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FIXED POINT RESULTS IN PARTIALLY ORDERED METRIC SPACES USING WEAK CONTRACTIVE INEQUALITIES

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Abstract. In recent times control functions have been used in several problems of metric fixed point theory. Again, after the establishing of the weak contraction principle, weak contractive inequalities have been considered in a number of works on fixed points in metric spaces. There has been a rapid development of fixed point theory in partially ordered metric spaces in recent times. In this paper we establish fixed point results for mappings of partially ordered metric spaces satisfying some weak contractive inequalities involving more than one control functions. An illustrative example is given.

1. Introduction

The Banach contraction mapping principle is widely recognized as the source of metric fixed point theory. Its generalizations over the years have remained heavily investigated. Some recent works of this kind are noted in [2, 6, 9, 12]. In particular, in [1] Alber and Guerre-Delabriere introduced the concept of the weak contraction in Hilbert spaces. Rhoades [10] had shown that the result which Alber et al. had proved in Hilbert spaces [9] is also valid in complete metric spaces. Weak contraction principle, its generalizations and extensions and other fixed point results for mappings satisfying weak contractive type inequalities have been considered in a number of recent works. Some of these works are noted in [3, 4, 5, 7, 11, 13]. Khan et al. [8] introduced the use of a control function in fixed point problems. This function was referred to as 'Altering distance function' by the authors of [8]. This function and its extensions have been used in several problems of fixed point theory. Particularly, in [7] more than one control functions have been used. In recent times fixed point theory has developed rapidly in partially ordered metric spaces, that is, in metric spaces endowed with a partial ordering.

The purpose of the paper is to establish two fixed point theorems in partially ordered complete metric spaces for mappings satisfying certain weak contractive inequalities each of which involves more than one control functions (as in [7]). In

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section 2 we have proved a common fixed point result for two mappings. The result has a corollary. In section 3 we have established a fixed point theorem for a single mapping. Finally in section 4 we have given an illustrative example.
Throughout this paper (X, d) stands for a complete metric space and $' \preceq'$ is a partial order on X.
A mapping $T : X \to X$ is said to be non-decreasing if $Tx \preceq Ty$ whenever $x \preceq y$. In this paper we shall assume that X has the following property:

If a non-decreasing sequence $\{x_n\}_{n\geqslant 0}$ converges to $z \in X,$ then$x_n \preceq z$ for each $n \geqslant 0$. (1.1)

2. A Common Fixed Point Result

This section is devoted to common fixed point results for two self mappings T, S defined on a partially ordered metric space (X, d). We begin by recalling the following theorem proved by D. Doric in [5]. Then we shall extend this result to complete partially ordered metric spaces.

Theorem 2.1. *([5]). Let (X, d) be a complete metric space and let T, $S : X \to X$ be two self mappings such that for all $x, y \in X$*

$$\Psi(d(Tx, Sy)) \leq \Psi(M(x, y)) - \Phi(M(x, y)), \tag{2.1}$$

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\}, \tag{2.2}$$

and $\Psi : [0, \infty) \to [0, \infty)$ is a continuous monotone non-decreasing function with $\Psi(t) = 0$ if and only if $t = 0$, and $\Phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\Phi(t) = 0$ if and only if $t = 0$. Then there exists a unique point $u \in X$ such that $Su = Tu = u$.

The following theorem is a generalization of the above result in partially ordered metric spaces.

Theorem 2.2. *Let (X, d) be a complete partially ordered metric space with a partial order $' \preceq'$ and having the property described in (1.1). Let T, $S : X \to X$ be two self mappings such that for all comparable $x, y \in X$ with*

$$\Psi_1(d(Tx, Sy)) \leq \Psi_2(M(x, y)) - \Phi(M(x, y)) \tag{2.3}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\}, \tag{2.4}$$

$\Psi_1, \Psi_2 : [0, \infty) \to [0, \infty)$ are continuous monotone non-decreasing functions, and $\Phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function which satisfies $\Psi_1(t) - \Psi_2(t) + \Phi(t) > 0$. If X has the property (1.1) and if there exists a point $x_0 \in X$ satisfying $x_0 \preceq Sx_0 \preceq TSx_0 \preceq S(TS)x_0 \preceq (TS)^2x_0 \preceq \cdots$. Then there exists a point $u \in X$ such that $Su = Tu = u$.

Proof. We consider the following sequence:

$$x_1 = Sx_0, \quad x_2 = Tx_1, \quad x_3 = Sx_2, \quad x_4 = Tx_3, \cdots, \tag{2.5}$$

and in general, for all $n \geq 0$, $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$.

