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APPLYING THE ALGORITHM OF LAGRANGE MULTIPLIERS IN DIGITAL IMAGE RESTORATION *

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Abstract. A method for digital image restoration, based on the algorithm of Lagrange multipliers, has many practical applications. We apply the method to remove blur in an image caused by uniform linear motion. This method assumes that linear motion corresponds to an integral number of pixels. The main contributions of the method were found on the Improvement in Signal to Noise Ration (ISNR) that have been increase significantly compared to the classic techniques, also the parameter Mean Square Error (MSE) has lower values and computational time that has been decreased considerably with respect to the other methods. We give an implementation in the MATLAB programming package.

1. Introduction

Motion blur is an effect you will see in photographs of scenes where objects are moving. It is mostly noticeable when the exposure is long, or if objects in the scene are moving rapidly. The field of image restoration is concerned with the reconstruction or estimation of the uncorrupted image from a blurred one. Essentially, it tries to perform an operation on the image that is the inverse of the imperfections in the image formation system. In the use of image restoration methods, the characteristics of the degrading system are assumed to be known a priori. This paper present a method for removing blur from recorded sampled images. There are many excellent overview articles, journal papers, and textbooks on the subject of image restoration and identification [1, 2, 8, 10, 11].

The method, based on algorithm of Lagrange multipliers, is applied for the removal of blur in an image caused by uniform linear motion. This method assumes that linear motion corresponds to an integral number of pixels. For comparison, we used two commonly used filters from the collection of least-squares filters, namely

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Wiener filter and the constrained least-squares filter [2, 3]. Also we used in comparison the iterative nonlinear restoration based on the Lucy-Richardson algorithm [5, 6].

This paper is organized as follows. In the second section we present process of image formation and problem formulation. In Section 3 we describe a method for the restoration of the blurred image. We observe certain enhancement in the parameters: $ISNR$ and MSE, compared with other standard methods for image restoration, which is confirmed by the numerical examples reported in the last section.

2. Image Formation Process

We assume that the blurring function acts as a convolution kernel or point-spread function $h(n_1, n_2)$ and the image restoration methods that are described here fall under the class of linear spatially invariant restoration filters. It is also assumed that the statistical properties (mean and correlation function) of the image do not change spatially. Under these conditions the restoration process can be carried out by means of a linear filter of which the point-spread function (PSF) is spatially invariant. These modeling assumptions can be mathematically formulated as follows. If we denote by $f(n_1, n_2)$ the desired ideal spatially discrete image that does not contain any blur or noise, then the recorded image is modeled as [3]:

$$\begin{aligned} g(n_1, n_2) &= h(n_1, n_2) * f(n_1, n_2) \\ &= \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{M-1} h(k_1, k_2) f(n_1 - k_1, n_2 - k_2). \end{aligned} \tag{2.1}$$

Mathematical model of this process can be simply described as a matrix equation. Suppose that $F \in \mathbb{R}^{r \times n}$ is the matrix corresponding to the original image and $G \in \mathbb{R}^{r \times m}$, with picture elements $f_{i,j}$ for $i = 1, \dots, r,\ j = 1, \dots, n$, is the matrix corresponding to the degraded image. Let us denote the degradation matrix by $H \in \mathbb{R}^{m \times n}$. For each row of the matrices F and G we consider an equation of the form:

$$g = Hf, \quad g \in \mathbb{R}^m,\ f \in \mathbb{R}^n,\ H \in \mathbb{R}^{m \times n}, \tag{2.2}$$

where $f = f_i^T$ and f_i corresponds to ith row of the original image F, $g = g_i^T$ and g_i corresponds to ith row of the blurred image G. The procedure is repeated for each row of the degraded image. In this way, we describe an underdetermined system of m simultaneous equations (one for each element of the vector g) and $n = m + l - 1$ unknowns (one for each element of the vector f).

From the previous presumptions, it is easy to determine the PSF and consequently, imposing zero boundary conditions, the degradation matrix H. Let l be an integer indicating the length of the linear motion blur in pixels.

In practice the degradation (index l) is rarely known exactly, and the index l must be identified from the blurred image itself. To estimate the index l, two different cepstral methods can be used: one dimensional or two dimensional cepstral method [9].

The problem of restoring an image that has been blurred by uniform linear motion, usually results of camera panning or fast object motion can be expressed as, consists of solving the underdetermined system of the form (2.2).

