qquad \begin{aligned} &||(w,v)_{\mathbb{C}}(v,v)^{-1}(v,\bar{w})_{\mathbb{C}}|| \leq \\ &||(w,v)_{\mathbb{C}}(v,v)^{-1}(v,\bar{w})_{\mathbb{C}} - \frac{1}{2}(w,\bar{w})_{\mathbb{C}}|| + \frac{1}{2}||(w,\bar{w})_{\mathbb{C}}|| \leq \frac{1}{2}[||w||||\bar{w}|| + ||(w,\bar{w})_{\mathbb{C}}||]. \end{aligned}$$

If we assume that $w = x + iy \in \mathcal{H}_{\mathbb{C}}$ then, we have

$$\begin{aligned} (w,v)_{\mathbb{C}} &= (x+iy,v)_{\mathbb{C}} = (x,v) + i(y,v) = (v,x) + i(v,y) \\ (v,\bar{w})_{\mathbb{C}} &= (v, x - iy)_{\mathbb{C}} = (v,x) + i(v,y) = (w,v)_{\mathbb{C}} = (v,w)^*_{\mathbb{C}} \end{aligned}$$

and

$$\begin{aligned} (w,w)_{\mathbb{C}} &= (x,x) + (y,y) \\ (\bar{w},\bar{w})_{\mathbb{C}} &= (x,x) + (y,y) \end{aligned}$$

then $||w||_{\mathbb{C}} = ||\bar{w}||_{\mathbb{C}}$. Therefore, by (4.7), we deduce the desired results (4.6). □

If $\mathbb{A}$ is a real unitary commutative C^*-algebra, we have the following result.

Theorem 4.4. *For each $v \in \mathcal{H}$ and $w \in \mathcal{H}_{\mathbb{C}}$, one has the double inequality*

$$(4.8) \qquad \begin{aligned} ||(w,v)_{\mathbb{C}}||^2 \leq \; & ||(w,v)^2_{\mathbb{C}} - \frac{1}{2}(v,v)(w,\bar{w})_{\mathbb{C}}|| + \frac{1}{2}\,||(w,\bar{w})_{\mathbb{C}}||\,||v||^2 \leq \\ & \frac{1}{2}\,||v||^2\,[||w||^2 + ||(w,\bar{w})||] \end{aligned}$$

Proof. By similar argument. □

As a consequence of this inequality, we obtain the following two inequalities.

Let $\mathbb{A}$ be a unitary commutative C^*-algebra with unit e, and $(\rho_n)_{n\geq 1}$ be a sequence of positives reals numbers so that

$$\sum_{n=1}^{+\infty} \rho_n = 1.$$

If $\mathcal{H} = l^2_\rho(\mathbb{A})$, then we have the following discrete inequality

Corollary 4.3. *If $a=(a_n)_{n\geq 1}$ is a sequence of hermitian elements of $\mathbb{A}$ so that $a\in l^2_\rho(\mathbb{A})$, then for any $z=(z_n)_{n\geq 1}\in l^2_\rho(\mathbb{A})$, one has the double inequality*

$$
\begin{aligned}
(4.9)\qquad ||\sum_{n=1}^{+\infty}\rho_n a_n z_n||^2 &\leq ||(\sum_{n=1}^{+\infty}\rho_n a_n z_n)^2-\frac{1}{2}(\sum_{n=1}^{+\infty}\rho_n a_n^2)(\sum_{n=1}^{+\infty}\rho_n (z_n^*)^2)|| \\
+\frac{1}{2}||\sum_{n=1}^{+\infty}\rho_n a_n^2||||\sum_{n=1}^{+\infty}\rho_n z_n^2|| &\leq \frac{1}{2}||\sum_{n=1}^{+\infty}\rho_n a_n^2||[||\sum_{n=1}^{+\infty}\rho_n z_n z_n^*||+||\sum_{n=1}^{+\infty}\rho_n z_n^2||].
\end{aligned}
$$

Let (S,Σ,μ) be a positive measure space and φ be a $\mathbb{A}$-valued function $S\longrightarrow\mathbb{A}$ so that $\varphi(t)$ is hermitian and positive in $\mathbb{A}$ for which $t\in S$, and

$$\int_S ||\varphi(t)||^2 d\mu(t)\langle +\infty \quad\text{and}\quad \int_S \varphi(t)d\mu(t)=e.$$

Similarly, if $\mathcal{H}=\mathbb{L}^2_\varphi(S,\Sigma,\mu,\mathbb{A})$ the linear space of all $\mathbb{A}$-valued functions $f:S\longrightarrow\mathbb{A}$ such as

$$\int_S ||\varphi(t)||||f(t)||^2 d\mu(t)\langle +\infty$$

then, we have the following integral inequality

Corollary 4.4. *For $a\in\mathbb{A}^2_\varphi(S,\Sigma,\mu,\mathbb{A})$ such that $a(t)$ is hermitian in $\mathbb{A}$ for which $t\in S$, and any $f\in\mathbb{L}^2_\varphi(S,\Sigma,\mu,\mathbb{A})$, we have the inequality*

$$
\begin{aligned}
(4.10)\qquad & ||\int_S \varphi(t)a(t)f(t)d\mu(t)||^2 \\
\leq\; & ||(\int_S \varphi(t)a(t)f(t)^* d\mu(t))^2-\frac{1}{2}(\int_S \varphi(t)a^2(t)d\mu(t))(\int_S \varphi(t)(f(t)^*)^2 d\mu(t))|| \\
& +\frac{1}{2}||\int_S \varphi(t)a^2(t)d\mu(t)||||\int_S \varphi(t)f^2(t)d\mu(t))|| \\
\leq\; & \frac{1}{2}||\int_S \varphi(t)a^2(t)d\mu(t)||[||\int_S \varphi(t)f(t)f(t)^* d\mu(t)||+||\int_S \varphi(t)f^2(t)d\mu(t))||]
\end{aligned}
$$

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