Then, from a condition of the theorem, it follows that

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots x_n \preceq x_{n+1} \preceq \cdots. \tag{2.6}$$

We now prove that
(i) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$;
(ii) The sequence $\{x_n\}$ is Cauchy, so that $x_n \to z$ for some $z \in X$;
(iii) $Sz = Tz = z$;
(iv) The common fixed point z is unique.
Let $n = 2k + 1$ be an odd number. Then, from (2.5), $x_n = Sx_{n-1}$, $x_{n+1} = Tx_n$. We now have

$$M(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))\}.$$

If $d(x_n, x_{n+1}) > d(x_n, x_{n-1})$, then $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$.
Then, putting $x = x_n$, and $y = x_{n-1}$ in (2.3), in view of (2.6), we obtain

$$\Psi_1(d(x_{n+1}, x_n)) \leq \Psi_2(d(x_{n+1}, x_n)) - \Phi(d(x_{n+1}, x_n)) \tag{2.7}$$

which is a contradiction with $d(x_n, x_{n+1}) > d(x_n, x_{n-1}) \geq 0$. Therefore

$$d(x_n, x_{n+1}) \leqslant d(x_{n-1}, x_n). \tag{2.8}$$

Similar argument shows that (2.8) is valid for even natural numbers. Hence the sequence $\{d(x_n, x_{n+1})\}$ is decreasing, and the limit $r = \lim_{n\to\infty} d(x_n, x_{n+1}) \geqslant 0$ exists. If $r > 0$, by putting $x = x_n$, $y = x_{n+1}$ in (2.3) and taking limit as $n \to \infty$, it follows that $\Psi_1(r) \leqslant \Psi_2(r) - \Phi(r)$ which is a contradiction with our assumption. Hence $r = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.9}$$

In view of (2.9), to show that $\{x_n\}$ is a Cauchy sequence, it suffices to verify that $\{x_{2n}\}$ is Cauchy. If not, then there is an $\epsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ such that $n(k)$ is the smallest integer corresponding to $m(k)$, for which $n(k) > m(k) > k$ and

$$d(x_{2m(k)}, x_{2n(k)}) \geqslant \epsilon \tag{2.10}$$

Hence, for all $k > 0$,

$$d(x_{2m(k)}, x_{2n(k)+2}) < \epsilon. \tag{2.11}$$

Then, for all $k \geq 0$,

$$\begin{aligned} \epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2n(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2m(k)}) \end{aligned}$$

and

$$d(x_{2n(k)-1}, x_{2m(k)+1}) \leq d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)+1}).$$

Taking $k \to \infty$ in the above inequality, and using (2.9), we have

$$\lim_{k\to\infty} M(x_{2n(k)-1}, x_{2m(k)+1}) = \epsilon. \tag{2.12}$$

Similarly, we deduce the following limits.

$$\lim_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)}) = \epsilon. \tag{2.13}$$

$$\lim_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)}) = \epsilon. \tag{2.14}$$

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \epsilon. \tag{2.15}$$

Again, by virtue of (2.6), putting $x = x_{2n(k)-1}$ and $y = x_{2m(k)}$ in (2.3), and then letting $k \to \infty$, using (2.9),(2.12),(2.13), (2.14) and (2.15), we obtain $\Psi_1(\epsilon) \leqslant \Psi_2(\epsilon) - \Phi(\epsilon)$, which is a contradiction. Therefore, such an $\epsilon > 0$ can not exist. Hence the sequence $\{x_n\}$ is Cauchy. Let $x_n \to z \in X$. By our assumption (1.1), $x_n \preceq z$ for each n. Then following the same steps as in the proof of step 3 of Theorem 2.1 in [10] we conclude that

$$\Psi_1(d(Tz, z)) \leqslant \Psi_2(d(Tz, z)) - \Phi(d(Tz, z)),$$

which is a contradiction unless $d(Tz, z) = 0$ or $Tz = z$. We now have

$$\Psi_1(d(Sz, z)) = \Psi_1(d(Sz, Tz)) \leqslant \Psi_2(M(z, z)) - \Phi(M(z, z)) = \Psi_2(d(z, Sz)) - \Phi(d(z, Sz)),$$

which leads to a contradiction, unless $Sz = z$. Thus we have, $Tz = Sz = z$. This completes the proof of the theorem. $\square$

By putting $S = T$ in the above theorem, we have the following result.