Arbitrary ith row of the blurred image can be expressed using ith row of the original image as:

$$\begin{bmatrix} g_{i,1} \\ g_{i,2} \\ g_{i,3} \\ \vdots \\ g_{i,m} \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_l & 0 & 0 & 0 & 0 \\ 0 & h_1 & \cdots & h_l & 0 & 0 & 0 \\ 0 & 0 & h_1 & \cdots & h_l & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_1 & \cdots & h_l \end{bmatrix} \begin{bmatrix} f_{i,1} \\ f_{i,2} \\ f_{i,3} \\ \vdots \\ f_{i,n} \end{bmatrix}. \tag{2.3}$$

The element of matrix H are defined by $h_i = 1/l$ for $i = 1, \ldots, l$.

The objective is to estimate an original, row per row F (contained in the vectors f_i^T), given each row of a blurred image G (contained in the vectors g_i^T) and a priori knowledge of the degradation phenomenon H. The matrix $G \in \mathbb{C}^{r\times m}$ as a simulated blurred image can be calculated by

$$g_{i,j} = \frac{1}{l}\sum_{k=0}^{l-1} f_{i,j+k},\ i = 1, \ldots, r, j = 1, \ldots, m. \tag{2.4}$$

Equation (2.4) can be written in matrix form as

$$G = \left(HF^T\right)^T = FH^T,\ \ G \in \mathbb{R}^{r\times m},\ H \in \mathbb{R}^{m\times n},\ F \in \mathbb{R}^{r\times n}. \tag{2.5}$$

There is an infinite number of exact solutions for f that satisfy the equation (2.2) or (2.5), an additional criterion that find a sharp restored matrix is required.

The matrix form of the vertical motion blurring process is given by

$$G = HF,\ \ G \in \mathbb{R}^{m\times n},\ H \in \mathbb{R}^{m\times r},\ F \in \mathbb{R}^{r\times n}, r = m + l - 1. \tag{2.6}$$

Let us first consider a case where the blurring of the columns in the image is independent of the blurring of the rows. When this is the case, then there exist two matrices H_c and H_r , such that we can express the relation between the original and blurred images as:

$$G = H_c F_r^T,\ \ G \in \mathbb{R}^{m_1\times m_2},\ H_c \in \mathbb{R}^{m_1\times r},\ F \in \mathbb{R}^{r\times n},\ H_r \in \mathbb{R}^{m_2\times n}, \tag{2.7}$$

where $n = m_2 + l_1 - 1$, $r = m_1 + l_2 - 1$, l_1 is linear horizontal motion blur in pixels and l_2 is linear vertical motion blur in pixels.

3. Method for the Image Restoration

We illustrate a method which is developed based on the algorithm of Lagrange multipliers [4, 12]. The main purpose of the method is to remove the blur solution in images caused by uniform linear motion. This method assumes that linear movement corresponds to an integral number of pixels. Resolution of the restored image remains a very high level. The main contributions of the method are: increasing of $ISNR$, decreasing of MSE and reduction of computation time compared to other methods.

Solution also is defined as the vector in the solution space of the underdetermined system $g = Hf$ (2.2) whose first m components has the minimum distance to the measured data, i.e. $\|\hat{f} - g\| \to \min$, where $\hat{f}$ are the first m elements of f. We can express vector $\hat{f}$ as $\hat{f} = Pf$, with P a $m \times n$ matrix which projects the vector f on the support of g [4, 12] as in the following

$$P = \left[\begin{array}{c|c} I_m & O \end{array}\right], \tag{3.1}$$

where O denotes $m \times (l-1)$ zero block.
The original optimization problem is now defined as

$$\min_f \|Pf - g\|, \tag{3.2}$$

subject to constraint $\|Hf - g\|^2 = 0$. Applying the technique of Lagrange multipliers, this problem can be alternatively formulated as an optimization problem without constraints:

$$V(f) = \lambda\|Hf - g\|^2 + \|Pf - g\|^2 \to \min. \tag{3.3}$$

If λ is large enough, the solution of this problem is easy computing the partial derivative of criterion V respect to the unknown f:

$$\frac{\partial}{\partial f} V(f) = 2\lambda H^T(Hf - g) + 2P^T(Pf - g) = 0 \tag{3.4}$$

$$\hat{f} = \left(\lambda H^T H + P^T P\right)^{-1} (\lambda H + P)^T g. \tag{3.5}$$

