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SOME REMARKS OF MEDIAL GROUPOIDS *

Petar V. Protić

Abstract. Medial groupoids have been considered in quite a number of papers, especially in [3]. In this paper we describe the natural partial order on a medial band, define some subclasses of the class of medial groupoids, and describe medial band decompositions of medial groupoids.

1. Introduction

A groupoid $(G, \cdot)$ is a *medial* (or *entropic*) groupoid if the following holds:

$$(\forall a, b, c, d \in G)\ (ab)(cd) = (ac)(bd).$$

A groupoid G is a *band* if for all $a \in G$ holds $a^2 = a$. Hence, a groupoid G is a *medial band* if it is a medial groupoid and a band. Let Y be a band. Then a groupoid G is a *band Y of groupoids G_α, $\alpha \in Y$*, if

$$G = \bigcup_{\alpha \in Y} G_\alpha, \qquad G_\alpha \cap G_\beta = \emptyset, \text{ for } \alpha \neq \beta, \quad \text{and} \quad G_\alpha G_\beta \subseteq G_{\alpha\beta}.$$

A congruence ρ on G is called a *band congruence* if G/ρ is a band.

An associative band G is *rectangular* if for all $a, b \in G$ we have $a = aba$. The well-known result of the semigroup theory says that an associative band is a semilattice of rectangular bands [2].

In [4] the authors introduced the notion of an *antirectangular Abel-Grassman's band*. Here we generalize this notion.

Definition 1.1. A band G is *antirectangular* if for every $a, b \in G$ holds $a = (ba)b$.

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Remark 1.1. If G is an antirectangular medial band, $a, b \in G$, then

$$a = (ba)b = (ba)(bb) = (bb)(ab) = b(ab).$$

Example 1.1. Let G be a groupoid given by the following table:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 4 & 2 & 3 \\ 2 & 3 & 2 & 4 & 1 \\ 3 & 4 & 1 & 3 & 2 \\ 4 & 2 & 3 & 1 & 4 \end{array}.$$

Then G is an antirectangular medial band. Also, G is not associative since, for example, $1 = (12)3 \neq 1(23) = 3$.

Remark 1.2. If G is an associative antirectangular band, and $a, b \in G$ such that $a = bab$ and $b = aba$, then

$$a = bab = abaab = abab = bb = b.$$

Hence, a nontrivial associative antirectangular band does not exist.

1. About medial bands

In this section we describe a natural partial order and give some decompositions of a medial band.

Example 1.2. Let an AG-groupoid G be given by the following table:

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 2 & 2 & 5 & 6 & 4 \\ 2 & 2 & 2 & 2 & 5 & 6 & 4 \\ 3 & 2 & 2 & 3 & 5 & 6 & 4 \\ 4 & 6 & 6 & 6 & 4 & 2 & 5 \\ 5 & 4 & 4 & 4 & 6 & 5 & 2 \\ 6 & 5 & 5 & 5 & 2 & 4 & 6 \end{array}.$$

Then G is a medial band.

Theorem 1.1. *Let G be a medial band, then the relation $\leqslant$ defined on G by*

$$e \leqslant f \quad \Leftrightarrow \quad e = ef = fe$$

is a natural partial order on S and it is compatible.

Proof. Clearly, the relation $\leqslant$ is reflexive and symmetric.

If $e \leqslant f,\ f \leqslant g$, then $e = ef = fe,\ f = fg = gf$. Now,

$$eg = (ef)g = (ef)(gg) = (eg)(fg) = (eg)f = (eg)(ff) = (ef)(gf) = ef = e,$$
$$ge = g(fe) = (gg)(fe) = (gf)(ge) = f(ge) = (ff)(ge) = (fg)(fe) = fe = e,$$

what is equivalent to $e \leqslant g$. Hence, $\leqslant$ is transitive, and so it is a partial order on G.

