DRAGOMIR’S, BUZANO’S AND KERUPA’S INEQUALITIES IN HILBERT $C^*$-MODULES

Ahmed Roukbi

Abstract. In this paper we prove a type of Dragomir’s, Buzano’s and Kurepa’s inequalities in Hilbert $C^*$-modules. Some applications for discrete and integral inequalities improving the Cauchy-Schwartz result are given.

1. Introduction

In ([1]), M.L. Buzano obtained the following extension of the Cauchy-Schwartz’s inequality in a real or complex Hilbert space $(H, (, ,))$

$$|(a, x)(x, b)| \leq \frac{1}{2}||a|| ||b|| + ||(a, b)|| ||x||^2,$$

for any $a, b, x \in H$. It is clear that for $a = b$, the above inequality becomes the standard Cauchy-Schwartz’s inequality

$$|(a, x)|^2 \leq ||a||^2 ||x||^2 : \forall a, x \in H,$$

with equality if and only if there exists a scalar $\lambda \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) such that $x = \lambda a$.

It might be useful to observe that, out of (1.1) one may get the following discrete inequality

$$\left| \sum_{i=1}^{n} p_i a_i \bar{x}_i \sum_{i=1}^{n} p_i x_i b_i \right| \leq \frac{1}{2} \left( \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 \right)^{\frac{1}{2}} + \left| \sum_{i=1}^{n} p_i a_i \bar{b}_i \right| \sum_{i=1}^{n} p_i |x_i|^2,$$

where $p_i \geq 0, a_i, b_i, x_i \in \mathbb{C}, i \in \{1, \ldots, n\}$.

If one takes in (1.1) $b_i = \bar{a}_i$ for $i \in \{1, \ldots, n\}$, then one obtains

$$\left| \sum_{i=1}^{n} p_i a_i \bar{x}_i \sum_{i=1}^{n} p_i a_i x_i \right| \leq \frac{1}{2} \sum_{i=1}^{n} p_i |a_i|^2 + \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i |x_i|^2,$$

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for any \( p_i \geq 0, a_i, x_i \in \mathbb{C}, i \in \{1, \ldots, n\} \).

Note that, if \( x_i, i \in \{1, \ldots, n\} \) are real numbers, then out of (1.4), we may deduce the Buijn’s inequality (2)

\[
| \sum_{i=1}^{n} p_i x_i z_i |^2 \leq \frac{1}{2} \sum_{i=1}^{n} p_i | z_i |^2 + \frac{1}{2} \sum_{i=1}^{n} p_i z_i^2 \sum_{i=1}^{n} p_i x_i^2 ;
\]

where \( z_i \in \mathbb{C}, i \in \{1, \ldots, n\} \). In this way, Buzano’s inequality (1.1) may be regarded as a generalization of the Buijn’s inequality.

In ([4]) S.S, Dragomir established the following refinement of Buzano’s inequality

\[
| (a, x)(x, b) | \leq \frac{||b||}{||a||} \left[ ||b - 1||^2 + ||a||^2 - ||(a, x)^2 | \right]
\]

where \( a, b, x \in H, x \neq 0 \) and \( \alpha \in \mathbb{K} \). The case of equality holds in (1.6) if and only if there exists a scalar \( \lambda \in \mathbb{K} \) such that

\[
\alpha \frac{(a, x)}{||x||^2} x = a + \lambda b.
\]

The goal of this paper is to show some related as well as a extension of Buzano’s and Dragomir’s inequality (1.1) and (1.6) to Hilbert \( C^* \)-module. We can obtain various particular inequalities in Hilbert \( C^* \)-module. In section 3, we are given a extension of Kurepa’s and Dragomir’s inequality to Hilbert \( C^* \)-module, the corresponding applications for discrete and integral inequalities are also provided.

2. Preliminaries in Hilbert \( C^* \)-modules

In this section we briefly recall the definitions and examples of Hilbert \( C^* \)-modules. For information about Hilbert \( C^* \)-module, we refer to ([5,6,7,9,11]). Our references for \( C^* \)-algebras are ([3,13]).

Let \( \mathbb{A} \) be a \( C^* \)-algebra (not necessarily unitary) and \( \mathcal{H} \) be a complex linear space.

