FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 27, No 1 (2012), 67–82

NEW INEQUALITIES OF OSTROWSKI'S TYPE FOR $S-$CONVEX FUNCTIONS IN THE SECOND SENSE WITH APPLICATIONS

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Abstract. In this paper, we establish some new inequalities of Ostrowski's type for functions whose derivatives in absolute value are the class of s-convex. Some applications for special means of real numbers are also provided. Finally, some error estimates for the midpoint formula are obtained.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [12]

Theorem 1.1. *Let $f : I \subset [0, \infty] \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I , such that $f' \in L[a, b]$ where a , $b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

Recently, Ostrowski's inequality has been the subject of intensive research. In particular, many generalizations , improvements , and applications for the Ostrowski's inequality can be found in the literature ([1]-[3],[7]-[9],[12] and [14]) and the references therein.

In [1], Alomari and Darus obtained inequalities for differentiable convex mappings which are connected with Ostrowski's inequality, and they used the following lemma to prove them. We have corrected by writting $(a - b)$ instead of $(b - a)$ in the right side of this lemma.

Received May 10, 2012.
2010 *Mathematics Subject Classification.* Primary 26A51; Secondary 26D10
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Lemma 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)du = (a-b)\int_0^1 p(t) f'(ta + (1-t)b)dt \tag{1.2}$$

for each $t \in [0, 1]$, where

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{b-x}{b-a}\right] \\ t-1, & t \in \left(\frac{b-x}{b-a}, 1\right] \end{cases},$$

for all $x \in [a, b]$.

Definition 1.1. [4] A function $f : [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0,\infty)$, $\alpha \in [0, 1]$ and for some fixed $s \in (0, 1]$. This class of s-convex functions is usually denoted by K_s^2.

An $s-$convex function was introduced in Breckner's paper [4] and a number of properties and connections with $s-$convexity in the first sense are discussed in paper [5]. Of course, $s-$convexity means just convexity when $s = 1$.

In [6], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense:

Theorem 1.2. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0,\infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

In [2], Alomari et al. proved the following inequality of Ostrowski type for functions whose derivative in absolute value are $s-$convex in the second sense.

Theorem 1.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s-$convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a, b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{M}{(1+p)^{\frac{1}{p}}} \left(\frac{2}{s+1}\right)^{\frac{1}{q}} \left\{ \frac{(x-a)^2 + (b-x)^2}{(b-a)} \right\} \tag{1.4}$$

for each $x \in [a, b]$.

In [10], some inequalities of Hermite-Hadamard's type for differentiable convex mappings were presented as follows:

Theorem 1.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds,*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{4}\left[\frac{|f'(a)| + |f'(b)|}{2}\right]. \tag{1.5}$$

Theorem 1.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I^{\circ}$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then we have*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(|f'(a)|^{p/(p-1)} + 3\,|f'(b)|^{p/(p-1)}\right)^{(p-1)/p}\right. \\ & \left. + \left(3\,|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\right)^{(p-1)/p}\right]. \end{aligned} \tag{1.6}$$

Theorem 1.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I^{\circ}$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(|f'(a)| + |f'(b)|\right). \tag{1.7}$$

The main purpose of this paper is to establish new Ostrowski's type inequalities for the class of functions whose derivatives in absolute value at certain powers are $s-$convex in the second sense. Also, using these results we note some consequent applications to special means and to estimates of the error term in the midpoint formula.

2. Main Results

The next theorem gives a new result of the Ostrowski's inequality for s-convex functions:

Theorem 2.1. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is $s-$convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$
\begin{aligned}
&\leq \frac{b-a}{(s+1)(s+2)} \\
&\quad\times\left\{\left[2(s+1)\left(\frac{b-x}{b-a}\right)^{s+2}-(s+2)\left(\frac{b-x}{b-a}\right)^{s+1}+1\right]\left|f'(a)\right|\right. \\
&\quad\left.+\left[2(s+1)\left(\frac{x-a}{b-a}\right)^{s+2}-(s+2)\left(\frac{x-a}{b-a}\right)^{s+1}+1\right]\left|f'(b)\right|\right\}
\end{aligned}
$$

for each $x \in [a,b]$.