Let $e \leqslant f$ and $g \in G$. Then from $e = ef = fe$ it follows that

$$eg = (ef)(gg) = (eg)(fg),$$
$$eg = (fe)(gg) = (fg)(eg)$$

and so $eg \leqslant fg$. Similarly, $ge \leqslant gf$. Hence, $\leqslant$ is compatible. □

Theorem 1.2. *Let G be a medial band. Then G is a medial band of antirectangular (in the general case nontrivial) medial bands.*

Proof. On a medial band G we define the relation ρ by

$$(\forall a, b \in G) \quad a\,\rho\,b \Longleftrightarrow a = (ba)b,\ b = (ab)a.$$

Clearly, the relation ρ is reflexive and symmetric. Let $a\,\rho\,b,\ b\,\rho\,c$, then by definition of ρ and using Remark 1.1 we have

$$a = (ba)b = (ba)((cb)c) = (b(cb))(ac) = c(ac) = (ca)c,$$
$$c = (bc)b = (bc)((ab)a) = (b(ab))(ca) = a(ca) = (ac)a,$$

so $a\,\rho\,c$ and ρ is a transitive relation. Hence, ρ is an equivalence relation.

Let $a\,\rho\,b,\ c\,\rho\,d$, then

$$ac = ((ba)b)((dc)d) = ((ba)(dc))(bd) = ((bd)(ac))(bd,$$
$$bd = ((ab)a)((cd)c) = ((ab)(cd))(ac) = ((ac)(bd))(ac),$$

whence ρ is a congruence relation. Since G is a medial band, we have that ρ is a medial band congruence relation.

By definition of ρ it follows that ρ-classes are antirectangular bands. □

In Example 1.1 we have $\rho = G \times G$, and in Example 1.2 we have $G = G_\alpha \cup G_\beta \cup G_\gamma$, where $G_\alpha = \{1\},\ G_\beta = \{3\},\ G_\gamma = \{2, 4, 5, 6\}$. Also, if G is an associative band then, by Remark 1.2, the congruence ρ is an identity relation.

2. Some band decompositions of medial groupoids

If a medial groupoid G has an idempotent, then by $E(G)$ we denote the set of all idempotents of G.

Let G be a medial groupoid, then we define the relation μ on G with following:

$$(\forall a, b \in G) \quad a\,\mu\,b \quad \Leftrightarrow \quad a^2 = b^2$$

Theorem 1.3. *The relation μ defined on a medial groupoid G is a congruence relation on G. If G has idempotents then μ is an idempotent-separating congruence on G.*

Proof. Obviously, μ is an equivalence relation. If $a, b, c, d \in G$ and $a \,\mu\, b, c \,\mu\, d$, then

$$(ac)^2 = ac \cdot ac = a^2 \cdot c^2 = b^2 d^2 = (bd)^2,$$

and so $ac \,\mu\, bd$. Hence, μ is a congruence relation.

If $E(G) \neq \emptyset$ and $e, f \in E(G)$, then

$$e \,\mu\, f \quad \Leftrightarrow \quad e = e^2 = f^2 = f,$$

and so μ is an idempotent separating congruence on G. $\square$

Example 1.3. Let G be a groupoid given by the following table

	a	b	c	d	e	f	g	h
a	a	b	g	h	c	d	e	f
b	b	a	h	g	d	c	f	e
c	e	f	c	d	g	h	a	b
d	f	e	d	c	h	g	b	a
e	g	h	a	b	e	f	c	d
f	h	e	b	a	f	e	d	c
g	c	d	e	f	a	b	g	h
h	d	c	f	e	b	a	h	g.

We can easily verify that G is a medial groupoid, but G is not a semigroup, for example $(cb)h = c$ and $c(bh) = g$. Also, G is a quasigroup clearly. Notice that $E(G) = \{a, c, e, g\}$ and for all $x \in G$ holds $x^2 \in E(G)$.

Example 1.4. Let $(G, \cdot)$ be a commutative inverse semigroup and $a, b \in G$ arbitrary elements. We define the operation $*$ on G by $a * b = ba^{-1}$. Then

$$(a * b) * (c * d) = (ba^{-1}) * (dc^{-1}) = dc^{-1}(ba^{-1})^{-1} = dc^{-1}ab^{-1},$$
$$(a * c) * (b * d) = (ca^{-1}) * (db^{-1}) = db^{-1}(ca^{-1})^{-1} = db^{-1}ac^{-1}.$$

By the above and commutativity we conclude that $(G, *)$ is a medial groupoid.