**Definition 2.1.** A pre-Hilbert \( \mathbb{A} \)-module is a right \( \mathbb{A} \)-module \( \mathcal{H} \) equipped with a sesquilinear map \((.,.) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{A} \) satisfying

(i) \((x, x) \geq 0; (x, x) = 0 \) if and only if \( x = 0 \) for all \( x \) in \( \mathcal{H} \),
(ii) \((x, \alpha y + \beta z) = \alpha (x, y) + \beta (x, z) \) for all \( x, y, z \) in \( \mathcal{H} \), \( \alpha, \beta \) in \( \mathbb{C} \),
(iii) \((x, y) = (y, x)^* \) for all \( x, y \) in \( \mathcal{H} \),
(iv) \((x, y.a) = (x, y)a \) for all \( x, y \) in \( \mathcal{H} \), \( a \) in \( \mathbb{A} \).

The map \((.,.) \) is called an \( \mathbb{A} \)-valued inner product of \( \mathcal{H} \), and for \( x \in \mathcal{H} \), we define \( ||x|| = ||(x, x)||^{1/2} \).
Proposition 2.1. Let \( \mathcal{H} \) be a pre-Hilbert \( A \)-module, then

(i) \(|\cdot|\) is a norm on \( \mathcal{H} \),
(ii) \(|x.a| \leq |x||a|\) for all \( x \in \mathcal{H} \), \( a \in A \),
(iii) \((x, y)(y, x) \leq |y|^2 (x, x)\) for all \( x, y \in \mathcal{H} \),
(iv) \(|(x, y)| \leq |x||y|\) for all \( x, y \in \mathcal{H} \).

It is clear that (iii) and (iv) are a generalization of Cauchy-Schwartz’s inequality to a pre-Hilbert \( A \)-module. The equality holds in (iv) if there exists \( \lambda \in \mathbb{C} \) so that \( y = \lambda x \).

Definition 2.2. The completion of a pre-Hilbert \( A \)-module with respect to the norm induced by the \( A \)-valued inner product is called a Hilbert \( A \)-module.

Example 2.1.

1. Let \( A \) be a \( C^* \)-algebra. \( A \) is Hilbert \( A \)-module if an \( A \)-valued inner product is defined as \((x, y) = x^* y\) for all \( x, y \in A \). Any closed right ideal of \( A \) is sub-\( A \)-module under the above \( A \)-valued inner product.

2. Let \( \{ \mathcal{H}_i : i \in I \} \) be a finite family of Hilbert \( A \)-modules. Then \( \bigoplus_{i \in I} \mathcal{H}_i \) is a Hilbert \( A \)-module with its \( A \)-valued inner product is defined as

\[
((x_i), (y_i)) = \sum_{i \in I} (x_i, y_i).
\]

When \( \{ \mathcal{H}_i : i \in I \} \) is an infinite family of Hilbert \( A \)-modules we define

\[
\bigoplus_{i \in I} \mathcal{H}_i = \{(x_i)_{i \in I} : \sum_i (x_i, x_i) \text{ converges in norm in } A \}.
\]

Thus \( \bigoplus_{i \in I} \mathcal{H}_i \) is a Hilbert \( A \)-module with its \( A \)-valued inner product is defined as

\[
((x_i), (y_i)) = \sum_{i \in I} (x_i, y_i).
\]

3. Let \( A \) be a \( C^* \)-algebra and \( n \) be a integer \( \geq 1 \), then \( A^n \cong \bigoplus_{i=1}^{n} A \) is a Hilbert \( A \)-module with its \( A \)-valued inner product is defined as

\[
(a, b) = \sum_{i=1}^{n} a_i^* b_i \quad \text{for all } a = (a_1, ..., a_n), \ b = (b_1, ..., b_n) \in A^n.
\]

4. Let \( A \) be a \( C^* \)-algebra. Denote by \( l^2(A) \) the linear space of all sequences \( z = (z_n)_{n \geq 1} \) of elements \( A \) so that

\[
\sum_{n=1}^{+\infty} |z_n|^2 < +\infty.
\]

Then \( l^2(A) \) is a Hilbert \( A \)-module with its \( A \)-valued inner product is defined as

\[
(z, z') = \sum_{n=1}^{+\infty} z_n^* z'_n \quad \text{for all } z = (z_n)_{n \geq 1}, z' = (z'_n)_{n \geq 1} \in l^2(A).
\]
5. Let $A$ be a unitary $C^*$-algebra with unit $e$, and $(\rho_n)_{n \geq 1}$ be a sequence of positives reals numbers so that
\[ \sum_{n=1}^{+\infty} \rho_n = 1. \]
Denote by $l_\rho^2(A)$ the linear space of all sequences $z = (z_n)_{n \geq 1}$ of elements $A$ so that
\[ \sum_{n=1}^{+\infty} \rho_n \|z_n\|^2 = 1. \]
Then $l_\rho^2(A)$ is a Hilbert $A$-module with its $A$-valued inner product is defined as
\[ (z, z') = \sum_{n=1}^{+\infty} \rho_n z_n^* z'_n \quad \text{for all} \quad z = (z_n)_{n \geq 1}, z' = (z'_n)_{n \geq 1} \in l_\rho^2(A). \]

