FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 27, No 1 (2012), 13–26

INVERSE AND SATURATION THEOREMS FOR LINEAR COMBINATIONS OF A NEW CLASS OF POSITIVE LINEAR OPERATORS

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Abstract. The inverse and saturation theorems for the linear combinations of a class of positive linear operators of convolution type have been proved in this paper. This class contains a number of well known positive linear operators as special cases. The results make use of one of the Peetre's K-functionals. The analogues of inverse and saturation theorems in simultaneous approximation have also been proved.

1. Introduction

During past few decades a number of authors [1], [2], [6], [10], [11], [14] and [15] etc. have made an extensive study of the problems related to the inverse and saturation for different classes and sequences of the linear positive operators. In the present paper we study the inverse and saturation problems for the linear combinations of a new class of linear positive operators L_n. This class contains several well- known sequences of linear positive operators as special cases [8] in particular Gamma operators of Muller, Post-Widder and the Modified Post-Widder operators.

Let $M(R^+)$ be the class of complex valued measurable functions on R^+, $M_b(R^+)$ the subset of $M(R^+)$ consisting of the functions essentially bounded on R^+. We define

$$L_n(f:x) = D(m,n,\alpha)x^{mn+\alpha-1}\int_0^\infty u^{-mn-\alpha}e^{-n(\frac{x}{u})^m}f(u)du \tag{1.1}$$

where

$$D(m,n,\alpha) = \frac{mn^{n+\frac{\alpha-1}{m}}}{\Gamma(n+\frac{\alpha-1}{m})}, \quad m,x,n \in R^+, \quad \alpha \in R, \quad f \in M(R^+).$$

Clearly, (1.1) defines a class of positive linear operators.

Received February 10, 2012.
2010 *Mathematics Subject Classification.* Primary 41A40; Secondary 41A27

1.1. Basic Definitions and Preliminary Results

Definition 1.1. Let $\Omega(>1)$ be a continuous function defined on IR^+. We call Ω, a bounding function if for each compact $K \subseteq IR^+$, there exist positive numbers n_k and M_k such that

$$L_{n_k}(\Omega; x) < M_k, \quad x \in K.$$

For our operators the bounding function is

$$\Omega(u) = u^{-a} + e^{bu^m} + u^c, \quad \text{where } a, b, c > 0.$$

For this bounding function Ω, we define

$$D_\Omega = \{f \in Loc(IR^+)$$

such that

$$\limsup_{u \longrightarrow 0} \frac{f(u)}{\Omega(u)} \quad \text{and} \quad \limsup_{u \longrightarrow \infty} \frac{f(u)}{\Omega(u)}$$

exist.

Definition 1.2. Let f be a continuous function on $[a, b] \subseteq R^+$ and $\delta \geq o$. The $p-$ modulus of continuity of f is defined by

$$\omega_p(f, \delta) = \lim_{\substack{|h| < \delta \\ x, x+ph \in [a,b]}} \left| \sum_{j=0}^{p} (-1)^{p-j} \begin{pmatrix} p \\ j \end{pmatrix} f(x + jh) \right| \tag{1.2}$$

for $p = 1$, $\omega_p(f, \delta)$ is simply written as $\omega(f, \delta)$. If $\omega(f, \delta) \leq M\delta^\beta$, $(0 < \beta \leq 1)$, where M is a constant, we say that $f \in Lip_M^\beta$.

We define

$$Lip(\beta; a, b) = \cup_{M>0} Lip_M^\beta,$$

$$L_\infty[a, b] = \{f : f \text{ is essentially bounded on } [a, b]\},$$

$$AC[a, b] = \{f : f \text{ is absolutely continuous on } [a, b]\},$$

$$Lip(p, \beta; a, b) = \{f : f^{(k)} \in AC[a, b], k = 0, 1, 2, ..., (p-1) \text{ and } f^{(p)} \in Lip(\beta; a, b)\},$$

For $0 < \beta \leq 2$ and some constant M,

$$Liz(p, \beta; a, b) = \{f : \ \omega_{2p}(f, \delta) \leq M\delta^{\beta k}, k = 0, 1, 2, ..., (p-1)\}$$

for $p = 1$, $Liz(p, \beta; a, b)$ reduces to $Lip^*(1; a, b)$.

