# THE REFLEXIVE EXTREMAL RANK SOLUTIONS TO THE MATRIX EQUATION $A X=B$ 

Qingfeng Xiao, Xiyan Hu and Lei Zhang*


#### Abstract

This paper establishes some necessary and sufficient conditions for the existence of the reflexive extremal rank solutions to the matrix equation $A X=B$. Several representations for the reflexive solutions are given. Then, the explicit expression for the nearest matrix to a given matrix, that belongs to the corresponding minimal rank solution set of the equation, in the Frobenius norm has been provided. An algorithm provided for it and the numerical example show that the algorithm is practicable.


## 1. Introduction

Let us propose some notations that we will use in this paper. Let $\mathbb{C}^{m \times n}$ be the set of all $m \times n$ complex matrices, $\mathbb{U} \mathbb{C}^{n \times n}$ be the set of all $n \times n$ unitary complex matrices. Denote by $I_{n}$ the identity matrix with order $n$. For matrix $A, A^{*},\|A\|_{F}$ and $r(A)$ represent its conjugate transpose, Frobenius norm and rank, respectively.

Definition 1.1. A matrix $P \in \mathbb{C}^{n \times n}$ is said to be a generalized reflection matrix if $P^{*}=P, P^{2}=I_{n}$.

In this paper, without special statement, we assume that $P$ is a given generalized reflection matrix.

Definition 1.2. A matrix $X \in \mathbb{C}^{n \times n}$ is said to be a reflexive (anti-reflexive) if $X=P X P(X=-P X P)$.

The reflexive (anti-reflexive) matrix with respect to a generalized reflection matrix $P$ has many specific properties and widely used in engineering and scientific computations (see, e.g. [1-4]).

[^0]We know that investigating minimal ranks of matrix expressions has many immediate motivations in matrix analysis and applications. For example, the classical matrix equation $A X=B$ is consistent if and only if

$$
\min _{X} \operatorname{rank}(B-A X)=0
$$

The two consistent matrix equations $A_{1} X_{1} B_{1}=C_{1}, A_{2} X_{2} B_{2}=C_{2}$ where $X_{1}$ and $X_{2}$ have the same size, have a common solution if and only if

$$
\min _{X_{1}, X_{2}} \operatorname{rank}\left(X_{1}-X_{2}\right)=0
$$

In 1972, Mitra [5] considered solutions with fixed ranks for the matrix equations $A X=B$ and $A X B=C$. In 1984, Mitra [6] gave common solutions of minimal rank of the pair of complex matrix equations $A X=C, X B=D$. In 1987, Uhlig [11] presented the extremal ranks of solutions to the matrix equation $A X=B$. In 1990, Mitra studied the minimal ranks of common solutions to the pair of matrix equations $A_{1} X_{1} B_{1}=C_{1}$ and $A_{2} X_{2} B_{2}=C_{2}$ over a general field in [7]. In 2003, Tian ( $\left.[9,10]\right)$ investigated the extremal rank solutions to the complex matrix equation $A X B=C$ and gave some applications. Xiao et al. [12] in 2009 considered the symmetric minimal rank solution to equation $A X=B$. Recently, the anti-reflexive extremal rank solutions to the matrix equation $A X=B$ was derived by Xiao et al.[13]. In this paper, however, the similar problems for reflexive matrices will be considered. Thus we extend our previous results in [13].

In this paper, we consider the reflexive extremal rank solutions of the matrix equation

$$
\begin{equation*}
A X=B \tag{1.1}
\end{equation*}
$$

where $A$ and $B$ are given matrices in $\mathbb{C}^{m \times n}$.
We also consider the matrix nearness problem

$$
\begin{equation*}
\min _{X \in S_{m}}\|X-\tilde{X}\|_{F} \tag{1.2}
\end{equation*}
$$

where $\tilde{X}$ is a given matrix in $\mathbb{C}^{n \times n}$ and $S_{m}$ is the minimal rank solution set of Eq. (1.1).

The remainder of this paper is organized as follows: In Section 2, we firstly review some properties of the reflexive matrix, then we will propose decompositions of $A_{1}, A_{2}, B_{1}, B_{2}$. Some necessary and sufficient conditions for the existence of and the expressions for the reflexive extremal rank solutions to the matrix equation $A X=B$ are obtained by applying these decompositions and the generalized singular value decomposition(GSVD). In corresponding the minimal rank solution set of the equation, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm has been provided. Finally, an algorithm for computing the optimal approximate solution is given and a numerical example show that it is feasible. Also we give some brief concluding remarks in Section 3 to end this paper.