6. Let $A$ be a unitary commutative $C^*$-algebra with unit $e$, $(S, \Sigma, \mu)$ be a positive measure space and $\varphi$ be a $A$-valued function $S \rightarrow A$ so that $\varphi(t)$ is hermitian positive in $A$ for which $t \in S$, and
\[ \int_S |\varphi(t)|^2 d\mu(t) < +\infty \quad \text{and} \quad \int_S \varphi(t) d\mu(t) = e. \]
Denote by $L_\varphi^2(S, \Sigma, \mu, A)$ the linear space of all $A$-valued functions $f : S \rightarrow A$ such that
\[ \int_S ||\varphi(t)|| ||f(t)||^2 d\mu(t) < +\infty. \]
Then $L_\varphi^2(S, \Sigma, \mu, A)$ is a Hilbert $A$-module with its $A$-valued inner product is defined as
\[ (f, g)_\varphi = \int_S \varphi(t) f(t)^* g(t) d\mu(t) \quad \text{for all} \quad f, g \in L_\varphi^2(S, \Sigma, \mu, A). \]

3. Buzano’s and Dragomir’s inequality in Hilbert $C^*$-modules

We let $A$ be a unitary $C^*$-algebra with unit $e$ and $\mathcal{H}$ be a Hilbert $C^*$-module over $A$. The following results may be stated. It is a generalization of Dragomir’s result (1.6).

**Theorem 3.1.** For all $x, y, z \in \mathcal{H}$ so that $(x, x)$ is invertible in $A$ and for each invertible $a \in A$, one has the inequality
\[
 \|[(y, x)(x, x)^{-1}(x, z) - a^{-1}(y, z)]\|
 \leq \frac{\|z\| \|(a - e)(y, x)(x, x)^{-1}(x, y)(a^* - e) + (y, y) - (y, x)(x, x)^{-1}(x, y)\|^{\frac{1}{2}}.}
\]

The case of equality holds in (3.1) if there exists $\lambda \in \mathbb{C}$ such that
\[
 x.(x, x)^{-1}(y, x)a^* = y + \lambda z.a^*.
\]
Proof. Using Cauchy-Schwartz’s inequality (Proposition 2(iii)), we have that
\[
(x(x,x)^{-1}(x,y) - y.a^{-1})^* (z, x(x,x)^{-1}(x,y) - y.a^{-1})^*)
\]
\[
\leq ||z||^2 (x(x,x)^{-1}(x,y) - y.a^{-1})^*, x(x,x)^{-1}(x,y) - y.a^{-1})^*,
\]
and since
\[
(x(x,x)^{-1}(x,y) - y.a^{-1})^*, x(x,x)^{-1}(x,y) - y.a^{-1})^*)
\]
\[
= (y, x(x,x)^{-1}(x,y) - y.a^{-1})^*, x(x,x)^{-1}(x,y) - y.a^{-1})]^* - a^{-1}(y, x(x,x)^{-1}(x,y) - y.a^{-1})^* + a^{-1}(y, x(x,x)^{-1}(x,y) - y.a^{-1})^*\]
\[
= a^{-1}[(a-e)(y, x(x,x)^{-1}(x,y) - y.a^{-1})^* + (y, y) - (y, x(x,x)^{-1}(x,y) - y.a^{-1})]^*\]
and
\[
(x(x,x)^{-1}(x,y) - y.a^{-1})^*, z, x(x,x)^{-1}(x,y) - y.a^{-1})^*)
\]
\[
= [(y, x(x,x)^{-1}(x,y) - y.a^{-1})^*, x(x,x)^{-1}(x,y) - y.a^{-1})]^* - a^{-1}(y, y) - a^{-1}(y, y)\]
\[
= a^{-1}[(a-e)(y, x(x,x)^{-1}(x,y) - y.a^{-1})^* + (y, y) - (y, x(x,x)^{-1}(x,y) - y.a^{-1})]^*\]
Using (3.3) we get that
\[
||[(y, x(x,x)^{-1}(x,y) - y.a^{-1})^*, z, x(x,x)^{-1}(x,y) - y.a^{-1})^*)||^2
\]
\[
\leq ||z||^2 a^{-1}[(a-e)(y, x(x,x)^{-1}(x,y) - y.a^{-1})^* + (y, y) - (y, x(x,x)^{-1}(x,y) - y.a^{-1})]^*\]
Passing to the norm in \(\mathcal{A}\), we have that
\[
||[(y, x(x,x)^{-1}(x,y) - a^{-1}(y, z)]^||^2
\]
\[
\leq ||z||^2 a^{-1}[(a-e)(y, x(x,x)^{-1}(x,y) - a^{-1}(y, z)]^*\]
This proves (3.1).