We introduce some more classes of functions:

$$C_0(R^+) = \{f : f \text{ is continuous on } R^+ \text{ and has compact support in } R^+\},$$

$$C^{(k)}(R^+) = \{f : f \text{ is } k-\text{times continuously differentiable on } R^+\},$$

$$C_0^{(k)}(R^+) = \{f \in C^{(k)}(R^+) : f \text{ is compactly supported on } R^+\},$$
$$C_b^{(m)}(R^+) = \{f : f \in C^{(m)}(R^+) \text{ and } f^{(k)}, k = 0,..,m \text{ are bounded on } R^+\}.$$

For any fixed set of positive constants $\alpha_i, i = 0, 1, 2, ..., p$ following [13] we define the linear combination $L_{n,p}$ of the operators $L_n$by

$$L_{n,p}(f;x) = \frac{1}{\triangle}\begin{vmatrix} L_{\alpha_0 n}(f;x) & \alpha_0^{-1} & \alpha_0^{-2} & ... & ... & \alpha_0^{-p} \\ L_{\alpha_1 n}(f;x) & \alpha_1^{-1} & \alpha_1^{-2} & ... & ... & \alpha_1^{-p} \\ .... & ... & ... & ... & ... & ... \\ .... & ... & ... & ... & ... & ... \\ L_{\alpha_p n}(f;x) & \alpha_p^{-1} & \alpha_p^{-2} & ... & ... & \alpha_p^{-p} \end{vmatrix} \tag{1.3}$$

where $\triangle$ is the determinant obtained by replacing the operator column by the entries $'1'$. Clearly

$$L_{n,p} = \sum_{j=0}^{p} C(j,p) L_{\alpha_j n}, \tag{1.4}$$

for constants $C(j,p), j = 0, 1, 2, ..., p$ which satisfy

$$\sum_{j=1}^{p} C(j,p) = 1.$$

$L_{n,p}$ is called a linear combination of order p. $L_{n,0}$ denotes the operator L_n itself.

Let $[a', b'] \subset (a, b)$ with $\zeta = \{g : g \in C_0^{(2p+2)}, \textit{supp } g \subset [a', b']\}$, for $f \in C_0(R^+)$ with *supp* $f \subset [a', b']$ we define

$$K(\xi, f) = \inf_{g\in\zeta}\{\|f - g\| + \xi(\|g\| + \left\|g^{(2p+2)}\right\|)\}$$

where $0 < \xi \leq 1$ and the norms are the Chebyshev norms on $[a', b']$.

A function $f \in C_0(R^+)$ with *supp* $f \subset [a', b']$ is said to belong to the intermediate space $C_0(\beta, p+1; a', b'), (0 < \beta \leq 2)$ if

$$\|f\|_\beta = \sup_{0<\xi<1}\{\xi^{-\frac{\beta}{2}} K(\xi, f)\} < \infty.$$

For a detailed account of Peetre's $K-$ functional and the intermediate spaces, we refer [5].

We state the following results on the spaces $C_0(\beta, p+1; a', b')$ and $Liz(\beta, p+1; a', b')$ by employing $K(\xi, f)$ in the proofs of inverse and saturation theorems.

Lemma 1.1. *Let* $0 < a < a' < a'' < b'' < b' < b < \infty$. *If* $f \in C_0(R^+)$ *with supp* $f \subset [a'', b'']$, *then* $f \in C_0(\beta, p+1; a', b')$ *iff* $f \in Liz(\beta, p+1; a, b)$.

Lemma 1.2. *Let $0 < \beta < 2$ and $0 < a < b < \infty$. The following statements are equivalent:*

(i) $f \in Liz(\beta, p+1; a, b)$

(ii) **(a)** *if* $m < \beta(p+1) < m+1, m = 0, 1, ..., (2p+1)$, $f^{(m)}$ *exists and belongs to* $Lip(\beta(p+1) - m; a, b)$ *and*

(b) *if* $m + 1 = \beta(p+1), (m = 0, 1, 2, ..., 2p)$ $f^{(m)}$ *exists and belongs to* $Lip^*(1; a, b)$

Lemma 1.3. *If for $\xi, \eta \in (0, 1)$ and a constant M there holds*

$$K(\xi, f) \leq M \left| \eta^{\frac{\beta}{2}} + (\frac{\xi}{\eta}) K(\eta, f) \right|, \tag{1.5}$$

where $0 < \beta < 2$, then there exists a constant M' such that

$$K(\xi, f) \leq M' \xi^{\frac{\beta}{2}}.$$

Throughout the paper $\{\lambda_n : n \in N\}$ denotes an increasing sequence of positive numbers such that

(1) $n_s \to \infty$ as $s \to \infty$, and

(2) for some constant $C > 0$, $\frac{n_{s+1}}{n_s} \leq C, \ \ s \in N$.

2. Inverse theorems (ordinary approximation)

Let $K(\xi; f)$ denotes the Peetre's $K-$functionals. We first prove:

Lemma 2.1. *Let $0 < a < a' < a'' < b'' < b' < b < \infty$. If $f \in M_b(R^+)$ with supp $f \subset [a'', b'']$ and*

$$\sup_{x \in [a,b]} |L_{n_s,p}(f; x) - f(x)| = o(n_s^{\frac{-\beta(p+1)}{2}}), \quad (s \to \infty) \tag{2.1}$$

where $0 < \beta < 2$ and p is a non negative integer, then $f \in C_0(R^+)$ and $n \geq 1$, there holds

$$K(\xi; f) \leq M \left| n^{-\frac{\beta(p+1)}{2}} + n^{p+1} \xi K(n^{-(p+1)}; f) \right|, \tag{2.2}$$

where M is a constant.