## 2. Main results

Lemma 2.1. ${ }^{[13]}$ Let $P$ be the $n \times n$ generalized reflection matrix. Then there exists an $n \times n$ unitary matrix $U$ such that

$$
P=U\left[\begin{array}{cc}
I_{r} & 0  \tag{2.1}\\
0 & -I_{n-r}
\end{array}\right] U^{H}
$$

Lemma 2.2. ${ }^{[8]}$ Let $A \in \mathbb{C}^{n \times n}$ and the spectral decomposition of the $n \times n$ generalized reflection matrix $P$ be given as (2.1). Then $A$ is reflexive if and only if

$$
A=U\left[\begin{array}{cc}
M & 0  \tag{2.2}\\
0 & N
\end{array}\right] U^{H}
$$

where $M \in \mathbb{C}^{r \times r}, N \in \mathbb{C}^{(n-r) \times(n-r)}$ are arbitrary.
Assume the given generalized reflection matrix $P$ has the form of (2.1). Let

$$
\begin{equation*}
A U=\left[A_{1}, A_{2}\right], \quad B U=\left[B_{1}, B_{2}\right] \tag{2.3}
\end{equation*}
$$

where $A_{1}, B_{1} \in \mathbb{C}^{m \times r}, A_{2}, B_{2} \in \mathbb{C}^{m \times(n-r)}$, and the generalized singular value decomposition of the matrix pair $\left[A_{1}, B_{1}\right],\left[A_{2}, B_{2}\right]$ are, respectively,

$$
\begin{align*}
& A_{1}=M_{1} \Sigma_{A_{1}} U_{1}, \quad B_{2}=M_{1} \Sigma_{B_{1}} V_{1}  \tag{2.4}\\
& A_{2}=M_{2} \Sigma_{A_{2}} U_{2}, \quad B_{2}=M_{2} \Sigma_{B_{2}} V_{2} \tag{2.5}
\end{align*}
$$

where $U_{1}, V_{1} \in \mathbb{U} \mathbb{C}^{r \times r}, U_{2}, V_{2} \in \mathbb{U} \mathbb{C}^{(n-r) \times(n-r)}$, nonsingular matrices $M_{1}, M_{2} \in$ $\mathbb{C}^{m \times m}, k_{1}=r\left(\left[A_{1}, B_{1}\right]\right), r_{1}=r\left(A_{1}\right), s_{1}=r\left(A_{1}\right)+r\left(B_{1}\right)-r\left(\left[A_{1}, B_{1}\right]\right)$, and $k_{2}=$ $r\left(\left[A_{2}, B_{2}\right]\right), r_{2}=r\left(A_{2}\right), s_{2}=r\left(A_{2}\right)+r\left(B_{2}\right)-r\left(\left[A_{2}, B_{2}\right]\right)$,

$$
\begin{aligned}
& \Sigma_{A_{1}}=\left[\begin{array}{ccc}
I & 0 & 0 \\
0 & S_{A_{1}} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \begin{array}{c}
r_{1}-s_{1} \\
s_{1} \\
k_{1}-r_{1} \\
m-k_{1}
\end{array}, \quad \Sigma_{B_{1}}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & S_{B_{1}} & 0 \\
0 & 0 & I \\
0 & 0 & 0
\end{array}\right] \begin{array}{c}
r_{1}-s_{1} \\
s_{1} \\
k_{1}-r_{1} \\
m-k_{1} \\
\Sigma_{A_{2}}=\left[\begin{array}{ccc}
I & 0 & 0 \\
0 & S_{A_{2}} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \begin{array}{c}
r_{2}-s_{2} \\
s_{2} \\
k_{2}-r_{2} \\
m-k_{2}
\end{array},
\end{array} \quad \Sigma_{B_{2}}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & S_{B_{2}} & 0 \\
0 & 0 & I \\
0 & 0 & 0
\end{array}\right] \begin{array}{c}
r_{2}-s_{2} \\
s_{2} \\
k_{2}-r_{2} \\
m-k_{2}
\end{array},
\end{aligned}
$$