The case of equality holds in (3.3) if there exists \(\lambda \in \mathbb{C}\) so that \(x(x,x)^{-1}(x,y) - y.a^{-1})^* = \lambda z\). Or equivalently
\[
x(x,x)^{-1}(x,y) a^* = y + \lambda z.a^*.
\]

\(\square\)

If \(\mathcal{A}\) is a commutative \(C^*\)-algebra, we obtain the following result.

**Corollary 3.1.** For all \(x, y, z \in \mathcal{H}\) so that \((x, x)\) is invertible in \(\mathcal{A}\) and for each invertible \(a \in \mathcal{A}\), one has the inequality
\[
||\frac{(y, x(x,x))}{x(x,x)} - \frac{y, z}{a}||
\]
\[
\leq \frac{||z||}{||a||^2 ||x||} \|(a-e)(a^* - e)(y, x)(x, y) + (x, y)(y)(y) - (y, x)(x, y)||^2,
\]
where $\frac{1}{a} = a^{-1}$ is the inverse of $a$ in $\mathcal{A}$. The case of equality holds in (3.4) if there exists $\lambda \in \mathbb{C}$ such that

\[(3.5) \quad x.a\frac{(y, x)}{(x, x)} = y + \lambda z.\]

The following result also holds.

**Proposition 3.1.** For all $x, y, z \in \mathcal{H}$ such that $(x, x)$ is invertible in $\mathcal{A}$, and for each invertible $a \in \mathcal{A}$, one has the double inequality

\[(3.6) \quad \frac{||a^{-1}(y, z)|| - ||z||}{||a||} \leq \frac{1}{2} \left[ \frac{||(a - e)(y, x)(x, x)^{-1}(x, y)(a^* - e) + (y, y) - (y, x)(x, x)^{-1}(x, y)||}{\frac{1}{2}} \right].\]

**Proof.** Using the continuity of $C^*$-norm in $\mathcal{A}$, we get that

\[||a^{-1}(y, z)|| - \frac{||z||}{||a||} \leq \frac{1}{2} \left[ \frac{||(a - e)(y, x)(x, x)^{-1}(x, y)(a^* - e) + (y, y) - (y, x)(x, x)^{-1}(x, y)||}{\frac{1}{2}} \right].\]

Using (3.1) we deduce the double inequality (3.6).

The following result is generalization of Buzano’s inequality (1.1) to Hilbert $C^*$-module.

**Corollary 3.2.** For all $x, y, z \in \mathcal{H}$ so that $(x, x)$ is invertible in $\mathcal{A}$, we have the inequality

\[(3.7) \quad \frac{||(y, x)(x, x)^{-1}(x, z)||}{\frac{1}{2}} \leq \frac{1}{2} \left[ ||y|| ||z|| + ||(y, z)|| \right].\]

**Proof.** In (3.6) we put $a = 2e$, then we get

\[\frac{||(y, x)(x, x)^{-1}(x, z)||}{\frac{1}{2}} \leq \frac{1}{2} \left[ ||y|| ||z|| + \frac{1}{2} ||(y, z)|| \right],\]

this proves (3.7).

It is obvious that, out of (3.1) and (3.4), we can obtain various particular inequalities. A class of these which is
**Corollary 3.3.** For all $x, y, z \in H$ so that $(x, x)$ is invertible in $\mathfrak{A}$ and for each $\eta \in \mathbb{C}$ with $|\eta| = 1$ and $\Re(\eta) \neq -1$, we have the following inequality

\[(3.8) \quad ||(y, x)(x, x)^{-1}(x, z) - \frac{1}{1 + \eta}(y, z)|| \leq \frac{1}{\sqrt{2\sqrt{1 + \Re(\eta)}}}||y|| ||z||.\]