Now, for $a \in G$ we have $a * a = aa^{-1} \in E(G)$. Hence, for each $a \in G$ holds $a^2 = a * a \in E(G)$. Also

$$(a * a) * a = (aa^{-1}) * a = a(aa^{-1})^{-1} = aaa^{-1} = a,$$
$$a * (a * a) = a * (aa^{-1}) = aa^{-1}a^{-1} = a^{-1}.$$

By the above, $(G, *)$ is not a semigroup.

Theorem 1.4. *Let G be a medial groupoid such that $a^2 \in E(G)$, for every $a \in G$. Then μ is the maximal idempotent-separating congruence on S.*

Proof. By the above theorem, the relation μ is an idempotent-separating congruence on G. Let ρ be an arbitrary idempotent-separating congruence on G and let $a, b \in G$ such that $a\,\rho\,b$. By compatibility of ρ we have $a^2\,\rho\,b^2$. Since ρ is idempotent-separating and $a^2, b^2 \in E(G)$, we conclude that $a^2 = b^2$, and therefore, $a\,\mu\,b$. Hence, $\rho \subseteq \mu$ and so μ is the maximal idempotent-separating congruence on G. $\square$

Theorem 1.5. *If G is a medial groupoid such that $a^2 \in E(G)$, for every $a \in G$, then G is a medial band Y of groupoids S_α, $\alpha \in Y$, and for each $x \in S_\alpha$ holds $x^2 = e_\alpha$ where e_α is the unique idempotent of S_α.*

Proof. By the above theorem we have that the relation μ is a maximal idempotent-separating congruence on G. If $a \in G$, then $a^2 = e \in E(G)$ and $(a^2)^2 = a^2 \cdot a^2 = ee = e$ and so $a^2\,\mu\,a$, and hence, μ is a band congruence on G. Therefore, $G = \bigcup_{\alpha\in Y} G_\alpha$ where G_α are μ-classes of elements of G, and Y is a medial band. Classes G_α are unipotent since μ is a idempotent separating congruence. If $a \in G_\alpha$ and e_α is an idempotent in G_α, then $a\,\mu\,e_\alpha$, so $a^2 = e_\alpha^2 = e_\alpha$. $\square$

Lemma 1.1. *Let G be a medial groupoid. Then the relation ρ defined on G by*

$$a\,\rho\,b \quad\Leftrightarrow\quad ab = ba$$

is reflexive, symmetric and compatible.

Proof. The relation ρ is reflexive and symmetric clearly. Let $a\rho b$ and $c\rho d$, then

$$(ac)(bc) = (ab)(cc) = (ba)(cc) = (bc)(ac),$$
$$(ca)(cb) = (cc)(ab) = (cc)(ba) = (cb)(ca).$$

Hence, $ac\,\rho\,bc$ and $ca\,\rho\,cb$. $\square$

Definition 1.2. A medial groupoid G is *transitive commutative* if for $a, b, c \in G$ from $ab = ba$ and $bc = cb$ it follows that $ac = ca$.

The grupoids which are given in Examples 1.1 and 1.2 are transitive commutative medial bands.

Example 1.5. Let $(G, \cdot)$ be an Abelian group with identity e and $a, b \in G$ arbitrary elements. We define the operation $*$ on G with $a * b = ba^{-1}$. Then, by Example 1.4, $(G, *)$ is a medial groupoid. Let for $a, b, c \in G$ holds $a * b = b * a, b * c = c * b$ then $ba^{-1} = ab^{-1}, cb^{-1} = bc^{-1}$ and so

$$\begin{aligned} a * c = ca^{-1} &= cea^{-1} = cb^{-1}ba^{-1} = (bc^{-1})^{-1}(ab^{-1})^{-1} \\ &= (cb^{-1})^{-1}(ba^{-1})^{-1} = bc^{-1}ab^{-1} = abb^{-1}c^{-1} = ac^{-1} = c * a. \end{aligned}$$

Hence, $(G, *)$ is a transitive commutative medial groupoid.