Proof. Due to the condition $\frac{n_{s+1}}{n_s} \le C$ it is sufficient to prove (2.2) with n replaced by n_s where s is sufficiently large. Since $G(u) = u^m e^{-u^m}$ is infinitely differentiable. Therefore for some $\delta > 0, G(u)$ is $(2p+2)-$times continuously differentiable on $(1-2\delta, 1+2\delta)$. Here δ can be chosen so small that $0 < 2\delta < \min\{1 - \frac{a'}{a"}, \frac{b'}{b"} - 1\}$. It is obvious that we can find a function $G^* \in C_0^{2p+2}(R^+)$ s.t.

$$G^*(u) = \left\{ \begin{array}{cc} G(u), & |u-1| \le \delta \\ 0, & u \le \frac{a'}{a''} \text{ or } u \ge \frac{b'}{b''} \end{array} \right\} \tag{2.3}$$

Then, if L_n^* denotes the operator in (1.1) obtained by introducing (2.3), in view of (2.1) we also have

$$\sup_{x \in [a,b]} \left| L_{n_s,p}^*(f;x) - f(x) \right| \le M' n_s^{-\frac{\beta(p+1)}{2}}, \quad (s \to \infty) \tag{2.4}$$

where M' is some positive constant and $L_{n_s,p}^*$ are the linear combinations corresponding to the operator L_n^*. Here we notice that $L_n^*(f;x) \in C_0^{(2p+2)}(R^+)$ with supp $L_n^*(f;x) \subset [a', b']$ for all $n \in R^+$. In view of (2.4) it is clear that $f \in C_0(R^+)$ and

$$K(\xi; f) \le M n_s^{-\frac{\beta(p+1)}{2}} + \xi\{\left\| L_{n_s,p}^*(f;x) \right\|_{C[a',b']} + \left\| L_{n_s,p}^{*(2p+2)}(f;x) \right\|_{C[a',b']}\}. \tag{2.5}$$

Next, we assert that for each $g \in \zeta = \{g : g \in C_0^{(2p+2)}(R^+)$, supp $g \subset [a', b']\}$ there holds the inequality

$$\left\| L_n^{*(2p+2)}(g;x) \right\|_{C[a',b']} \le A_1 n^{p+1} \|g\|_{C[a',b']} \tag{2.6}$$

where A_1 is a constant. We have

$$\left| L_n^{*(2p+2)}(g;x) \right| \le C_1 \|g\|_\infty \sum_{j=0}^{2p+2} \sum_{\nu=0}^{[p+1-\frac{j}{2}]} n^{\nu+j} \frac{D^{**}(m,n,\alpha)}{D^*(m,n,\alpha)} L_n^{**}(|u-1|^j; 1), \tag{2.7}$$

where C_1 is a constant, L_n^{**} is the operator defined by (1), with $G(u) = u^m e^{-u^m}$ replaced by $G^*(u)$ and α by $\alpha + j$ and $D^{**}(m,n,\alpha)$ [3] is the corresponding $D(m,n,\alpha)$.

Now, in view of (2.7) and the fact that supp $g \subset [a', b']$, (2.6) is clear. Also, for every $g \in \zeta$, it is clear that

$$\left\| L_n^{*(2p+2)}(g;x) \right\|_{C[a',b']} \le A_2 \left\| g^{(2p+2)} \right\|_{C[a',b']}, \tag{2.8}$$

where A_2 is a constant. Using (2.7) and (2.8) for every $g \in \zeta$ we have

$$\begin{aligned} &\left\| L_{n_s,p}^*(f;x) \right\|_{C[a',b']} + \left\| L_{n_s,p}^{*(2p+2)}(f;x) \right\|_{C[a',b']} \le \\ &\le \quad n_s^{(p+1)} M" [\|f-g\|_{C[a',b']} + n_s^{-(p+1)}\{\|g\|_{C[a',b']} + \left\| g^{(2p+2)} \right\|_{C[a',b']}\}], \end{aligned} \tag{2.9}$$

where $M"$ is a constant . Hence, by (2.5) and (2.9)with $M = \max\{M', M"\}$ and for every $g \in \zeta$, we have

$$K(\xi; f) \leq M[n_s^{-\frac{\beta(p+1)}{2}} + n_s^{-(p+1)}\xi \|f - g\|_{C[a',b']} + n_s^{-(p+1)}\{\|g\|_{C[a',b']} + \left\|g^{(2p+2)}\right\|_{C[a',b']}\}]. \tag{2.10}$$

Taking the infimum on the right hand side of (2.10) we get (2.2). This completes the proof of the lemma. □

Now we are in position to prove the main result of this section.