In particular for $\eta = 1$, we have

\[(3.9) \quad ||(y, x)(x, x)^{-1}(x, z) - \frac{1}{2}(y, z)|| \leq \frac{1}{2}||y|| ||z||.\]

**Proof.** In Theorem 1, on choosing $a = (1 + \eta)e$, we get that

\[||(y, x)(x, x)^{-1}(x, z) - \frac{1}{1 + \eta}(y, z)|| \leq \frac{1}{|1 + \eta|}||z||||(y, y)||^{\frac{1}{2}},\]

then

\[||(y, x)(x, x)^{-1}(x, z) - \frac{1}{1 + \eta}(y, z)|| \leq \frac{1}{\sqrt{2\sqrt{1 + \Re(\eta)}}}||y|| ||z||.\]

Finally, if $\eta = 1$ we have the inequality (3.9). \(\square\)

If $\mathfrak{A}$ is a commutative $C^*$-algebra, we have the following result

**Corollary 3.4.** For all $x, y, z \in H$ so that $(x, x)$ is invertible in $\mathfrak{A}$ and for all unitary element $b \in \mathfrak{A}$ so that $\Re(b) \neq -e$, we have the following inequality

\[(3.10) \quad ||(y, x)(x, x) - (y, z)\frac{b}{e + b}|| \leq \frac{||y|| ||z||}{\sqrt{2\sqrt{2 + \Re(b)}}}.\]

In particular, for $b = e$, we have the inequality

\[(3.11) \quad ||(y, x)(x, x) - (y, z)\frac{b}{e + b}|| \leq \frac{1}{2}||y|| ||z||.\]

**Proof.** Using Corollary 1, on choosing $a = e + b$, we get that

\[||(y, x)(x, x) - (y, z)\frac{b}{e + b}|| \leq \frac{||z||}{||e + b||} ||b^*(y, x)(x, y) + (x, x)(y, y) - (y, x)(x, y)||^{\frac{1}{2}}.\]

\[\iff \quad ||(y, x)(x, x) - (y, z)\frac{b}{e + b}|| \leq \frac{||z||}{||e + b||} ||(x, x)(y, y)||^{\frac{1}{2}}.\]

\[\iff \quad ||(y, x)(x, x) - (y, z)\frac{b}{e + b}|| \leq \frac{||z|| ||y||}{||e + b||}.\]

The inequality (3.10) result by using the fact that

\[||e + b||^2 = ||(e + b)(e + b^*)|| = ||2e + 2\Re(b)|| = 2||e + \Re(b)||.\]

Finally, if $b = e$ we have the inequality (3.11). \(\square\)
Remark 3.1. Using the continuity of $C^*$-norm in $\mathbb{A}$, we get from (3.9) that
\[ ||(y, x)(x, x)^{-1}(x, z)|| \leq \frac{||x|| ||y|| + ||(y, z)||}{\sqrt{2} \sqrt{1 + \text{Re}(\eta)}}, \]
for each $\eta \in \mathbb{C}$ such that $|\eta| = 1$ and $\text{Re}(\eta) \neq -1$.

Let $n$ be a integer $\geq 1$, $(\rho_i)_{1 \leq i \leq n}$ be a finite family of positives reals numbers and $\mathcal{H} = \mathbb{A}^n \cong \bigoplus_{i=1}^n \mathbb{A}$ the Hilbert $\mathbb{A}$-module with the $\mathbb{A}$-valued inner product
\[ (a, b) = \sum_{i=1}^n \rho_i a_i^* b_i \quad \text{for all } a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{A}^n. \]

It might be useful to observe that, out of (3.7), we obtain the following discrete inequality
\[ ||\sum_{i=1}^n \rho_i a_i x_i \sum_{i=1}^n \rho_i x_i^* b_i || \leq \frac{1}{2} (\sum_{i=1}^n \rho_i ||a_i||^2 \sum_{i=1}^n \rho_i ||b_i||^2)^{\frac{1}{2}} + \sum_{i=1}^n \rho_i a_i b_i^* ||, \]
where $a_i, b_i, x_i \in \mathbb{A}$, $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n \rho_i x_i^* x_i$ is invertible in $\mathbb{A}$.

If one takes in (3.13) $b_i = a_i^*$ for $i \in \{1, \ldots, n\}$, then we obtain
\[ ||\sum_{i=1}^n \rho_i a_i x_i \sum_{i=1}^n \rho_i x_i^* a_i || \leq \sum_{i=1}^n \rho_i ||a_i||^2. \]

Note that, if $x_i$ is hermitian for $i \in \{1, \ldots, n\}$ then, out of (3.13), we may deduce the following generalization of Bruijn’s inequality (1.5) (see [2]).
\[ ||\sum_{i=1}^n \rho_i a_i x_i \sum_{i=1}^n \rho_i x_i^* a_i || \leq \frac{1}{2} (\sum_{i=1}^n \rho_i ||a_i||^2) + \sum_{i=1}^n \rho_i a_i^2 ||. \]

We closed this section by noting the following result.