Theorem 2.3. *Let (X, d) be a complete metric space with a partial order $'\preceq'$ and having the property described in (1.1). Let $T : X \to X$ be a self mapping which is non-decreasing and satisfies the following inequality:*

$$\Psi_1(d(Tx, Ty)) \leq \Psi_2(N(x, y)) - \Phi(N(x, y)) \tag{2.16}$$

for all comparable $x, y \in X$, where, $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(y, Tx) + d(x, Ty))\}$, and $\Psi_1, \Psi_2, \Phi : [0, \infty) \to [0, \infty)$ are such that Ψ_1 and Ψ_2 are continuous, Φ is lower semi-continuous, and $\Psi_1(t) - \Psi_2(t) + \Phi(t) > 0$ for all $t > 0$. If X has the property described in (1.1) and if there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

3. Fixed Point of a Single Mapping

Theorem 3.1. *Let (X, d) be a complete metric space with a partial order $' \preceq'$ and $T : X \to X$ be a self mapping which is non-decreasing and satisfies the following inequality:*

$$\Psi_1(d(Tx, Ty)) \leq \Psi_2(N(x, y)) - h(Q(x, y)) \tag{3.1}$$

for $x, y \in X$, where x and y are comparable, $x \neq y$,

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(y, Tx) + d(x, Ty))\}, \tag{3.2}$$

$$Q(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(y, Tx) + d(x, Ty))\}, \tag{3.3}$$

$\Psi_1, \Psi_2, h : [0, \infty) \to [0, \infty)$ are such that Ψ_1 and Ψ_2 are continuous, $h : [0, \infty) \to [0, \infty)$ is monotone decreasing in $(0, \infty)$, lower semi-continuous in $(0, \infty)$ with $h(t) > 0$ for all $t > 0$ and

$$\Psi_1(s) - \Psi_2(s) + h(s) > 0, \quad s > 0. \tag{3.4}$$

Further, for all $x \in X$, we assume

$$d(x, T^2x) \geq 2d(Tx, T^2x) \tag{3.5}$$

If X has the property described in (1.1) and if there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Proof. Starting with $x_0 \in X$, we construct the sequence $\{x_n\}$ as

$$x_n = Tx_{n-1}, n \geq 1 \tag{3.6}$$

Then

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq .... \preceq x_n.... \tag{3.7}$$

Also when $x_n = x_{n+1}$,T has a fixed point. So we assume

$$x_n \neq x_{n+1} \text{ for all } n \geq 0 \tag{3.8}$$

By virtue of (3.7) and (3.8), putting $x = x_n$ and $y = x_{n+1}$ in (3.1), (3.2) and (3.3), for all $n \geq 0$, we obtain

$$\Psi_1(d(x_{n+1}, x_{n+2})) \leq \Psi_2(N(x_n, x_{n+1})) - h(Q(x_n, x_{n+1})) \tag{3.9}$$

where

$$\begin{aligned} N(x_n, x_{n+1}) = \max\{&d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\frac{1}{2}(d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2}))\} \end{aligned} \tag{3.10}$$

and

$$Q(x_n, x_{n+1}) = \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tag{3.11}$$
$$\frac{1}{2}(d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2}))\}.$$

Let, if possible, for some n, $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$.
Then, by the triangle inequality, $0 < d(x_{n+1}, x_{n+2}) - d(x_n, x_{n+1}) \leq d(x_n, x_{n+2})$.
By the above two inequalities, from (3.10) and (3.11), for all $n \geq 0$, we have

$$M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2}) \tag{3.12}$$

and

$$0 < Q(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2}). \tag{3.13}$$

Then, from (3.9), (3.12) and (3.13) and by the monotone decreasing property of the h-function, we have

$$\Psi_1(d(x_{n+1}, x_{n+2})) \leq \Psi_2(d(x_{n+1}, x_{n+2})) - h(d(x_{n+1}, x_{n+2})).$$

The above inequality implies that $d(x_{n+1}, x_{n+2}) = 0$ which contradicts (3.8).
Hence

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}). \tag{3.14}$$

In view of (3.14), for all $n \geq 0$, we obtain

$$M(x_n, x_{n+1}) = d(x_n, x_{n+1}) \tag{3.15}$$

and in view of (3.5) and (3.14), we have

$$\begin{aligned} 0 < Q(x_n, x_{n+1}) &= \frac{1}{2}(d(x_n, x_{n+2})) \\ &\leq \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) \leq d(x_n, x_{n+1}) \end{aligned} \tag{3.16}$$