Matrix form of the solution of (3.5) is:

$$\hat{F} = G\,(\lambda H + P)\left((\lambda H^T H + P^T P)^{-1}\right)^T. \tag{3.6}$$

The equation (3.6) is solution for the restored image when we have horizontal blurring. In the case of process of vertical blurring solution for the restored image, taking into account equations (2.6) and (3.5), is:

$$\hat{F} = \left(\lambda H^T H + P^T P\right)^{-1} (\lambda H + P)^T G. \tag{3.7}$$

When we have a separable two-dimensional blurring process, the restored image is given by:

$$\hat{F} = \left(\lambda H^T H + P^T P\right)^{-1} (\lambda H + P)^T G[\lambda H + P] \left((\lambda H^T H + P^T P)^{-1}\right)^T . \tag{3.8}$$

4. Numerical Results

In this section we have tested the method based on Lagrange multipliers of images and present numerical results and compare with two standard methods for image restoration called least-squares filters: Wiener filter and constrained least-squares filter and the iterative method called Lucy-Richardson algorithm. In the tests we use the value of $\lambda = 10^6$.

The experiments have been performed using MATLAB programming language on an Intel(R) Core(TM)2 Duo CPU T5800 @ 2.00 GHz 32-bit system with 2 GB of RAM memory running on the Windows Vista Business Operating System.

In image restoration the improvement in quality of the restored image over the recorded blurred one is measured by the signal-to-noise ratio (*SNR*) improvement is defined as follows in decibels:

$$ISNR = 10 \log_{10} \left(\frac{\sum_{n1,n2} (G(n_1, n_2) - F(n_1, n_2))^2}{\sum_{n1,n2} (\tilde{F}(n_1, n_2) - F(n_1, n_2))^2} \right). \tag{4.1}$$

The improvement in *SNR* is basically a measure that expresses the reduction of disagreement with the ideal image when comparing the distorted and restored image. Note that all of the above signal-to-noise measures can only be computed in case the ideal image is available, i.e., in an experimental setup or in a design phase of the restoration algorithm.

The simplest and most widely used full-reference quality metric is the mean squared error (MSE) [13], computed by averaging the squared intensity differences of restored and reference image pixels, along with the related quantity of peak signal-to-noise ratio ($PSNR$). These are appealing because they are simple to calculate, have clear physical meanings, and are mathematically convenient in the context of optimization. The advantages of MSE and $PSNR$ are that they are very fast and easy to implement. However, they simply and objectively quantify the error signal. With $PSNR$ greater values indicate greater image similarity, while with MSE greater values indicate lower image similarity. Below MSE, $PSNR$ are defined:

$$MSE = \frac{1}{rm} \sum_{i=1}^{r} \sum_{j=1}^{m} |f_{i,j} - \hat{f}_{i,j}|^2, \tag{4.2}$$

$$PSNR = 20 \log_{10} \left(\frac{MAX}{\sqrt{MSE}} \right) (dB), \tag{4.3}$$

where MAX is the maximum pixel value.

4.1. Restoration of images blurred from horizontal motion

The X-ray image making provides a crucial method of diagnostic by using the image analysis. Figure 4.1, Original Image, shows such a deterministic original X-ray image. Figure 4.1, Degraded Image, presents the degraded X-ray image for $l = 40$. Finally, from Figure 4.1, Lagrange multipliers Image, Wiener Restored Image, Constrained LS Restored Image and Lucy-Richardson Restored Image, it is clearly seen that the details of the original image have been recovered. These figures demonstrate four different methods of restoration, the method of Lagrange multipliers, Wiener filter, Constrained least-squares (LS) filter, and Lucy-Richardson algorithm, respectively.