Corollary 3.5. Let $\mathcal{H}$ be a Hilbert $C^*$-module over a unitary $C^*$-algebra $\mathbb{A}$. For all $x, y, z \in \mathcal{H}$ such that $(x, x)$ is invertible in $\mathbb{A}$, we have the following double inequality
\[ ||(y, x)(x, x)^{-1}(x, z)|| \leq ||(y, x)(x, x)^{-1}(x, z) - \frac{1}{2} (y, z)|| + \frac{1}{2} ||(y, z)|| \]
\[ \leq \frac{1}{2} (||y|| ||z|| + ||(y, z)||). \]

Proof. Using (3.9) we get for all $x, y, z \in \mathcal{H}$ that
\[ ||(y, x)(x, x)^{-1}(x, z) - \frac{1}{2} (y, z)|| \leq \frac{1}{2} ||y|| ||z||. \]
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the continuity of $C^*$-norm in $A$ implies that
\[\left|\|(y, x)(x, x)^{-1}(x, z)|| - \frac{1}{2}||y, z)||\right| \leq \frac{1}{2}||y, z)|| - \frac{1}{2}||y, z)||,\]
then
\[\|(y, x)(x, x)^{-1}(x, z)|| \leq \|(y, x)(x, x)^{-1}(x, z) - \frac{1}{2}(x, x)(y, z)|| + \frac{1}{2}||y, z)|| \leq \frac{1}{2} \left[||y|| \cdot |z| + ||y, z)||\right].\]

Remark 3.2. In (3.4) on choosing $a = e$, we get that
\[(3.17) \quad ||(y, x)(x, x)^{-1}(x, z) - (y, z)|| \leq ||y, y)|| - (y, x)(x, x)^{-1}(y, y)||^\frac{1}{2},\]
where $x, y, z \in H$ and $(x, x)$ is invertible in $A$.

4. Kurepa’s and Dragomir’s inequality in Hilbert $C^*$-module

In 1960, N.G. Bruijn ([2]), obtained the following refinement of the Cauchy-Bunyakovsky-Schwartz inequality
\[(4.1) \quad \left|\sum_{i=1}^{n} a_i z_i\right|^2 \leq \frac{1}{2} \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} |z_i|^2 + \frac{1}{2} \sum_{i=1}^{n} |z_i|^2\]
provided that $a_i$ are real numbers while $z_i$ are complex for each $i \in \{1, ..., n\}$.

In an effort to extend this result to Hilbert space, S. Kurepa ([9]) obtained the following results

**Theorem 4.1.** Let $(H; (,))$ be a real Hilbert space and $(H_C; (,)_C)$ its complexification. Then for any $a \in H$ and $z \in H_C$, one has the following refinement of Cauchy-Schwartz’s inequality
\[(4.2) \quad |(a, z)_C|^2 \leq \frac{1}{2} ||a||^2 \left[ ||z||^2 + ||(z; \bar{z})|| \right] \leq ||a||^2 ||z||^2,\]
where $\bar{z}$ denote the conjugate of $z \in H_C$.

As consequences of these results, S. Kerupa noted the following integral, respectively, discrete inequality.

**Corollary 4.1.** Let $(S, \Sigma, \mu)$ be a positive measure space and let $a, z \in L_2(S, \Sigma, \mu)$, the Hilbert space of complex-valued $2\mu$-integrable functions defined on $S$. If $a$ is a real function and $z$ is a complex function, then
\[(4.3) \quad \left|\int_{S} a(t)z(t)d\mu(t)\right|^2 \leq \frac{1}{2} \int_{S} a^2(t)d\mu(t) \left( \int_{S} |z(t)|^2d\mu(t) + \int_{S} z^2(t)d\mu(t) \right).\]
Corollary 4.2. If \(a_1, ..., a_n\) are real numbers, \(z_1, ..., z_n\) are complex numbers and \(A = (\alpha_{i,j})\) is a positive definite real matrix of order \(n \times n\), then

\[
|\sum_{i,j=1}^{n} \alpha_{i,j}a_ia_jz_j|^2 \leq \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{i,j}a_ia_j \left( \sum_{i,j=1}^{n} \alpha_{i,j}z_i\bar{z}_j + \sum_{i,j=1}^{n} \alpha_{i,j}z_i\bar{z}_j \right).
\]

In [4] S.S. Dragomir proved the following refinement of Kurepa’s results.