Proof. By Lemma 1.1 and since $|f'|$ is $s-$convex on $[a,b]$, then we have

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\quad+(b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left(t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|\right)dt \\
&\quad+(b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left(t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&= (b-a)\left\{\left|f'(a)\right|\int_0^{\frac{b-x}{b-a}} t^{s+1}dt+\left|f'(b)\right|\int_0^{\frac{b-x}{b-a}} t(1-t)^s dt\right. \\
&\quad\left.+\left|f'(a)\right|\int_{\frac{b-x}{b-a}}^1\left(t^s-t^{s+1}\right)dt+\left|f'(b)\right|\int_{\frac{b-x}{b-a}}^1(1-t)^{s+1}dt\right\} \\
&= \frac{b-a}{(s+1)(s+2)}\left\{\left[2(s+1)\left(\frac{b-x}{b-a}\right)^{s+2}-(s+2)\left(\frac{b-x}{b-a}\right)^{s+1}+1\right]\left|f'(a)\right|\right. \\
&\quad\left.+\left[s\left(\frac{x-a}{b-a}\right)^{s+2}-(s+2)\frac{b-x}{b-a}\left(\frac{x-a}{b-a}\right)^{s+1}+1\right]\left|f'(b)\right|\right\}
\end{aligned}
$$

where we use the facts that

$$
\int_0^{\frac{b-x}{b-a}} t^{s+1}dt=\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}
$$

$$
\int_0^{\frac{b-x}{b-a}} t(1-t)^s dt=\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}-\frac{1}{(s+1)}\left(\frac{x-a}{b-a}\right)^{s+1}+\frac{1}{(s+1)(s+2)}
$$

$$\int_{\frac{b-x}{b-a}}^{1}\left(t^{s}-t^{s+1}\right)dt=\frac{1}{(s+1)(s+2)}-\frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1}+\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}$$

$$\int_{\frac{b-x}{b-a}}^{1}(1-t)^{s+1}dt=\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}$$

which completes the proof. □

Corollary 2.1. *In Theorem 2.1, if we choose $x=\frac{a+b}{2}$, then we have the following midpoint inequality:*

$$\begin{aligned} &\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_{a}^{b}f(u)du\right| \\ \leq\ & \frac{b-a}{(s+1)(s+2)}\left(1-\frac{1}{2^{s+1}}\right)\left[|f'(a)|+|f'(b)|\right]. \end{aligned} \tag{2.1}$$

Remark 2.1. In Corollary 2.1, if $s=1$, then we have

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\leq\frac{(b-a)}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$

which is (1.5).

Theorem 2.2. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{q}$ is $s-$convex on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality holds:*

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right| \\ \leq\ & (b-a)\frac{1}{(p+1)^{\frac{1}{p}}}\frac{1}{(s+1)^{\frac{1}{q}}} \\ &\times\left\{\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}}\left(\left(\frac{b-x}{b-a}\right)^{s+1}|f'(a)|^{q}+\left[1-\left(\frac{x-a}{b-a}\right)^{s+1}\right]|f'(b)|^{q}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}}\left(\left[1-\left(\frac{b-x}{b-a}\right)^{s+1}\right]|f'(a)|^{q}+\left(\frac{x-a}{b-a}\right)^{s+1}|f'(b)|^{q}\right)^{\frac{1}{q}}\right\} \end{aligned}$$

for each $x\in[a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Suppose that $p>1$. From Lemma 1.1 and using the Hölder inequality, we have

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right| \tag{2.2}$$

$$
\begin{aligned}
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\quad +(b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad +(b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Using the $s-$convexity of $|f'|^q$, we obtain

$$
\begin{aligned}
&\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt \qquad (2.3)\\
&\leq \int_0^{\frac{b-x}{b-a}}\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt \\
&= \frac{1}{s+1}\left\{\left(\frac{b-x}{b-a}\right)^{s+1}\left|f'(a)\right|^q+\left[1-\left(\frac{x-a}{b-a}\right)^{s+1}\right]\left|f'(b)\right|^q\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt \qquad (2.4)\\
&\leq \int_{\frac{b-x}{b-a}}^1\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt \\
&= \frac{1}{s+1}\left\{\left[1-\left(\frac{b-x}{b-a}\right)^{s+1}\right]\left|f'(a)\right|^q+\left(\frac{x-a}{b-a}\right)^{s+1}\left|f'(b)\right|^q\right\}.
\end{aligned}
$$

Further, we have

$$
\int_0^{\frac{b-x}{b-a}} t^p dt=\frac{1}{(p+1)}\left(\frac{b-x}{b-a}\right)^{p+1} \qquad (2.5)
$$

and

$$
\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt=\frac{1}{(p+1)}\left(\frac{x-a}{b-a}\right)^{p+1}. \qquad (2.6)
$$

A combination of (2.3)-(2.6) gives the required inequality (2.2). □

Remark 2.2. In Theorem 2.2, if we choose $x=\frac{a+b}{2}$ and $s=1$, then we have

$$
\begin{aligned}
&\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(\left|f'(a)\right|^q+3\left|f'(b)\right|^q\right)^{1/q}+\left(3\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right]
\end{aligned}
$$

which is (1.6).