Theorem 2.1. *Let $f \in D_\Omega$. If $0 < r < 2p+2, p \in N^0$ (set of non-negative integers) and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, then in the following statements the implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).*

(i) $\sup_{x\in[a_1,b_1]} |L_{n_s,\ p}(f;x) - f(x)| = o(n_s^{-\frac{r}{2}}) \quad (n_s \to \infty)$;

(ii) *if $r \neq [r], f^{([r])}$ exists and belongs to $Lip(r - [r]; a_2, b_2)$ and if $r = [r], f^{(r-1)}$ exists and belongs to $Lip^*(1; a_2, b_2)$;*

(iii) $\sup_{x\in[a_3,b_3]} |L_{n,\ p}(f;x) - f(x)| = O(n^{-\frac{r}{2}}) \qquad (n \to \infty)$.

Proof. Since $0 < r < 2p+2$, we write $r = \beta(p+1)$ for some $\beta \in (0,2)$. We first prove that $(ii) \Rightarrow (iii)$. Assuming (ii) and using $Lemma 2$ $a_2 < a_2^* = a' < a_2' < a_2" < a_3 < b_3 < b_2" < b_2' < b' = b_2^* < b_2$ and $g_0 \in C_0^\infty(R^+)$ be such that $g_0(u) = 1$ for $u \in [a_2", b_2"]$ and supp $g_0 \subset [a_2', b_2']$. Then, since $f \in Liz(\beta, (p+1); a_2, b_2)$ also $f^* = fg_0 \in Liz(\beta, p+1; a_2, b_2)$ and supp $f^* \subset [a_2', b_2']$. Hence by Lemma 1.1

$$\begin{aligned} |L_{n,p}(f;x) - f(x)| &\leq |L_{n,p}(f - f^*;x)| + |L_{n,p}(f^*;x) - f^*(x)| \leq \\ &\leq |L_{n,p}(f^*;x) - f^*(x)| + B_1 n^{-\frac{r}{2}}, \end{aligned} \tag{2.11}$$

where B_1 is a constant independent of n and x. Now for any $g \in \zeta$ and $x \in [a_2^*, b_2^*]$, we have

$$\begin{aligned} |L_{n,p}(f^*;x) - f(x)| &\leq |L_{n,p}(f^* - g;x)| + |L_{n,p}(g;x) - g(x)| + |g(x) - f^*(x)| \\ &\leq B_2 \|f^* - g\|_{C[a_2^*,b_2^*]} + |L_{n,p}(g;x) - g(x)| \end{aligned}$$

where B_2 is a constant.

By a mean value theorem,

$$g(u) - g(x) = \sum_{j=1}^{2p+1} \frac{g^{(j)}(x)}{j!}(u-x)^j + \frac{(u-x)^{2p+2}}{(2p+2)!} g^{(2p+2)}(\xi_u)$$

for all $u \in R^+$, where $\xi_u \in (u, x)$. Hence,

$$\begin{aligned} L_{n,p}(g(u);x) - g(x) &= \sum_{j=1}^{2p+1} \frac{g^{(j)}(x)}{j!} L_{n,p}((u-x)^j;x) + L_{n,p}(\frac{(u-x)^{2p+2}}{(2p+2)!} g^{(2p+2)}(\xi_u);x) \\ &= \sum_1 + \sum_2 \quad \text{(say)}. \end{aligned}$$

By the definition of $L_{n,p}$,

$$\left|\sum_1\right| \leq B_3 n^{-(p+1)} \sum_{j=1}^{2p+1} \left\| g^{(j)} \right\|_{C[a_2^*,b_2^*]}, \text{ for large } n \text{ and } x \in [a_2^*, b_2^*]. \tag{2.12}$$

Also,

$$\begin{aligned} \left|\sum_2\right| &\leq \frac{\left\| g^{(2p+2)} \right\|_{C[a_2^*,b_2^*]}}{(2p+2)!} \sum_{j=0}^{p} |C(j,p)|\, L_{\alpha_j n}((u-x)^{2p+2}; x) \\ &\leq B_4 n^{-(p+1)} \left\| g^{(2p+2)} \right\|_{C[a_2^*,b_2^*]}, \end{aligned} \tag{2.13}$$

where B_3, B_4 are constants. Hence if $B_5 = \max(B_3, B_4)$, we have

$$|L_{n,p}(g;x) - g(x)| \leq B_5 n^{-(p+1)} \sum_{j=1}^{2p+1} \left\| g^{(j)} \right\|_{C[a_2^*,b_2^*]}. \tag{2.14}$$

Since there exist a constant B_6 such that

$$\sum_{j=1}^{2p+2} \left\| g^{(j)} \right\|_{C[a_2^*,b_2^*]} \leq B_6 \{ \|g\|_{C[a_2^*,b_2^*]} + \left\| g^{(2p+2)} \right\|_{C[a_2^*,b_2^*]} \}.$$