Similarly to the proof of Theorem 2.6 in [13], we can prove the following theorem.
Theorem 2.1. Let $A, B \in \mathbb{C}^{m \times n}$ and a generalized reflection matrix $P$ of size $n$ be known. Suppose the generalized reflection matrix $P$ is represented by (2.1), $A U, B U$ have the partition forms of (2.3), and the generalized singular value decompositions
of the matrix pair $\left[A_{1}, B_{1}\right]$ and $\left[A_{2}, B_{2}\right]$ are given by (2.4) and (2.5). Then the equation (1.1) has a reflexive solution $X$ if and only if

$$
\begin{equation*}
r\left(\left[A_{1}, B_{1}\right]\right)=r\left(A_{1}\right), \quad r\left(\left[A_{2}, B_{2}\right]\right)=r\left(A_{2}\right) \tag{2.6}
\end{equation*}
$$

and its general solution can be expressed as

$$
X=U\left[\begin{array}{cc}
U_{1}^{*}\left[\begin{array}{cc}
0 & 0 \\
0 & S_{A_{1}}^{-1} S_{B_{1}} \\
Z_{31} & Z_{32}
\end{array}\right] V_{1} & 0 \\
& 0
\end{array}\right.
$$

where $Z_{31} \in \mathbb{C}^{\left(r-r_{1}\right) \times\left(r-s_{1}\right)}, Z_{32} \in \mathbb{C}^{\left(r-r_{1}\right) \times s_{1}}$, $W_{31} \in \mathbb{C}^{\left(n-r-r_{2}\right) \times\left(n-r-s_{2}\right)}$, $W_{32} \in$ $\mathbb{C}^{\left(n-r-r_{2}\right) \times s_{2}}$ are arbitrary.

The following theorem can be proved in a similar way as Theorem 3.1 in [13].
Theorem 2.2. Suppose that the matrix equation (1.1) has a reflexive solution $X$ and $\Omega$ is the set of all reflexive solutions of (1.1). Then the extreme ranks of $X$ are as follows:
(1) The maximal rank of $X$ is
$\max _{X \in \Omega} r(X)=\min \left\{r, r-r\left(A_{1}\right)+r\left(B_{1}\right)\right\}+\min \left\{n-r, n-r-r\left(A_{2}\right)+r\left(B_{2}\right)\right\}$.
and its general solution can be expressed as

$$
X=U\left[\begin{array}{cc}
U_{1}^{*}\left[\begin{array}{cc}
0 & 0 \\
0 & S_{A_{1}}^{-1} S_{B_{1}} \\
Z_{31} & Z_{32}
\end{array}\right] V_{1} & 0 \\
0 & U_{2}^{*}\left[\begin{array}{cc}
0 & 0 \\
0 & S_{A_{2}}^{-1} S_{B_{2}} \\
W_{31} & W_{32}
\end{array}\right] V_{2}
\end{array}\right] U^{*}
$$

where $Z_{31} \in \mathbb{C}^{\left(r-r_{1}\right) \times\left(r-s_{1}\right)}$, $W_{31} \in \mathbb{C}^{\left(n-r-r_{2}\right) \times\left(n-r-s_{2}\right)}$ are chosen such that $r\left(Z_{31}\right)=$ $\min \left(r-r_{1}, r-s_{1}\right), r\left(W_{31}\right)=\min \left(n-r-r_{2}, n-r-s_{2}\right), Z_{32} \in \mathbb{C}^{\left(r-r_{1}\right) \times s_{1}}$, $W_{32} \in \mathbb{C}^{\left(n-r-r_{2}\right) \times s_{2}}$ are arbitrary.
(2) The minimal rank of $X$ is

$$
\min _{X \in \Omega} r(X)=r\left(B_{1}\right)+r\left(B_{2}\right)
$$

and its general solution can be expressed as

$$
X=U\left[\begin{array}{cc}
U_{1}^{*}\left[\begin{array}{cc}
0 & 0 \\
0 & S_{A_{1}}^{-1} S_{B_{1}} \\
0 & Z_{32}
\end{array}\right] V_{1} & 0 \\
& 0
\end{array} \quad U_{2}^{*}\left[\begin{array}{cc}
0 & 0 \\
0 & S_{A_{2}}^{-1} S_{B_{2}} \\
0 & W_{32}
\end{array}\right] V_{2}\right] U^{*}
$$