Theorem 2.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p>1$, then the following inequality holds:*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\ \leq \; & \frac{1}{(b-a)} \frac{1}{(p+1)^{\frac{1}{p}}} \\ & \times \left\{ (b-x)^2 \left(\frac{|f'(x)|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} + (x-a)^2 \left(\frac{|f'(a)|^q + |f'(x)|^q}{s+1} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

for each $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $p>1$. From Lemma 1.1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\ \leq \; & (b-a) \int_0^{\frac{b-x}{b-a}} t \left| f'(ta + (1-t)b) \right| dt \\ & + (b-a) \int_{\frac{b-x}{b-a}}^1 |t-1| \left| f'(ta+(1-t)b) \right| dt \\ \leq \; & (b-a) \left(\int_0^{\frac{b-x}{b-a}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} \left| f'(ta+(1-t)b) \right|^q dt \right)^{\frac{1}{q}} \\ & + (b-a) \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 \left| f'(ta+(1-t)b) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is $s-$convex, by (1.3) we have

$$\int_0^{\frac{b-x}{b-a}} \left| f'(ta+(1-t)b) \right|^q dt \leq \frac{b-x}{b-a} \left(\frac{|f'(x)|^q + |f'(b)|^q}{s+1} \right) \tag{2.7}$$

and

$$\int_{\frac{b-x}{b-a}}^1 \left| f'(ta+(1-t)b) \right|^q dt \leq \frac{x-a}{b-a} \left(\frac{|f'(a)|^q + |f'(x)|^q}{s+1} \right). \tag{2.8}$$

Therefore,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{1}{(b-a)}\frac{1}{(p+1)^{\frac{1}{p}}}$$
$$\times\left\{(b-x)^2\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+(x-a)^2\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Also, we note that

$$\int_0^{\frac{b-x}{b-a}} t^p dt = \frac{1}{p+1}\left(\frac{b-x}{b-a}\right)^{p+1}$$

and

$$\int_{\frac{b-x}{b-a}}^1 (1-t)^p\, dt = \frac{1}{p+1}\left(\frac{x-a}{b-a}\right)^{p+1}.$$

This completes the proof. □

Remark 2.3. We choose $|f'(x)| \leq M$, $M>0$ in Theorem 2.3, then we recapture the inequality (1.4).

Corollary 2.2. *In Theorem 2.3, if we choose $x=\frac{a+b}{2}$, then*

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{b-a}{4}\frac{1}{(p+1)^{\frac{1}{p}}}\left\{\left(\frac{\left|f'(\frac{a+b}{2})\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+\left|f'(\frac{a+b}{2})\right|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.3. *In Corollary 2.2, if we choose $f'(a)=f'\left(\frac{a+b}{2}\right)=f'(b)$ and $s=1$, then*

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\frac{|f'(b)|+|f'(a)|}{4}\right). \tag{2.9}$$

Remark 2.4. We note that the obtained midpoint inequality (2.9) is better than the inequality (1.7).

Theorem 2.4. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is $s-$convex on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(b-a)}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{b-x}{b-a}\right)^{p+1}+\left(\frac{x-a}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}} \tag{2.10}$$

for each $x\in[a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Suppose that $p>1$. From Lemma 1.1 and using the Hölder inequality, we have

$$\begin{aligned}(2.11)\qquad &\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq (b-a)\left(\int_0^1 |p(t)|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(ta+(1-t)\,b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is $s-$convex, we have

$$\begin{aligned}\int_0^1 \left|f'\left(ta+(1-t)\,b\right)\right|^q dt &\leq \int_0^1 \left(t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right)dt \\ (2.12)\qquad &= \frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{s+1}\end{aligned}$$

and

$$\begin{aligned}\int_0^1 |p(t)|^p\, dt &= \int_0^{\frac{b-x}{b-a}} t^p dt+\int_{\frac{b-x}{b-a}}^1 (1-t)^p\, dt \\ (2.13)\qquad &= \frac{1}{p+1}\left[\left(\frac{b-x}{b-a}\right)^{p+1}+\left(\frac{x-a}{b-a}\right)^{p+1}\right].\end{aligned}$$