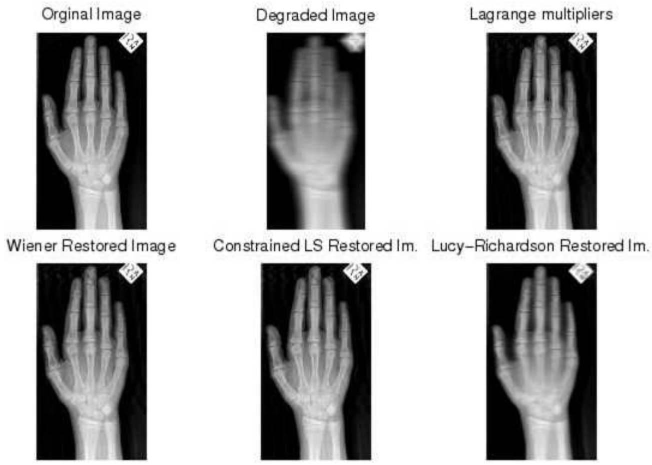

Figure 4.1. Restoration in simulated degraded X-ray image for length of the blurring process, $l = 40$

The difference in quality of restored images between the three methods is insignificant, and can hardly be seen by human eye. For this reason, the $ISNR$ and MSE have been chosen in order to compare the restored images obtained by the proposed method, the Wiener filter methods, the Constrained least-squares filter method and the Lucy-Richardson algorithm. Figure 4.2 shows the corresponding $ISNR$ and MSE value for restored images as a function of l for the proposed method and the mentioned classical methods.

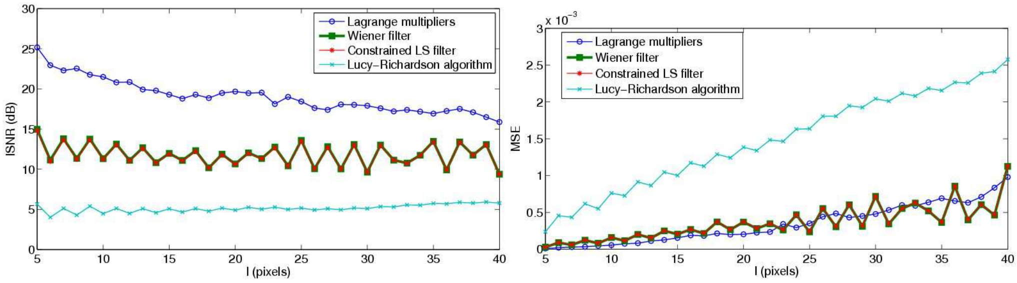

Figure 4.2. (left) Improvement in signal-to-noise-ratio vs. length of the blurring process (right) Mean squared error vs. length of the blurring process in pixels.

The figures illustrate that the quality of the restoration is as satisfactory as the classical methods or better from them ($l < 40$ pixels). Realistically speaking, large motions do not occur frequently in radiography.

4.2. Restoration of images blurred from vertical motion

We can consider another practical example with images of ANPR (Automatic Number Plate Recognition) system. Results are shown concerning when we have vertical blurring of images.

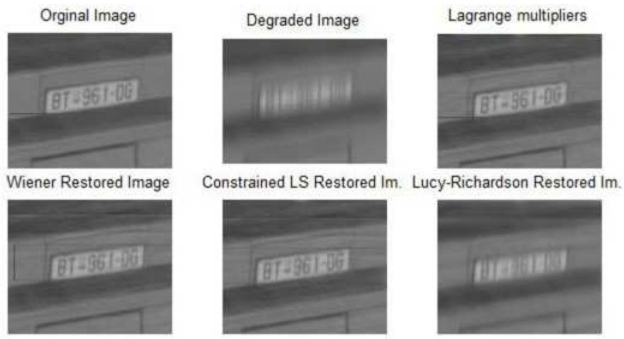

Figure 4.3. Restoration in vertical degraded image for length $l = 40$ of the blurring

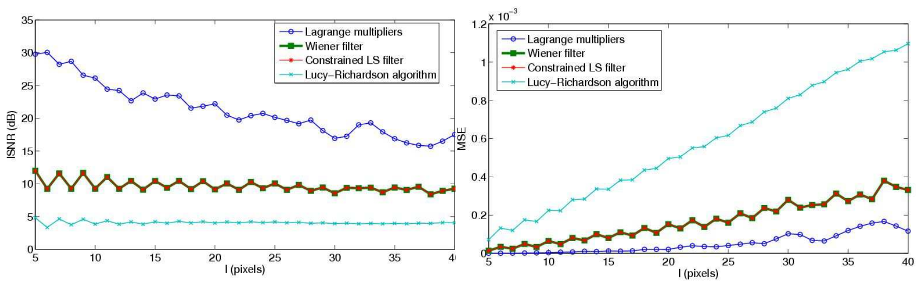

Figure 4.4. (left) Improvement in signal-to-noise-ratio vs. length of the blurring process (right) Mean squared error vs. length of the blurring process in pixels.