It follows from (2.10)-(2.13) that for all sufficiently large n

$$\begin{aligned} &\sup_{x\in[a_3,b_3]} |L_{n,p}(f;x) - f(x)| \leq \\ &\leq M' \left| \|f^* - g\|_{C[a_2^*,b_2^*]} + n^{-(p+1)} \{ \|g\|_{C[a_2^*,b_2^*]} + \left\| g^{(2p+2)} \right\|_{C[a_2^*,b_2^*]} \} + n^{-\beta(p+1)} \right| \end{aligned} \tag{2.15}$$

where M' is some constant. Taking infimum over $g \in \zeta$ in (2.15) for sufficiently large n we have

$$\sup_{x\in[a_3,b_3]} |L_{n,p}(f;x) - f(x)| \leq M \left| n^{-\beta\frac{(p+1)}{2}} + K(n^{-(p+1)}; f^*) \right| \tag{2.16}$$

since $f^* \in C_0(\beta, p+1; a_2^*, b_2^*)$ and $a_2^* = a', b_2^* = b'$, we have

$$K(n^{-(p+1)}; f^*) \leq \acute{M}\text{"} n^{-\beta(p+1)} \tag{2.17}$$

where M" is a constant. Also, as $r = \beta(p+1)$, it follows from (2.16)-(2.17) that

$$\sup_{x\in[a_3,b_3]} |L_{n,p}(f;x) - f(x)| = O(n^{-\frac{r}{2}}).$$

This completes the proof of $(ii) \Rightarrow (iii)$.

To prove $(i) \Rightarrow (ii)$. Let us assume (i). If supp $f \subset (a_1, b_1)$ with $a = a_1, b = b_1$, we can choose $a', b', a", b"$ such that $0 < a_1 = a < a' < a" < b" < b' < b = b_1 < \infty$

and supp $f \subset [a", b"]$. By Lemma 2.1 we obtain

$$K(\xi; f) \leq Mn^{-\beta\frac{(p+1)}{2}} + n^{p+1}\xi K(n^{-(p+1)}; f), \quad (n \geq 1)$$

Hence by Lemma 1.3 we have (ii). When supp $f \subset [a_1, b_1]$ we proceed as follows:

If a_1^*, b_1^* are such that $a_1 < a_1^* < a_2 < b_2 < b_1^* < b_1$ and $f^* = f$ on $[a_1, b_1]$ and vanishes outside it then also

$$\sup_{x \in [a_1^*, b_1^*]} |L_{n_s,p}(f^*; x) - f^*(x)| = o(n_s^{-\frac{r}{2}}). \tag{2.18}$$

Let us first consider the case when $0 < r < 1$. Let $g \in C_0^\infty(R^+)$ with supp $g \subset [a", b"]$ and $g(u) = 1$ for $u \in [a_2, b_2]$ where $a_1 < a_1^* < a' < a" < b_2 < b" < b' < b_1^* < b_1$. Then,

$$\begin{aligned} &\sup_{x \in [a', b']} |L_{n_s,p}(f^* g; x) - f^*(x) g(x)| \\ &\leq \sup_{x \in [a', b']} |g(x) L_{n_s,p}(f^*(u) - f^*(x); x)| + \sup_{x \in [a', b']} |L_{n_s,p}(f^*(u)(g(u) - g(x)); x)| \\ &= I_1 + I_2 \qquad\qquad \text{(say)}. \end{aligned}$$

By (2.18)

$$I_1 = o(n_s^{-\frac{r}{2}}),$$

and by a simple computation

$$I_2 = o(n_s^{-\frac{r}{2}}).$$

Hence with $F = f^* g$, we have

$$\sup_{x \in [a', b']} |L_{n_s,p}(F; x) - F(x)| = o(n_s^{-\frac{r}{2}}) \tag{2.19}$$

from which since supp $F \subset [a', b']$ it follows that $F \in Liz(\beta, p+1; a_1, b_1)$ as before and $f \in Liz(\beta, p+1; a_2, b_2)$. Thus by Lamma 1.3 (ii) holds.

Next, we assume that assertion $(i) \Rightarrow (ii)$ holds when $0 < r < q - \delta$, where $0 < \delta < \frac{1}{2}$ is arbitrary and q takes one of the values of $1, 2, ..., 2p+1$. Since for $q = 1$ the result has already been proved. If we can establish it for $q - \delta \leq r < q + 1 - 2\delta$ the proof will be over. Hence let $q - \delta \leq r < q + 1 - 2\delta$. Then by the assumption that $f^{(p-1)}$ exists and belongs to $Lip^*(1 - \delta; a_2^*, b_2^*)$, where $[a_2^*, b_2^*] \subset (a_1, b_1)$ is any fixed interval. Let $a_2^* < a_1^* < a_1^{**} < a' < a" < a_2 < b_2 < b" < b' < b_1^{**} < b_1^* < b_2^*$. We choose g as before and write $F = f^* g$ after defining $f^* = f$ on $[a_2^*, b_2^*]$ and zero otherwise.