Using (2.12) and (2.13) in (2.11), we obtain (2.10). □

Corollary 2.4. *Under assumptation in Theorem 2.4 with $p=q=2$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{(b-a)}{\sqrt{3}}\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]^{\frac{1}{2}}\left(\frac{|f'(a)|^2+|f'(b)|^2}{s+1}\right)^{\frac{1}{2}}.$$

Corollary 2.5. *In Corollary 2.4, if we choose $x=\frac{a+b}{2}$ and $s=1$, then we have the following midpoint inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{(b-a)}{2}\left(\frac{|f'(a)|^2+|f'(b)|^2}{6}\right)^{\frac{1}{2}}.$$

Remark 2.5. We choose $|f'(x)|\leq M$, $M>0$ and $s=1$ in Corollary 2.4, then we recapture the following Ostrowski's type inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{M\,(b-a)}{\sqrt{3}}\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]^{\frac{1}{2}}.$$

Theorem 2.5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
\leq\ & (b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{b-x}{b-a}\right)^{2(1-1/q)} \left[\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} |f'(a)|^q \right.\right. \\
&\left.\left. + \left(\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)} \right) |f'(b)|^q \right]^{\frac{1}{q}} \right\} \\
& + (b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{x-a}{b-a}\right)^{2(1-1/q)} \left[\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2} |f'(b)|^q \right.\right. \\
&\left.\left. + \left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)} \right) |f'(a)|^q \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

for each $x \in [a,b]$.

Proof. Suppose that $q \geq 1$. From Lemma 1.1 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
\leq\ & (b-a)\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|\,dt \\
& + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\,|f'(ta+(1-t)b)|\,dt \\
\leq\ & (b-a)\left(\int_0^{\frac{b-x}{b-a}} tdt\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
& + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)^{1-\frac{1}{q}} \left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is $s-$convex, we have

$$\begin{aligned}
&\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|^q\,dt \\
\leq\ & \int_0^{\frac{b-x}{b-a}} t\left[t^s|f'(a)|^q + (1-t)^s|f'(b)|^q\right]dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\left|f'(a)\right|^{q}+\left[\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\right. \\
&\quad \left. -\frac{1}{(s+1)}\left(\frac{x-a}{b-a}\right)^{s+1}+\frac{1}{(s+1)(s+2)}\right]\left|f'(b)\right|^{q}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{b-x}{b-a}}^{1}(1-t)\left|f'(ta+(1-t)b)\right|^{q}dt \\
\leq\ & \int_{\frac{b-x}{b-a}}^{1}(1-t)\left[t^{s}\left|f'(a)\right|^{q}+(1-t)^{s}\left|f'(b)\right|^{q}\right]dt \\
=\ & \left(\frac{1}{(s+1)(s+2)}-\frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1}+\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\right)\left|f'(a)\right|^{q} \\
& +\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\left|f'(b)\right|^{q}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right| \\
\leq\ & (b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{b-x}{b-a}\right)^{2(1-1/q)}\left[\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\left|f'(a)\right|^{q}\right.\right. \\
& \left.\left.+\left(\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}-\frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1}+\frac{1}{(s+1)(s+2)}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right\} \\
& +(b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{x-a}{b-a}\right)^{2(1-1/q)}\left[\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\left|f'(b)\right|^{q}\right.\right. \\
& \left.\left.+\left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}-\frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1}+\frac{1}{(s+1)(s+2)}\right)\left|f'(a)\right|^{q}\right]^{\frac{1}{q}}\right\}
\end{aligned}
$$

which is required. $\square$

Corollary 2.6. *In Theorem 2.5, if we choose $x=\frac{a+b}{2}$ and $s=1$, then we have*

$$
\begin{aligned}
&\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_{a}^{b}f(u)du\right| \\
\leq\ & \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}+\left(3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

3. Applications To Special Means

Let $0 < s < 1$ and $u, v, w \in \mathbb{R}$. We define a function $f : [0, \infty) \to \mathbb{R}$

$$f(t) = \begin{cases} u & if \quad t = 0 \\ vt^s + w & if \quad t > 0. \end{cases}$$

If $v \geq 0$ and $0 \leq w \leq u$, then $f \in K_s^2$ (see [5]). Hence, for $u = w = 0$, $v = 1$, we have $f : [0,1] \to [0,1]$, $f(t) = t^s$, $f \in K_s^2$.