Using the above relations in (3.9), and by the monotone decreasing property of h, for all $n \geq 0$, we have

$$\Psi_1(d(x_{n+1}, x_{n+2})) < \Psi_2(d(x_n, x_{n+1})) - h(d(x_n, x_{n+1})). \tag{3.17}$$

Again, (3.14) implies that the sequence $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Let, if possible, $r \neq 0$. Taking $n \to \infty$ in (3.17) and using the above relation, by continuity of Ψ_1 and Ψ_2 and by lower semi-continuity of the h, we have $\Psi_1(r) \leq \Psi_2(r) - h(r)$ which contradicts (3.4). Hence,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.18}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. If otherwise, we can have some $\epsilon > 0$ and corresponding subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that for every natural number k, we have $n(k) > m(k) > k$,

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{3.19}$$

and

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{3.20}$$

Then, for each $k > 0$, by (3.19) and (3.20), we have

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) < \epsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Taking $k \to \infty$, by (3.18), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon, \tag{3.21}$$

Similarly we have the following limits.

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon, \tag{3.22}$$

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon, \tag{3.23}$$

and

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \tag{3.24}$$

From (3.19), for all $k \geq 1$, $d(Tx_{m(k)-1}, Tx_{n(k)-1}) \neq 0$, which implies that $x_{m(k)-1} \neq x_{n(k)-1}$.
Also, by (3.7), $x_{m(k)-1}$ and $x_{n(k)-1}$ are comparable.
Hence, putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (3.1), (3.2) and (3.3), we get

$$\Psi_1(d(x_{m(k)}, x_{n(k)})) \leq \Psi_2(N(x_{m(k)-1}, x_{n(k)-1})) - h(Q(x_{m(k)-1}, x_{n(k)-1})) \tag{3.25}$$

$$\begin{aligned} N(x_{m(k)-1}, x_{n(k)-1}) = \max\{& d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)}), \\ & \frac{1}{2}(d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)}))\}, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} Q(x_{m(k)-1}, x_{n(k)-1}) = \min\{& d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)})), \\ & \frac{1}{2}(d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)}))\}. \end{aligned} \tag{3.27}$$

Taking $k \to \infty$ in (3.26) and (3.27) and using (3.18), (3.22), (3.23) and (3.24), we obtain

$$\lim_{k\to\infty} N(x_{m(k)-1}, x_{n(k)-1}) = \epsilon \tag{3.28}$$

and

$$\lim_{k\to\infty} Q(x_{m(k)-1}, x_{n(k)-1}) = 0. \tag{3.29}$$

Then, by the properties of h,

$$\liminf_{k\to\infty} h(Q(x_{m(k)-1}, x_{n(k)-1})) = c > 0. \tag{3.30}$$

Further, taking liminf as $k \to \infty$ in (3.25), using (3.21), (3.28) and (3.30), and the continuities of Ψ_1 and Ψ_2 we obtain

$$\Psi_1(\epsilon) \leq \Psi_2(\epsilon) - c. \tag{3.31}$$

Next we note that the constructions of (3.19) and (3.20) are valid whenever ϵ is replaced by a smaller value. This is because of the fact that for any $p \in X$, $\{x : d(p, x) < \epsilon^{'}\} \subseteq \{x : d(p, x) < \epsilon\}$ whenever $\epsilon^{'} < \epsilon$. Hence (3.30) is also valid if ϵ is replaced by a smaller value.
Then taking $\epsilon \to 0$ in (3.30) we obtain that $c \leq 0$, which is a contradiction. This proves that $\{x_n\}$ is a Cauchy sequence and therefore is convergent in the complete metric space X. Let $x_n \to z$ as $n \to \infty$ Again, by (3.7), $\{x_n\}$ is a monotone increasing sequence. Hence by the property (1.1) we have $x_n \preceq z$ for all $n \geq 0$. Let, if possible, $d(z, Tz) \neq 0$. By (3.8), there exists a subsequences $\{x_{n(j)}\}$ of $\{x_n\}$ such that $z \neq x_{n(j)}$ for all $j \geq 1$. Substituting $x = x_{n(j)}$ and $y = z$ in (3.1), (3.2) and (3.3), we obtain

$$\Psi_1(d(x_{n(j)+1}, Tz)) \leq \Psi_2(M(x_{n(j)}, z)) - h(Q(x_{n(j)}, z)) \tag{3.32}$$

where

$$M(x_{n(j)}, z) = \max\{d(x_{n(j)}, z), d(x_{n(j)}, x_{n(j)+1}), d(z, Tz), \frac{1}{2}(d(x_{n(j)}, Tz) + d(z, x_{n(j)+1})\},$$

and

$$Q(x_{n(j)}, z) = \min\{d(x_{n(j)}, z), d(x_{n(j)}, x_{n(j)+1}), d(z, Tz), \frac{1}{2}(d(x_{n(j)}, Tz) + d(z, x_{n(j)+1})\}.$$