Then,

$$\begin{aligned}&\sup_{x\in[a',b']}|L_{n_s,p}(F;x)-F(x)|\le\sup_{x\in[a',b']}|g(x)L_{n_s,p}(f^*(u)-f^*(x);x)|+\\&+\sup_{x\in[a',b']}|L_{n_s,p}(f^*(u)-f^*(x)(g(u)-g(x));x)|+\sup_{x\in[a',b']}|f^*(x)L_{n_s,p}(g(u)-g(x);x)|\\&=J_1+J_2+J_3,\qquad\qquad say.\end{aligned}$$

Obviously,

$$J_1=o(n_s^{-\frac{r}{2}}),\qquad J_2=o(n_s^{-\frac{r}{2}})\qquad\text{and}\qquad J_3=o(n_s^{-\frac{r}{2}}).$$

Combining these estimates we have

$$\sup_{x\in[a',b']}|L_{n_s,p}(F;x)-F(x)|=o(n_s^{-\frac{r}{2}}).$$

Again. since supp $F\subset[a",b"]$, as before $F\in Liz(\beta,p+1;a_1^*,b_1^*)$ and (ii) follows. This completes the proof of the theorem. □

3. Saturation theorems (ordinary approximation)

If $f\in D_\Omega$, the following assymptotic relation for $L_{n,p}$ holds:

$$L_{n,p}(f;x)-f(x)=n^{-(p+1)}\sum_{i=1}^{2p+2}\frac{f^{(i)}(x)}{i!}x^i\gamma_{i,p+1}\frac{(-1)^p}{\alpha_0\alpha_1...\alpha_p}+o(n^{-(p+1)})\tag{3.1}$$

at any $x\in R^+$ where $f^{(2p+2)}$ exists. Moreover, if $f^{(2p+2)}$ exists and is continuous on an open interval containing $[a,b]$, (3.1) holds uniformly in $x\in[a,b]$. This asymptotic formula indicates a saturation behaviour of the linear combinations $L_{n,p}$. A more precise result is as follows:

Theorem 3.1. *Let $p\in N^0$,$f\in D_\Omega$. If $0<a_1<a_2<a_3<b_3<b_2<b_1<\infty$ in the following statements, the implications $(i)\Rightarrow(ii)\Rightarrow(iii)$ and $(iv)\Rightarrow(v)\Rightarrow(vi)$ hold.*

(i) $\sup_{x\in[a_1,b_1]}|L_{n_s,p}(f;x)-f(x)|=o(n_s^{-(p+1)}),\ (s\to\infty)$

(ii) $f^{(2p+1)}\in AC[a_2,b_2]$ *and* $f^{(2p+2)}\in L_\infty[a_2,b_2]$

(iii) $\sup_{x\in[a_3,b_3]}|L_{n,p}(f;x)-f(x)|=(n^{-(p+1)}),\ (n\to\infty)$

(iv) $\sup_{x\in[a_1,b_1]}|L_{n_s,p}(f;x)-f(x)|=o(n_s^{-(p+1)}),\ (s\to\infty)$

(v) $f\in C^{2p+2}[a_2,b_2]$ *and* $\sum_{i=1}^{2p+2}\frac{f^{(i)}(x)}{i!}x^i\gamma_{i,p+1}=0,\ x\in[a_2,b_2]$

(vi) $\sup_{x\in[a_3,b_3]} |L_{n,p}(f;x) - f(x)| = (n^{-(p+1)}),\ (n \to \infty)$.

Proof. Assume (i). Let L_n^* denote the operator as defined before . It is clear from $Theorem 1$ that $f^{(2p+1)}$ exists and is continuous on each closed subinterval of $[a_1, b_1]$. Then let $f^* \in C_0(R^+)$ be such that $f^* = f$ on $[a_1^*, b_1^*]$ where $a_1 < a_1^* < a_2$ and $b_1 < b_1^* < b_2$. Then we have

$$\sup_{x\in[a_2^*,b_2^*]} |L_{n_s,p}(f^*;x) - f^*(x)| = o(n_s^{-(p+1)}), \quad (s \to \infty),$$

where $a_1^* < a_2^* < a_2$ and $b_1^* < b_2^* < b_1$. Also, we have

$$\sup_{x\in[a_3^*,b_3^*]} n_s^{(p+1)} |L_{n_s,p}(L_n^*(f^*;u);x) - L_n(f;x)|$$
$$= \sup_{x\in[a_2^*,b_2^*]} n_s^{(p+1)} |L_n^*(L_{n_s,p}(f^*;u) - f^*(u);x)| = o(1)$$

where $a_2^* < a_3^* < a_2$ and $b_2 < b_3^* < b_2^*$. Hence by the uniformity assertion regarding (6) we have

$$\left\|\sum_{i=1}^{2p+2} \frac{x^i}{i!}\gamma_{i,p+1} L_n^*(f^*;x)\right\|_{C[a_3^*,b_3^*]} \le M,$$

where M is a constant. Hence for all n sufficiently large,

$$\left\|\gamma_{2p+2,p+1} L_n^{*(2p+2)}(f^*;x)\right\|_{C[a_3^*,b_3^*]} \le M_1$$