where $Z_{32} \in \mathbb{C}^{\left(r-r_{1}\right) \times s_{1}}, W_{32} \in \mathbb{C}^{\left(n-r-r_{2}\right) \times s_{2}}$ are arbitrary.
The following result is analogous to the results given in Theorem 4.1 from [13].
Theorem 2.3. Let the matrix $\tilde{X}$ be given and the notations and conditions from Theorem 2.2 be used. Let us partition matrices $U^{*} \tilde{X} U, U_{1} \tilde{X}_{11} V_{1}^{*}, U_{2} \tilde{X}_{22} V_{2}^{*}$ as follows:

$$
\begin{gather*}
U^{*} \tilde{X} U=\left[\begin{array}{cc}
\tilde{X}_{11} & \tilde{X}_{12} \\
\tilde{X}_{21} & \tilde{X}_{22}
\end{array}\right], \quad \tilde{X}_{11} \in \mathbb{C}^{r \times r}, \quad \tilde{X}_{22} \in \mathbb{C}^{(n-r) \times(n-r)}  \tag{2.7}\\
U_{1} \tilde{X}_{11} V_{1}^{*}=\left[\begin{array}{cc}
\tilde{Z}_{11} & \tilde{Z}_{12} \\
\tilde{Z}_{21} & \tilde{Z}_{22} \\
\tilde{Z}_{31} & \tilde{Z}_{32}
\end{array}\right], U_{2} \tilde{X}_{22} V_{2}^{*}=\left[\begin{array}{cc}
\tilde{W}_{11} & \tilde{W}_{12} \\
\tilde{W}_{21} & \tilde{W}_{22} \\
\tilde{W}_{31} & \tilde{W}_{32}
\end{array}\right] \tag{2.8}
\end{gather*}
$$

If $S_{m}$ is nonempty, then the problem (1.2) has a unique matrix $\hat{X}$ which can be represented as

$$
\hat{X}=U\left[\begin{array}{cc}
U_{1}^{*}\left[\begin{array}{cc}
0 & 0 \\
0 & S_{A_{1}}^{-1} S_{B_{1}} \\
0 & Z_{32}
\end{array}\right] V_{1} & 0  \tag{2.9}\\
& 0
\end{array} U_{2}^{*}\left[\begin{array}{cc}
0 & 0 \\
0 & S_{A_{2}}^{-1} S_{B_{2}} \\
0 & W_{32}
\end{array}\right] V_{2}\right] U^{*}
$$

where $\tilde{Z}_{32}, \tilde{W}_{32}$ are the same as in (2.6).
Now we give the procedures to compute the optimal approximate solution $\hat{X}$ and a numerical example.

Algorithm 2.1. (1) Input $A ; B ; P$ and $\tilde{X}$.
(2) Compute $r$ and $U$ according to Lemma 2.1.
(3) Compute $A_{i}$ and $B_{i}(i=1,2)$ according to (2.3).
(4) Compute $M_{i}, \Sigma_{A_{i}}, \Sigma_{B_{i}}, U_{i}$ and $V_{i}(i=1,2)$ according to (2.3) and (2.4), respectively.
(4) If (2.6) holds, then continue; otherwise, go to (1).
(5) Compute $\tilde{X}_{11}$ and $\tilde{X}_{22}$ according to (2.7).
(5) Compute $\tilde{Z}_{32}$ and $\tilde{W}_{32}$ according to (2.8).
(7) Calculate $\hat{X}$ according to (2.9).

Example 2.1. The matrices $P, A, B$ and $\tilde{X}$ are given as follows:

$$
A=\left[\begin{array}{cccc}
0 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 3 & 11 & 1
\end{array}\right], \quad B=\left[\begin{array}{cccc}
-2.4983 & -0.6166 & 3.3831 & 0.5421 \\
-0.9158 & -0.4262 & 1.5737 & 0.1043 \\
-1.5826 & -0.1904 & 1.8094 & 0.4377 \\
-19.2549 & 1.9545 & 17.9534 & 5.1777
\end{array}\right]
$$

$$
P=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right], \quad \tilde{X}=\left[\begin{array}{llll}
0.7948 & 0.1730 & 0.8757 & 0.8939 \\
0.9568 & 0.9797 & 0.7373 & 0.1991 \\
0.5226 & 0.2714 & 0.1365 & 0.2987 \\
0.8801 & 0.2523 & 0.0118 & 0.6614
\end{array}\right]
$$