4.3. Restoration of images blurred from separable two-dimensional motion

The results present in Figure 4.5, and Figure 4.6 refer when we have separable two-dimensional blurring process of X-ray image.

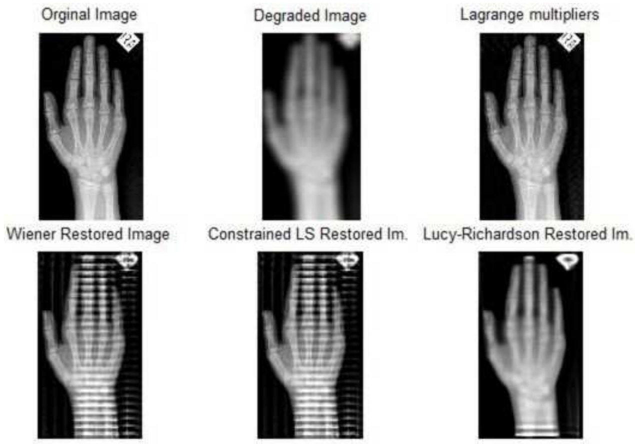

Figure 4.5. Restoration in degraded X-ray image for length of the blurring $l_1 = 35$ and $l_2 = 25$

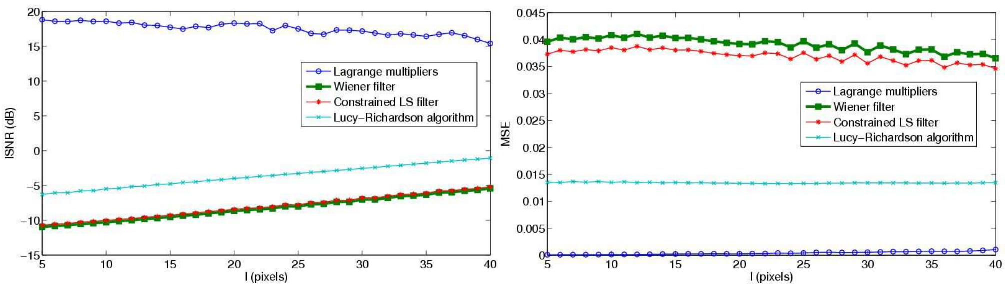

Figure 4.6. (left) Improvement in signal-to-noise-ratio vs.length of the blurring process in pixels $l_1 = 5, 6, ..., 40$, $l_2 = 20$
(right) Mean squared error vs.length of the blurring process in pixels $l_1 = 5, 6, \ldots, 40$, $l_2 = 20$.

On the next two figures are shown restoration of an ANPR image in case of two-dimensional blurring of the image.

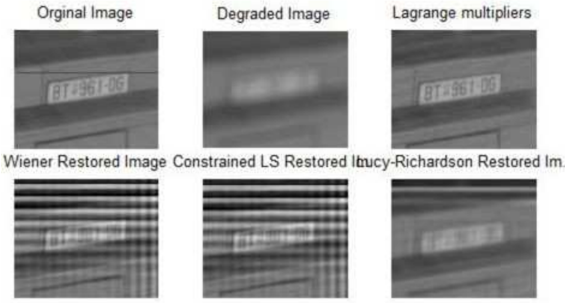

Figure 4.7. Restoration in simulated degraded ANPR image for length of the blurring process $l_1 = 25$ and $l_2 = 30$

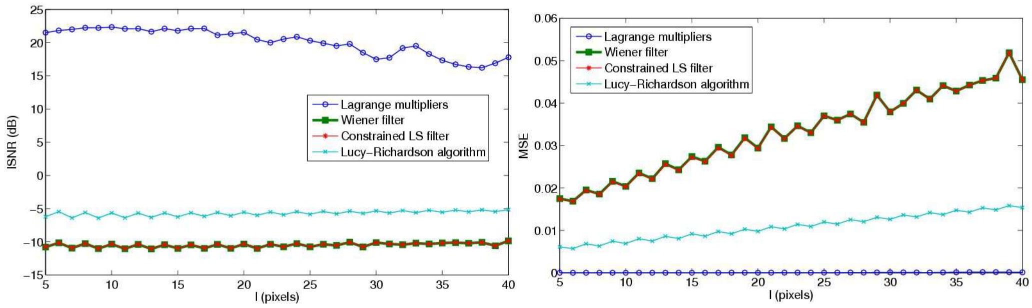

Figure 4.8. (left) Improvement in signal-to-noise-ratio vs.length of the blurring process in pixels $l_1 = 30$, $l_2 = 5, 6, ..., 40$
(right) Mean squared error vs.length of the blurring process in pixels $l_1 = 30$, $l_2 = 5, 6, ..., 40$.

