

SHARP MAXIMAL FUNCTION AND MORREY SPACES
ESTIMATES FOR MULTILINEAR COMMUTATOR OF SINGULAR
INTEGRAL OPERATORS WITH NON-SMOOTH KERNELS *

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Abstract. In this paper, we will study sharp maximal function estimates for some multilinear commutator of the singular integral operators with non-smooth kernels. As their applications, the boundedness of these operators on Lebesgue spaces and Morrey type spaces are obtained.

1. Introduction and Notations

Let T be the Calderón-Zygmund singular integral operator, a well known result of Coifman, Rochberg and Weiss states that the commutator $[b, T](f) = bT(f) - T(bf)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [2][4][7][11-14], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. In [3] and [10], the boundedness of the singular integral operators with non-smooth kernels and their commutators are obtained. The main purpose of this paper is to study the properties of multilinear commutator of singular integral operators with non-smooth kernels by means of the establishment of their sharp maximal function estimates.

First, let us introduce some notations. Throughout this paper, $Q = Q(x, d)$ will denote a cube of R^n with sides parallel to the axes, whose center is x and side length is d . For a locally integrable function b , the sharp function of b is defined by

$$b^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy,$$

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where, and in what follows, $b_Q = |Q|^{-1} \int_Q b(x)dx$. It is well-known that (see [6])

$$b^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |b(y) - c| dy.$$

We say that b belongs to $BMO(R^n)$ if $b^\#$ belongs to $L^\infty(R^n)$ and $\|b\|_{BMO} = \|b^\#\|_{L^\infty}$. For $0 < \beta < 1$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{Lip_\beta} = \sup_{\substack{x, y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 1.1 A family of operators $D_t, t > 0$ is said to be an "approximation to the identity" if, for every $t > 0$, D_t can be represented by the kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x, y) f(y) dy$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies:

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2}s(|x - y|^2/t),$$

where s is a positive, bounded and decreasing function satisfying

$$\lim_{r \rightarrow \infty} r^{n+\epsilon} s(r^2) = 0$$

for some $\epsilon > 0$.

Definition 1.2 A linear operator T is called the singular integral operators with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every continuous function f with compact support, and for almost all x not in the support of f .

(1) There exists an "approximation to the identity" $\{B_t, t > 0\}$ such that TB_t has associated kernel $k_t(x, y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1t^{1/2}} |K(x, y) - k_t(x, y)| dx \leq c_2 \quad \text{for all } y \in R^n.$$

(2) There exists an "approximation to the identity" $\{A_t, t > 0\}$ such that $A_t T$ has associated kernel $K_t(x, y)$ which satisfies

$$\left(\int_{2Q} |K_t(x, y)|^q dy \right)^{1/q} \leq C|2Q|^{-1/q'}$$

and

$$\left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x, y) - K_t(x, y)|^q dy \right)^{1/q} \leq C_k (2^k|x-x_0|)^{-n/q'} \\ \text{if } |x-y| \geq c_3 t^{1/2},$$

for $c_3 > 0$, where $\{C_k\}$ is a sequence of positive constant numbers for each $k \in N$ and (q, q') is a fix pair of positive numbers with $1 < q < \infty$ and $1/q + 1/q' = 1$.

Given some locally integrable functions b_j ($j = 1, \dots, m$). The multilinear operator associated to T is defined by

$$T_b(f)(x) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] K(x, y) f(y) dy.$$

For $b_j \in BMO(R^n)$ ($j = 1, \dots, m$), set

$$\|\vec{b}\|_{BMO} = \prod_{j=1}^m \|b_j\|_{BMO}.$$

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$. When $b_j \in Lip_\beta(R^n)$, we have the similar results.

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. The sharp maximal function $M_A(f)$ associated with the "approximations to the identity" $\{A_t, t > 0\}$ is defined by

$$M_A^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)| dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q .