Theorem 4.2. Let \((H; \langle ., . \rangle)\) be a real Hilbert space and \((H_C; \langle ., . \rangle_C)\) its complexification. Then for any \(v \in H\) and \(\omega \in H_C\), one has the following inequality

\[
|\langle \omega, v \rangle_C|^2 \leq |\langle \omega, v \rangle_C|^2 - \frac{1}{2} \langle \omega, \bar{\omega} \rangle_C |v|^2 + \frac{1}{2} \langle \omega, \bar{\omega} \rangle_C ||v||^2
\]

\[
\leq \frac{1}{2} ||v||^2 \left( ||\omega||_C^2 + ||\omega, \bar{\omega}||_C \right).
\]

In an effort to extend the Dragomir’s and Kurepa’s results to Hilbert \(C^*\)-module, I considered the following setting.

Let \(A\) be a real unitary \(C^*\)-algebra and \(A_C\) its complexification, it is clear that \(A_C\) is a unitary \(C^*\)-algebra. A element \(a \in A\) is sided positive if its positive in \(A_C\).

Definition 4.1. Let \(A\) be a real unitary \(C^*\)-algebra, a real pre-Hilbert \(A\)-module is a real vector space \(H\) which is an algebraic right \(A\)-module equipped with a bilinear map \((., .) : H \times H \rightarrow A\) satisfying

(i) \((x, x) \geq 0; (x, x) = 0\) if and only if \(x = 0\) for all \(x \in H\).
(ii) \((x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z)\) for all \(x, y, z \in H, \alpha, \beta \in \mathbb{R}\),
(iii) \((x, y) = \langle y, x \rangle^*\) for all \(x, y \in H\),
(iv) \((x, y.a) = (x, y)a\) for all \(x, y \in H, a \in A\).

The map \((., .)\) is called an \(A\)-valued inner product of \(H\), and for all \(x \in H\), we define a norm in \(H\) by \(||x|| = ||(x, x)||^{\frac{1}{2}}\). The completion of a real pre-Hilbert \(A\)-module with respect to the norm induced by the \(A\)-valued inner product is called a real Hilbert \(A\)-module.

Let \(H\) be a real Hilbert \(A\)-module with the \(A\)-valued inner product \((., .)\) and the norm \(||.||\). The complexification \(H_C = H \oplus iH\) of \(H\) endowed with the following operations

\[
(x + iy) + (x' + iy') = (x + x') + i(y + y') : x, x', y, y' \in H
\]
\[
(a + i\beta)(x + iy) = (ax - \beta y) + i(\beta x + \alpha y) : x, y \in H; \alpha, \beta \in \mathbb{C}
\]
\[
(x + iy)(a + ib) = (xa - yb) + i(xb + ya) : x, y \in H; a, b \in A
\]

is complex vector space and right \(A_C\)-module. On \(H_C\) one can consider the \(A_C\)-valued inner product defined by

\[
\langle z, z' \rangle_C = \langle x, x' \rangle + \langle y, y' \rangle + i[\langle y, x' \rangle - \langle x, y' \rangle]
\]
where \( z = x + iy \) and \( z' = x' + iy' \in \mathcal{H}_C \), then \( \mathcal{H}_C \) is Hilbert \( \mathbb{A}_C \)-module. We define the conjugate of a vector \( z = x + iy \in \mathcal{H}_C \) by \( \bar{z} = x - iy \).

The next results is a generalization of Dragomir’s results (Theorem 3) to Hilbert \( C^* \)-module

**Theorem 4.3.** For each \( v \in \mathcal{H} \) so that \((v, v)\) is invertible in \( \mathbb{A} \) and for each \( w \in \mathcal{H}_C \), one has the double inequality

\[
\| (w, v)_C (v, v)_C^{-1} (v, w)_C \| \leq \frac{1}{2} \| (w, \bar{w})_C \| + \frac{1}{2} \| (w, \bar{w})_C \| \leq \frac{1}{2} \| w \|^2 + \| (w, \bar{w})_C \|.
\]

**Proof.** By applying the corollary 12, for \( \mathcal{H}_C \) and \( x = v \), \( y = w \) and \( z = \bar{w} \), then we have

\[
\| (w, v)_C (v, v)_C^{-1} (v, w)_C \| \leq \frac{1}{2} \| (w, \bar{w})_C \| + \frac{1}{2} \| (w, \bar{w})_C \| \leq \frac{1}{2} \| w \|^2 + \| (w, \bar{w})_C \|.
\]

If \( \mathbb{A} \) is a real unitary commutative \( C^* \)-algebra, we have the following result.