From these figures related to separable two-dimensional motion, we can conclude that with using classical methods of image restoration, from the resulting image we cannot recognize number plate of the vehicle. Examples related to ANPR images lead to the same conclusions to which we were when we had X-ray images. The proposed method has better results for parameters $ISNR$ and MSE compared with other methods.

4.4. Motion blur estimation

The blurred image is modeled as a convolution between the original image and an unknown point-spread function. To estimate the extent of the motion blur, 2-D cepstral methods are employed. The cepstrum of a blurred image shows two significant negative peaks at a distance l from the origin. An estimate for the length of motion blur is this value l. We have to convert the image to Cepstrum domain. This is how we represent Cepstrum Domain [9]:

$$Cep(g(x,y)) = infFT\{\log(FT(g(x,y)))\} \tag{4.4}$$

The Figure 4.9 illustrate that when we used 2-D cepstral methods for motion blur estimation we have low level of error for estimation of the parameter l.

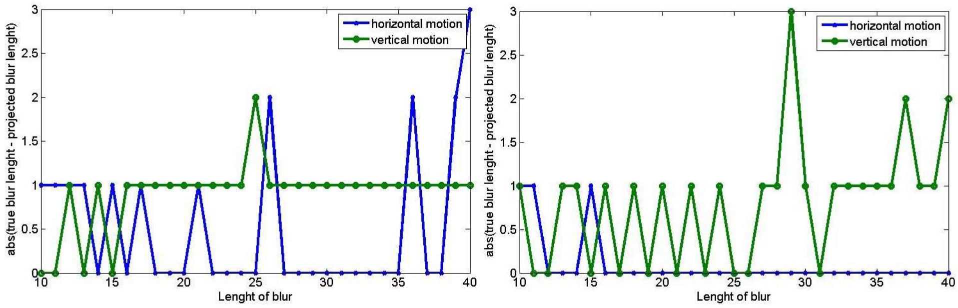

Figure 4.9. (left) Error of the estimated blur length with 2D cepstral method for the

X-ray image presented in Figure 4.1
(right) Error of the estimated blur length with 2D cepstral method for the standard MATLAB image Cameraman.

4.5. Time consuming

In this section we tested the method presented in Section 3., present numerical results for computational time and compare with three standard methods for image restoration: Wiener filter, Constrained least-squares filter and Lucy-Richardson algorithm. Tests were made for different values of the dimensions of the image $r \times n$, while the parameter l takes values from 5 to 40 with step 5 ($l = 5; 10; 15; 20; 25; 30; 35; 40$).

The following figures show the results when the dimension of the images are $r \times n$, and the dimension of the matrix H is $m \times n$. The results presented in Figure 4.10, Figure 4.11 and Figure 4.12 refer to computational time t(sec) for restored images as a function of $l < 40$ pixels for the proposed method and the mentioned other methods for various random matrices with dimensions 800×600, 1000×600 and 1200×600. Figure 4.10 compares the proposed method with all the mentioned other methods: Wiener filter, Constrained least-squares filter and Lucy-Richardson algorithm.

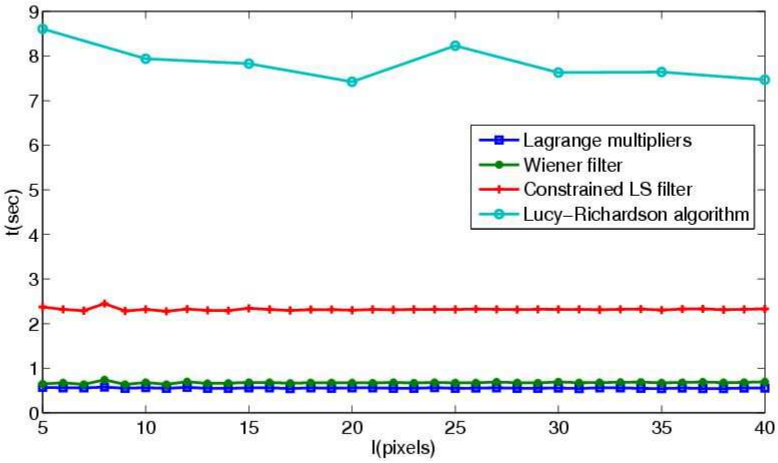

Figure 4.10. Computational time vs. length of the blurring process in pixels for

$r = 800$, $n = 600$

Since the Lucy-Richardson algorithm produces the worst results, it is eliminated from comparisons in Figure 4.11 and Figure 4.12.