Taking $j \to \infty$ in the above two expressions we obtain

$$\lim_{j\to\infty} M(x_{n(j)}, z) = d(z, Tz) \tag{3.33}$$

and

$$\lim_{j\to\infty} Q(x_{n(j)}, z) = \frac{1}{2} d(z, Tz). \tag{3.34}$$

Letting $j \to \infty$ in (3.32), using (3.33), (3.34), the continuities of Ψ_1 and Ψ_2 and the lower semi continuity of h, we obtain

$$\Psi_1(d(z, Tz)) \leq \Psi_2(d(z, Tz)) - h(\frac{1}{2} d(z, Tz)).$$

Since h is decreasing, the above inequality implies that

$$\Psi_1(d(z, Tz)) \leq \Psi_2(d(z, Tz)) - h(d(z, Tz))$$

which, by (3.4), contradicts an assumption that $d(z, Tz) \neq 0$. Hence $z = Tz$, that is, z is a fixed point of T. This completes the proof of the theorem. $\square$

4. An Example

Example 4.1. Let $X = \{0, 1, 2, 3, 4, ...\}$ and

$$d(x,y) = \begin{cases} x+y, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Then X is a complete metric space. Let a partial order be defined as $x \preceq y$ whenever $y \geq x$. Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x-1, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Let $\Psi_1, \Psi_2, \Phi : [0,\infty) \to [0,\infty)$ be defined as $\Psi_1(t)$= t for all $t \geq 0$,

$$\Psi_2(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 1; \\ t + \frac{1}{t}, & \text{if } t > 1. \end{cases}$$

and

$$\Phi(t) = \begin{cases} 1, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

We next verify that the function T satisfies the inequality (2.15). Without loss of generality we assume that $x > y$. Then we have the following cases:

Case-I. $x \in \{0, 1\}$. Then on the left hand side of (2.15) we have zero, and (2.15) is automatically satisfied in this case.

Case-II. $x = 2, y = 1$.
Then $\Psi_1(d(Tx, Ty)) = \Psi_1(d(1, 0)) = 1$.
N(x,y)=3. Therefore, $\Psi_2(N(x, y)) - \Phi(N(x, y)) = 3 + \frac{1}{3} - 1 = \frac{7}{3}$. Thus (2.1) is satisfied in this case.

Case-III. $x = 2, y = 0$. $\Psi_1(d(Tx, Ty)) = \Psi_1(d(1, 0)) = 1$. $N(x, y) = \frac{3}{2}$. Therefore, $\Psi_2(N(x, y)) - \Phi(N(x, y)) = \frac{3}{2} + \frac{2}{3} - 1 = \frac{7}{6}$. Thus (2.15) is satisfied in this case.

Case-IV. $x \geq 3$,$y > 0$. $N(x, y) = 2x - 1 \geq x + y$.
Then

$$\begin{aligned} \Psi_2(N(x,y)) - \Phi(N(x,y)) &= 2x - 1 + \frac{1}{2x-1} - 1 (\text{ since } x > y) \\ &\geq x + y - 2 = \Psi_1(d(Tx, Ty)) \end{aligned}$$

Thus (2.15) is satisfied in this case.

Case-V. $x \geq 3$,$y > 0$.
Then

$$\Psi_2(N(x,y)) - \Phi(N(x,y)) = 2x - 1 + \frac{1}{2x-1} - 1 \geq (x-1) = \Psi_1(d(Tx,Ty)).$$

Thus (2.15) is satisfied in this case also. Considering all the above cases we see that (2.15) is satisfied for all $x, y \in X$. By an application of corollary 2.2, T has a unique fixed point.

Here '0' is the unique fixed point of T. If we take $h(t) = \Phi(t)$ in the above example, then we see that the inequality (3.1) is satisfied in this case. The differences with the above cases arise only when y=0. It is seen that in these cases also satisfy (3.1). Further, T satisfies (3.5). Also, h satisfies all the requirements of Theorem 3.1. Then Theorem 3.1 is applicable to this example.

Remark 4.1. If we assume $Q(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$ then also the result of Theorem 3.1 is valid. Then (3.5) is no longer a required condition.

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