Theorem 1.6. *Let G be a transitive commutative medial groupoid. Then G is a disjoint union of commutative semigroups.*

Proof. Let ρ be the relation defined in the above lemma. If $a\,\rho\,b, b\,\rho\,c$, then $ab = ba$ and $bc = cb$, and since G is a transitive commutative semigroup we have $ac = ca$, and so $a\,\rho\,c$. By the above lemma it follows that ρ is a congruence on G. Hence, G is the union of ρ-classes which are a commutative semigroups. □

A groupoid G is *locally associative* if for every $a \in G$ holds $a \cdot a^2 = a^2 \cdot a$.

Remark 1.3. If G is a transitive commutative locally associative medial groupoid, then for every $a \in G$, by the above theorem, we have that $a\,\rho\,a^2$ and so ρ is a band congruence. Hence, a locally associative medial groupoid G is a band of commutative semigroups.

Let G be a groupoid given in Example 1.4. It is easy to verify that in G holds $a = a^2 \cdot a = a \cdot a^2$ for every $a \in G$. Now, $G = \bigcup_{\alpha \in Y} G_\alpha$, $Y = E(G) = \{a, c, e, g\}$ and $G_a = \{a, b\}$, $G_c = \{c, d\}$, $G_e = \{e, f\}$, $G_g = \{g, h\}$ are commutative semigroups.

3. Medial 3-bands

In the paper [5] the authors defined Abel-Grassmann's 3-bands. Here we generalize this notion.

Definition 1.3. Let G be a groupoid. An element $a \in G$ is *left* 3*-potent* if $a^2 \cdot a = a$, *right* 3*-potent* if $a \cdot a^2 = a$, and 3*-potent* if it is both left and right 3-potent.

For a groupoid G by $T(G)$ we denote the set of all 3-potents of G.

Lemma 1.2. *If G is a medial groupoid and $T(G) \neq \emptyset$, then $T(G)$ is a subgroupoid of G.*

Proof. Let $a, b \in T(G)$. Then

$$(ab)^3 = (ab \cdot ab)ab = (a^2 \cdot b^2)ab = (a^2 \cdot a)(b^2 \cdot b) = a^3 \cdot b^3 = ab,$$

and so $ab \in T(G)$. □

Definition 1.4. A groupoid G is a 3-band (left 3-band, right 3-band) if every element in G is 3-potent (left 3-potent, right 3-potent).

The groupoid $(G, *)$ given in Example 3 is a left 3-band, and the groupoids given in Examples 1 and 2 are 3-bands.

If a medial groupoid G is a 3-band, then we call it a *medial* 3*-band*.

Lemma 1.3. *Let G be a medial groupoid and $T(G) \neq \emptyset$. Then $T(G)$ is a 3-band and the set $B = \{b \in G \mid (\exists a \in T(G)\; b = a^2\}$ is a subgroupoid of G.*

Proof. By above lemma, $T(G)$ is a 3-band .

For $x, y \in B$ there exist $a, b \in T(G)$ such that $x = a^2$, $y = b^2$, so $xy = a^2 \cdot b^2 = (ab)^2$. Since $ab \in T(G)$, we have that $xy \in B$. □

Definition 1.5. A groupoid G is an *antirectangular groupoid* if for all $a, b \in G$ holds $a = (ba)b = b(ab)$

Theorem 1.7. *Let G be a medial 3-band. Then G is a medial band of antirectangular 3-bands.*

Proof. Let G be a medial 3-band. On G we define a relation η by

$$a\,\eta\,b \quad\Leftrightarrow\quad a = (ba)b = b(ab),\ b = (ab)a = a(ba).$$

We will prove that η is a band congruence on G.