Using the software MATLAB, we obtain the unique solution $\hat{X}$ of the matrix nearness problem

$$
\hat{X}=\left[\begin{array}{cccc}
0.3334 & -0.1179 & -0.1179 & -0.1667 \\
-1.2492 & 1.6916 & -0.3083 & 0.2710 \\
-1.2492 & -0.3083 & 1.6916 & 0.2710 \\
-1.7666 & 0.2710 & 0.2710 & 1.3833
\end{array}\right]
$$

## 3. Conclusions

The reflexive matrix have wide applications in many fields. Also matrix equation has important applications in stability analysis, in observers design, in output regulation with internal stability, and in the eigenvalue assignment. Recently, the extremal rank solutions (i.e. solutions of maximal and minimal ranks), of some matrix equations have found many applications in control theory, statistics and economics. In this paper, we obtained the the maximal and minimal rank solutions with reflexive to the matrix equation $A X=B$. We also obtained, in corresponding minimal rank solution set, the nearest solution to a given matrix in Frobenius norm. The solvability conditions and the explicit formula for the solution are given. Furthermore, an algorithm provided for it and the numerical example illustrate derived results.

## REFERENCES

1. H. C. Chen and A. Sameh: Numerical linear algebra algorithms on the cedar system, in: A.K. Noor (Ed.), Parallel Computations and Their Impact on Mechanics. The American Society of Mechanical Engineers, AMD, 86 (1987) 101-125.
2. H. C. Chen: The SAS Domain Decomposition Method for Structural Analysis, CSRD Tech. Report 754, Center for Supercomputing Research and Development, University of Illinois, Urbana, IL, 1988.
3. H. C. Chen: Generalized reflexive matrices: special properties and applications. SIAM J. Matrix Anal. Appl., 19 (1998) 140-153.
4. J. L. Chen and X. H. Chen: Special Matrices. Qinghua University Press, Beijing, China, 2001(in Chinese).
5. S. K. Mitra: Fixed rank solutions of linear matrix equations. Sankhya Ser.A., 35 (1972) 387-392.
6. S. K. Mitra: The matrix equation $A X=C, X B=D$ : Linear Algebra Appl., 59 (1984) 171-181.
7. S. K. Mitra: $A$ pair of simultaneous linear matrix equations $A_{1} X_{1} B_{1}=$ $C_{1}, A_{2} X_{2} B_{2}=C_{2}$ and a matrix programming problem. Linear Algebra Appl., 131 (1990) 107-123.
8. Z. Y. Peng and X. Y. Hu: The reflexive and anti-reflexive solutions of the matrix equation $A X=B$. Linear Algebra Appl., 375 (2003) 147-155.
9. Y. G. Tian: Ranks of solutions of the matrix equation $A X B=C$. Linear Multilinear Algebra, 51(2) (2003) 111-125.
10. Y. G. Tian and S. Cheng: The maximal and minimal ranks of $A-B X C$ with applications. New York J. Math. 9 (2003) 45-362.
11. F. Uhlig: On the matrix equation $A X=B$ with applications to the generators of controllability matrix. Linear Algebra Appl., 85 (1987) 203-209.
12. Q. F. Xiao, X. Y. Hu and L. Zhang: The symmetric minimal rank solution of the matrix equation $A X=B$ and the optimal approximation. Electron. J. Linear Algebra, 18 (2009) 264-271.
13. Q. F. Xiao, X. Y. Hu and L. Zhang: The anti-reflexive extremal rank solutions of the matrix equation $A X=B$. Functional Analysis, Approximation and Computation, 4:2 (2012), 15-22.

Qingfeng Xiao<br>Dongguan Polytechnic<br>Department of Basic Courses<br>523808 Dongguan, China<br>qfxiao@hnu.edu.cn

Xiyan Hu
Hunan University
College of Mathematics and Econometrics
410082 Changsha, China

Lei Zhang
Hunan University
College of Mathematics and Econometrics
410082 Changsha, China


[^0]:    Received April 24, 2012.
    2010 Mathematics Subject Classification. Primary 15A29; Secondary 65F30, 65F35

    * The authors were supported in part by Scientific Research Fund of Dongguan Polytechnic (2011a15)