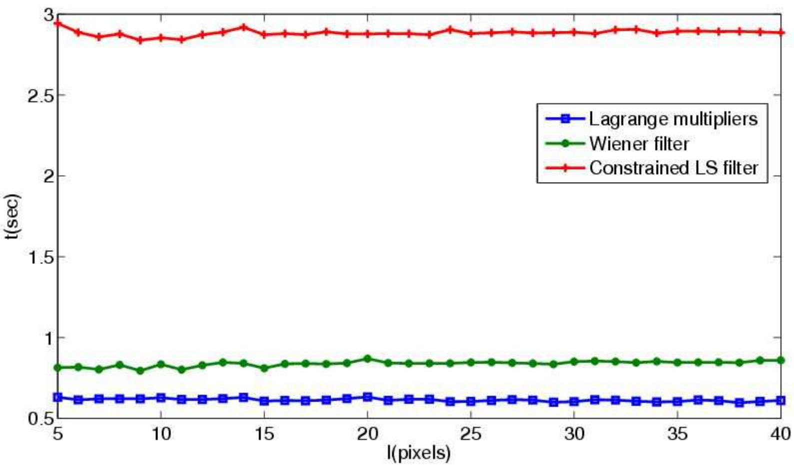

Figure 4.11. Computational time vs. length of the blurring process in pixels for

$r = 1000$, $n = 600$

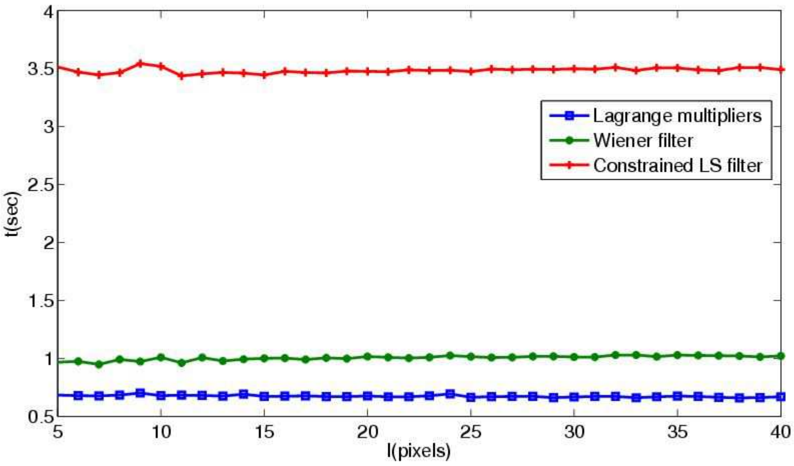

Figure 4.12. Computational time vs. length of the blurring process in pixels for

$r = 1200$, $n = 600$

By using the proposed method, the resolution of the restored image remains at a very high level, although the advantage of the method was found on the computational efficiency that has been lower considerably compared to the other methods and techniques. The computational time corresponding to our method is almost independent with respect to increasing of the parameter l.

5. Conclusion

We introduce a computational method, based on algorithm of Lagrange multipliers, to restore an image that has been blurred by uniform linear motion. We are motivated by the problem of restoring blurry images via well developed mathematical methods and techniques based on the Lagrange multipliers in order to obtain an approximation of the original image.

By using the proposed method, the resolution of the restored image remains at a very high level, although the main advantage of the method was found on the improvement $ISNR$ that has been increased considerably compared to the other methods and techniques.

We present the results by comparing our method and that of the Wiener filter, Constrained least-squares filter and Lucy-Richardson algorithm, a well established methods used for fast recovered and high resolution restored images. Obviously the proposed method is not restricted to restoration of X-ray images.

We can consider another practical example with images of ANPR (Automatic Number Plate Recognition) system. Results are shown concerning when we have horizontal, vertical and separable two-dimensional blurring of images.

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