Clearly, η is reflexive and symmetric. If $a, b, c \in G$ such that $a\,\eta\,b$ and $b\,\eta\,c$, then

$$a = (ba)b = b(ab),\ b = (ab)a = a(ba),$$
$$b = (ca)c = c(ac),\ c = (bc)b = b(cb),$$

so

$$a = b(ab) = (c(bc))(ab) = (ca)((bc)b) = (ca)c,$$
$$a = (ba)b = (ba)((cb)c) = (b(cb))(ac) = c(ac),$$
$$c = b(cb) = (a(ba))(cb) = (ac)((ba)b) = (ac)a,$$
$$c = (bc)b = (bc)((ab)a) = (b(ab))(ca) = a(ca),$$

and hence, $a\,\eta\,c$. Thus, η is a transitive relation, i.e., it is an equivalence relation.

Now, If $a, b, c, d \in G$ such that $a\,\eta\,b$ and $c\,\eta\,d$, then

$$a = (ba)b = b(ab),\ b = (ab)a = a(ba),$$
$$c = (dc)d = d(cd), d = (cd)c = c(dc),$$

whence

$$ac = ((ba)b)((dc)d) = ((ba)(dc))(bd) = ((bd)(ac))(bd),$$
$$ac = (b(ab))(d(cd)) = (bd)((ab)(cd)) = (bd)((ac)(bd)),$$
$$bd = ((ab)a)((cd)c) = ((ab)(cd))(ac) = ((ac)(bd))(ac),$$
$$bd = (a(ba))(c(dc)) = (ac)((ba)(dc)) = (ac)((bd)(ac)),$$

and therefore, $ac\,\eta\,bd$. Hence, η is a congruence relation.

Since G is a 3-band we have

$$a = (a^2 \cdot a)a^2 = a^2(a \cdot a^2), a^2 = (a \cdot a^2)a = a(a^2 \cdot a),$$

whence $a\,\eta\,a^2$ and so η is a band congruence on G.

Hence, G is a band of antirectangular 3-bands. □

According to Definition 1.5, a semigroup G is antirectangular if for all $a, b \in G$ holds $a = bab$, $b = aba$.

Theorem 1.8. *Let G be a 3-potent medial semigroup. Then G is a medial band of unipotent antirectangular 3-bands.*

Proof. Define a relation η on G by

$$a\,\eta\,b \quad\Leftrightarrow\quad a = bab,\ b = aba.$$

The η is reflexive and symmetric. Let $a, b, c \in G$ such that $a\,\eta\,b$ and $b\,\eta\,c$, i.e.,

$$a = bab,\ b = aba,\ b = cbc,\ c = bcb.$$

Then

$$a = bab = bacbc = bcbac = cac,\ c = bcb = bcaba = babca = aca,$$

so $a\,\eta\,c$. Hence, η is a transitive relation, so it is an equivalence relation.

Let $a, b, c, d \in G$ such that $a\,\eta\,b,\ c\,\eta\,d$. Then

$$ac = babdcd = bdacbd,\ bd = abacdc = acbdac,$$

whence $ac\,\eta\,bd$, so η is a congruence relation.

Moreover, for an arbitrary $a \in G$ by

$$a = a^2aa^2,\ a^2 = a^2aa^2$$

it follows that $a\,\eta\,a^2$, so η is a band congruence.

Let $a, b \in G$. Then $a^2, b^2 \in E(G)$, and if $a\,\eta\,b$, then $a^2\,\eta\,b$, i.e., $a^2 = ba^2b, b = a^2ba^2$, so

$$a^2 = ba^2b = ba^2a^2ba^2 = ba^2ba^2 = bb = b^2.$$

Hence, G is a band of unipotent antirectangular 3-bands. □

Example 1.6. Let a groupoid G be given by the following table.

	1	2	3	4
1	2	1	1	2
2	1	2	2	1
3	4	3	3	4
4	3	4	4	3

Then G is a medial 3-potent semigroup. Since $1 = 212, 2 = 121, 3 = 434, 4 = 343$, we have that $1\,\eta\,2, 3\,\eta\,4$. Hence $\{1, 2\}$ and $\{3, 4\}$ are unipotent antirectangular 3-bands.

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P. V. Protić
University of Niš
Faculty of Civil Engineering
Aleksandra Medvedeva 14
18000 Niš, Serbia
pvprotic@yahoo.com