where M_1 is a constant. But $\gamma_{2p+2,p+1} \ne 0$. Hence there exists a constant M_2 such that for all n sufficiently large, there holds

$$\left\|L_n^{*(2p+2)}(f^*;x)\right\|_{C[a_3^*,b_3^*]} < M_2.$$

Thus for all n sufficiently large,$L_n^{*(2p+2)}(f^*;x)$ are uniformly bounded and hence belong to $L_\infty[a_3^*, b_3^*]$. As $L_\infty[a_3^*, b_3^*]$ is the dual of$L_1[a_3^*, b_3^*]$.By weak-compactness there is an $h \in L_\infty[a_3^*, b_3^*]$ and subset $\{n_i\}$of $\{n\}$ such that $L_{n_i}^{*(2p+2)}(f^*;x)$ converges to h in weak topology. In particular, for any $g \in C_0^\infty(R^+)$ with supp $g \subset (a_3^*, b_3^*)$ we have

$$\int_{a_3^*}^{b_3^*} L_{n_i}^{*(2p+2)}(f^*;x)g(x)dx \to \int_{a_3^*}^{b_3^*} h(x)g(x)dx \quad (n_i \to \infty).$$

But by integration by parts,

$$\int_{a_3^*}^{b_3^*} L_{n_i}^{*(2p+2)}(f^*;x)g(x)dx = \int_{a_3^*}^{b_3^*} L_{n_i}^*(f^*;x)g^{(2p+2)}(x)dx$$

Hence,

$$\int_{a_3^*}^{b_3^*} h(x)g(x)dx = \lim_{i\to\infty}\int_{a_3^*}^{b_3^*} L_{n_i}^*(f;x)g^{(2p+2)}(x)dx = \int_{a_3^*}^{b_3^*} f^*(x)g^{(2p+2)}(x)dx$$

for every g as above. Hence $D^{2p+2}f^*(t) = h(t)$ is a generalized function. Thus $Df^{*(2p+2)}(t) = h(t) \in L_\infty[a_3^*, b_3^*]$, implying that $f^{*(2p+1)} \in AC[a_2, b_2]$ and $f^{*(2p+2)} \in L_\infty[a_1, b_1]$. But $f = f^*$ on $[a_2, b_2]$ and (ii) follows.

$(ii) \Rightarrow (iii)$ is obvious.

Now, let (iv) hold. Then, proceeding as in the proof of $(i) \Rightarrow (ii)$ we have for all sufficiently large n

$$\sum_{i=1}^{2p+2} \frac{x^i}{i!}\gamma_{i,p+1} L_n^{*(i)}(f^*; x) = 0, \quad x \in [a_3^*, b_3^*].$$

Thus, if $P(D)$ denotes the differential operator $\sum_{i=1}^{2p+2} \frac{x^i}{i!}\gamma_{i,p+1}D^i$ and $P^*(D)$ its adjoint, for any $g \in C_0^\infty(R^+)$ with supp $g \subset [a_3^*, b_3^*]$ we have for all n sufficiently large

$$0 = \int_{a_3^*}^{b_3^*} P(D)L_n^*(f^*; x)g(x)dx = \int_{a_3^*}^{b_3^*} L_n^*(f; x)P^*(D)g(x)dx$$

Taking limit as $n \to \infty$, we obtain

$$\int_{a_3^*}^{b_3^*} f^*(x)P^*(D)g(x)dx = 0.$$

Hence, $D^{2p+2}f^* \in C[a_3^*, b_3^*]$ and $P(D)f^*(x) = 0, x \in [a_3^*, b_3^*]$, and (v) follows since $f = f^*$ on $[a_2, b_2]$. Thus $(iv) \Rightarrow (v)$.

Lastly, $(v) \Rightarrow (vi)$ follows from the uniformity assertion for (6). This completes the proof of the theorem. □

4. Inverse and saturation theorems (simultaneous approximation)

The inverse and saturation theorems for the class of continuously differentiable functions are as follows:

Theorem 4.1. *Let $m \in N$ and $f \in D_\Omega$. If $0 < q < 2p+2, p \in N^0$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, then in the following statements the following implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold.*

(i) *$f^{(m)}$ exists on $[a_1, b_1]$ and*

$$\sup_{x\in[a_1,b_1]} \left| L_{n_s,p}^{(m)}(f; x) - f^{(m)}(x) \right| = o(n_s^{-\frac{q}{2}}), \quad (s \to \infty),$$

(ii) *If $q \neq [q]$ (the greatest integer not greater than q), $f^{([q]+m)}$ exists and belongs to $Lip(q - [q]; a_2, b_2)$ and*

(iii) *If $q=[q]$, $f^{(m+q-1)}$ exists and belongs to $Lip^*(1;a_2,b_2)$, and*

$$\sup_{x\in[a_3,b_3]}\left|L_{n,p}^{(m)}(f;x)-f^{(m)}(x)\right|=o(n^{-\frac{q}{2}}),\quad (n\to\infty).$$