**Theorem 4.4.** For each \( v \in \mathcal{H} \) and \( w \in \mathcal{H}_C \), one has the double inequality

\[
\| (w, v)_C \|^2 \leq \| (w, v)_C \|^2 - \frac{1}{2} (v, v)_C (v, \bar{v})_C + \frac{1}{2} \| (w, \bar{w})_C \| \| v \|^2 \leq \frac{1}{2} \| v \|^2 \| w \|^2 + \| (w, \bar{w})_C \|.
\]

**Proof.** By similar argument. \( \square \)

As a consequence of this inequality, we obtain the following two inequalities.

Let \( \mathbb{A} \) be a unitary commutative \( C^* \)-algebra with unit \( e \), and \( (\rho_n)_{n \geq 1} \) be a sequence of positives reals numbers so that

\[
\sum_{n=1}^{+\infty} \rho_n = 1.
\]

If \( \mathcal{H} = l^2_\rho(\mathbb{A}) \), then we have the following discrete inequality
Corollary 4.3. If \( a = (a_n)_{n \geq 1} \) is a sequence of hermitian elements of \( \mathcal{A} \) so that \( a \in L^2_\rho(\mathcal{A}) \), then for any \( z = (z_n)_{n \geq 1} \in L^2_\rho(\mathcal{A}) \), one has the double inequality

\[
(4.9) \quad \left\| \sum_{n=1}^{+\infty} \rho_n a_n z_n \right\|^2 \leq \left| \left( \sum_{n=1}^{+\infty} \rho_n a_n z_n \right)^2 - \frac{1}{2} \left( \sum_{n=1}^{+\infty} \rho_n a_n^2 \right) \left( \sum_{n=1}^{+\infty} \rho_n (z_n^*)^2 \right) \right| + \frac{1}{2} \left\| \sum_{n=1}^{+\infty} \rho_n a_n^2 \right\|. \]

Let \( (S, \Sigma, \mu) \) be a positive measure space and \( \varphi \) be a \( \mathcal{A} \)-valued function \( S \rightarrow \mathcal{A} \) so that \( \varphi(t) \) is hermitian and positive in \( \mathcal{A} \) for which \( t \in S \), and

\[
\int_S \left\| \varphi(t) \right\|^2 d\mu(t)(+\infty) \quad \text{and} \quad \int_S \varphi(t) d\mu(t) = e.
\]

Similarly, if \( \mathcal{H} = L^2_\varphi(S, \Sigma, \mu, \mathcal{A}) \) the linear space of all \( \mathcal{A} \)-valued functions \( f : S \rightarrow \mathcal{A} \) such as

\[
\int_S \left\| \varphi(t) \right\| \left\| f(t) \right\|^2 d\mu(t)(+\infty)
\]

then, we have the following integral inequality

Corollary 4.4. For \( a \in L^2_\varphi(S, \Sigma, \mu, \mathcal{A}) \) such that \( a(t) \) is hermitian in \( \mathcal{A} \) for which \( t \in S \), and any \( f \in L^2_\varphi(S, \Sigma, \mu, \mathcal{A}) \), we have the inequality

\[
(4.10) \quad \left\| \int_S \varphi(t) a(t) f(t) d\mu(t) \right\|^2 \leq \left| \left( \int_S \varphi(t) a(t) f(t) d\mu(t) \right) \left( \int_S \varphi(t)^* f(t)^* d\mu(t) \right) \right| + 2 \left| \int_S \varphi(t) a(t) f(t) d\mu(t) \right| \left| \int_S \varphi(t)^* f(t) d\mu(t) \right| \leq \frac{1}{2} \left| \int_S \varphi(t) a(t) f(t) d\mu(t) \right| \left| \int_S \varphi(t) f(t)^* d\mu(t) \right| + \left| \int_S \varphi(t) f(t)^* d\mu(t) \right| \left| \int_S \varphi(t) f(t) d\mu(t) \right|
\]

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Ahmed Roukbi
Faculty of Science
Department of Mathematics and Informatics
Ibn Tofail University,
BP:14000. Kenitra, Morocco
rroukbi.a2000@gmail.com