As in [15], we shall consider the means for arbitrary positive real numbers a, b, $a \neq b$. We take

(1) The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2},$$

(2) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & if \quad a = b \\ \frac{b-a}{\ln b - \ln a} & if \quad a \neq b \end{cases}.$$

(3) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \quad p \in \mathbb{R} \diagdown \{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality

$$L \leq A.$$

Now, using the results of Section 2, we give some applications to special means of positive real numbers.

Proposition 3.1. *Let $0 < a < b$ and $s \in (0,1)$. Then we have*

$$|A^s(a,b) - L_s^s(a,b)| \leq (b-a)\frac{s}{(s+1)(s+2)}\left(1 - \frac{1}{2^{s+1}}\right)\left[a^{s-1} + b^{s-1}\right].$$

Proof. The inequality follows from (2.1) applied to the $s-$convex function in the second sense $f : [0,1] \to [0,1]$, $f(x) = x^s$. The details are omitted. $\square$

Proposition 3.2. *Let* $0 < a < b$ *and* $s \in (0,1)$. *Then we have*

$$\begin{aligned} & \left|A^s(a,b) - L_s^s(a,b)\right| \\ \leq \; & s\frac{(b-a)}{4}\frac{1}{(p+1)^{1/p}}\left[\left(\frac{A^{q(s-1)}+b^{q(s-1)}}{s+1}\right)^{1/q}+\left(\frac{a^{q(s-1)}+A^{q(s-1)}}{s+1}\right)^{1/q}\right]. \end{aligned}$$

Proof. The proof is similar to that of Proposition 3.1, using Corollary 2.2. □

Proposition 3.3. *Let* $0 < a < b$ *and* $s \in (0,1)$. *Then we have*

$$\begin{aligned} & \left|A^s(a,b) - L_s^s(a,b)\right| \\ \leq \; & s\frac{(b-a)}{8}\left(\frac{2}{3}\right)^{\frac{1}{q}}\left\{\left[A\left(a^{q(s-1)},3b^{q(s-1)}\right)\right]^{1/q}+\left[A\left(3a^{q(s-1)},b^{q(s-1)}\right)\right]^{1/q}\right\}. \end{aligned}$$

Proof. The proof is similar to that of Proposition 3.1, using Corollary 2.6. □

4. The Midpoint Formula

As in [11] and [13], let d be a division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a,b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = T(f,d) + E(f,d) \tag{4.1}$$

where

$$T(f,d) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)$$

is the midpoint version and $E(f,d)$ denotes the associated approximation error.

In the following, we propose some new estimates for midpoint formula.

Proposition 4.1. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is convex on* $[a,b]$ *and* $p > 1$, *then in (4.1), for every division* d *of* $[a,b]$, *the midpoint error satisfy*

$$|E(f,d)| \leq \frac{1}{4(p+1)^{\frac{1}{p}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_{i+1})| + |f'(x_i)|\right].$$

Proof. On applying Corollary 2.3 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the division , we get

$$\left|(x_{i+1}-x_i)f\left(\frac{x_i+x_{i+1}}{2}\right) - \int_{x_i}^{x_{i+1}} f(x)dx\right| \leq \frac{(x_{i+1}-x_i)^2}{(p+1)^{1/p}}\left[\frac{|f'(x_{i+1})|+|f'(x_i)|}{4}\right].$$

Summing over i from 0 to $n-1$ and taking into account that $|f'|$ is convex, we obtain, by the triangle inequality, that

$$\begin{aligned}
&\left|\int_a^b f(x)dx - T(f,d)\right| \\
= &\left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right| \\
\leq &\sum_{i=0}^{n-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right| \\
\leq &\frac{1}{4(p+1)^{1/p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_{i+1})|+|f'(x_i)|\right]
\end{aligned}$$

which is completed the proof. □

Proposition 4.2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$, then in (4.1), for every division d of $[a,b]$, the midpoint error satisfy*

$$|E(f,d)| \leq \frac{1}{2\sqrt{6}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_{i+1})|^2+|f'(x_i)|^2\right]^{1/2}.$$

Proof. The proof uses Corollary 2.5 and is similar to that of Proposition 4.1. □

Proposition 4.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ and $q \geq 1$, then in (4.1), for every division d of $[a,b]$, the midpoint error satisfy*

$$\begin{aligned}
&|E(f,d)| \\
&\leq \frac{1}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left(|f'(x_i)|^q+3|f'(x_{i+1})|^q\right)^{\frac{1}{q}}+\left(3|f'(x_i)|^q+|f'(x_{i+1})|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The proof uses Corollary 2.6 and is similar to that of Proposition 4.1. □

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