Proof. Assume (i) and consider the function $G^*(u)$ defined in (8) implies that $f^{(m)}(x)$ is continuous on each open interval of $[a_1,b_1]$ and moreover that

$$\sup_{x\in[a_1^*,b_1^*]}\left|L_{n_s,p}^{*(m)}(f;x)-f^{(m)}(x)\right|=o(n_s^{-\frac{q}{2}}),\qquad (s\to\infty). \tag{4.1}$$

Next, if $f^*\in C_0^m(R^+)$ and coincides with f on $[a_2^*,b_2^*]\subset(a_1^*,b_1^*)$, it follows that

$$\sup_{x\in[a_3^*,b_3^*]}\left|L_{n_s,p}^{*(m)}(f^*;x)-f^{*(m)}(x)\right|=o(n_s^{-\frac{q}{2}}),\qquad (s\to\infty) \tag{4.2}$$

where $a_2^*<a_3^*<a_2<b_2<b_3^*<b_2^*$. But here (4.2) is equivalent to

$$\sup_{x\in[a_3^*,b_3^*]}\left|L_{n_s,p}^{*}(u^m f^{*(m)}(u);x)-x^m f^{*(m)}(x)\right|=(n_s^{-\frac{q}{2}}),\qquad (s\to\infty). \tag{4.3}$$

Thus, by Theorem 2.1, since $f^*=f$ on $[a_2,b_2]$, we have (ii). Next, assume that $f^*\in C_0^m(R^+)$ which coincides to f on $[a_2^*,b_2^*]\subset(a_1^*,b_1^*)$ then $(u^m f^{*(m)})^{([q])}\in Lip(q-[q];a_2^*,b_2^*)$ if $q\neq[q]$ and $(u^m f^{*(m)})^{(q-1)}\in Lip(1;a_2^*,b_2^*)$ if $q=[q]$. Hence by Theorem 2.1, if $a_2'<a_3'<a_3<b_3<b_3'<b_2'$

$$\sup_{x\in[a_3',b_3']}\left|L_{n,p}(u^m f^{*(m)}(u);x)-x^m f^{*(m)}(x)\right|=o(n^{-\frac{q}{2}}),\quad (n\to\infty).$$

But, this is equivalent to

$$\sup_{x\in[a_3',b_3']}\left|L_{n_s,p}^{(m)}(f^*(u);x)-f^{*(m)}(x)\right|=o(n^{-\frac{q}{2}}),\quad (n\to\infty). \tag{4.4}$$

Again by the coincidence of f^* and g on $[a_2',b_2']$ and (4.4) we have (iii). This completes the proof of the Theorem. □

Finally, we obtain an analogue of Theorem 1.2 in simultaneous approximation, this states

Theorem 4.2. *Let $m\in N, p\in N^0$ and $f\in D_\Omega$. If $0<a_1<a_2<a_3<b_3<b_2<b_1<\infty$ in the following statements the implications $(i)\Rightarrow(ii)\Rightarrow(iii)$ and $(iv)\Rightarrow(v)\Rightarrow(vi)$ hold.*

(i) *$f^{(m)}$ exists on $[a_1,b_1]$ and*

$$\sup_{x\in[a_1,b_1]}\left|L_{n_s,p}^{(m)}(f;x)-f^{(m)}(x)\right|=o(n_s^{-(p+1)}),\quad (s\to\infty)$$

(ii) $f^{(2p+m+1)} \in AC[a_2, b_2]$ *and* $f^{(2p+m+2)} \in L_\infty[a_2, b_2]$,

(iii) $\sup_{x\in[a_3,b_3]} \left| L_{n,p}^{(m)}(f; x) - f^{(m)}(x) \right| = o(n^{-(p+1)}),\ (n \to \infty)$

(iv) $f^{(m)}$ *exists on* $[a_1, b_1]$ *and*

$$\sup_{x\in[a_1,b_1]} \left| L_{n_s,p}^{(m)}(f; x) - f^{(m)}(x) \right| = o(n_s^{-(p+1)}),\ (s \to \infty)$$

(v) $f \in C^{2p+m+2}[a_2, b_2]$ *and* $\sum_{i=1}^{2p+2} (\frac{f^{(i)}(x)x^i}{i!})^{(m)} \gamma_{i,p+1} = 0, \quad x \in [a_2, b_2]$

(vi) $\sup_{x\in[a_3,b_3]} \left| L_{n,p}^{(m)}(f; x) - f^{(m)}(x) \right| = o(n^{-(p+1)}), \qquad (n \to \infty)$

Proof. The proof of this theorem follows along the similar lines, with some essential modifications as in the case of Theorems 3.1 and 4.1. □

Acknowledgement. The first author is highly thankful to" Council of Scientific and Industrial Research, India" for providing financial assistance for the research work.

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