Definition 1.3 For $1 \leq r < \infty$ and $0 < \beta < n$, the fractional maximal operator $M_{\beta,r}$ is defined by

$$M_{\beta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

2. Sharp Maximal Function Estimates

Theorem 2.1 Let T be a singular integral operator with non-smooth kernel as in Definition 1.2. If the sequence $\{C_k\} \in l^1$, then for any $1 < s < \infty$, there exists a constant $C > 0$ such that for all smooth functions f with compact support, and any $\tilde{x} \in R^n$,

$$M_A^\#(Tf)(\tilde{x}) \leq CM_s(f)(\tilde{x}), \text{a.e.}$$

Proof. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$,

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |T(f)(x) - A_{t_Q} T(f)(x)| dx \\ &= \frac{1}{|Q|} \int_Q |T(f_1)(x) + T(f_2)(x) - A_{t_Q} T(f_1)(x) - A_{t_Q} T(f_2)(x)| dx \\ &\leq \frac{1}{|Q|} \int_Q |T(f_1)(x)| dx + \frac{1}{|Q|} \int_Q |A_{t_Q} T(f_1)(x)| dx + \frac{1}{|Q|} \int_Q |T(f_2)(x) - A_{t_Q} T(f_2)(x)| dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For I_1 , by the Hölder's inequality and the L^s -boundedness of T , we get

$$\begin{aligned} I_1 &\leq \left(\frac{1}{|Q|} \int_Q |T(f\chi_{2Q})(x)|^s dx \right)^{1/s} \\ &\leq C \left(\frac{1}{|Q|} \int_{2Q} |f(y)|^s dy \right)^{1/s} \\ &\leq CM_s(f)(\tilde{x}). \end{aligned}$$

For I_2, I_3 , when $1/q + 1/q' = 1$ and $q' \leq s < \infty$, by (2) of Definition 1.2, we have

$$\begin{aligned} I_2 &\leq \frac{1}{|Q|} \int_Q \int_{R^n} |K_t(x, y)| |f(y)\chi_{2Q}(y)| dy dx \\ &\leq \frac{1}{|Q|} \int_Q \left(\int_{2Q} |K_t(x, y)|^q dy \right)^{1/q} \left(\int_{2Q} |f(y)|^{q'} dy \right)^{1/q'} dx \\ &\leq C \frac{1}{|Q|} \int_Q |2Q|^{-1/q'} \left(\int_{2Q} |f(y)|^{q'} dy \right)^{1/q'} dx \end{aligned}$$

$$\begin{aligned}
&\leq C \frac{1}{|Q|} \int_Q \left(\frac{1}{|2Q|} \int_{2Q} |f(y)|^{q'} dy \right)^{1/q'} dx \\
&\leq C \frac{1}{|Q|} \int_Q \left(\frac{1}{|2Q|} \int_{2Q} |f(y)|^s dy \right)^{1/s} dx \\
&\leq CM_s(f)(\tilde{x}).
\end{aligned}$$

$$\begin{aligned}
I_3 &\leq \frac{1}{|Q|} \int_Q \int_{(2Q)^c} |K(x, y) - K_t(x, y)| |f(y)| dy dx \\
&\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x, y) - K_t(x, y)| |f(y)| dy dx \\
&\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x, y) - K_t(x, y)|^q dy \right)^{1/q} \\
&\quad \times \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |f(y)|^{q'} dy \right)^{1/q'} dx \\
&\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} C_k (2^k|x-x_0|)^{-n/q'} \left(\int_{Q(x_0, 2^{k+1}|x-x_0|)} |f(y)|^{q'} dy \right)^{1/q'} dx \\
&\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} C_k \left(\frac{1}{|Q(x_0, 2^{k+1}|x-x_0|)|} \int_{Q(x_0, 2^{k+1}|x-x_0|)} |f(y)|^{q'} dy \right)^{1/q'} dx \\
&\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} C_k \left(\frac{1}{|Q(x_0, 2^{k+1}|x-x_0|)|} \int_{Q(x_0, 2^{k+1}|x-x_0|)} |f(y)|^s dy \right)^{1/s} dx \\
&\leq CM_s(f)(\tilde{x}) \sum_{k=1}^{\infty} C_k \\
&\leq CM_s(f)(\tilde{x}).
\end{aligned}$$

This completes the proof of the theorem .

Theorem 2.2 Let T be a singular integral operators with non-smooth kernels as in Definition 1.2. If the sequence $\{kC_k\} \in l^1$ and $b_j \in BMO(R^n)$ for $j = 1, \dots, m$, then for any $1 < s < \infty$, there exists a constant $C > 0$ such that for all smooth

functions f with compact support, and any $\tilde{x} \in R^n$,

$$M_A^\#(T_b(f))(\tilde{x}) \leq C\|\vec{b}\|_{BMO} \left(M_s(f)(\tilde{x}) + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} M_s(T_{b_{\sigma^c}}(f))(\tilde{x}) \right), \text{a.e.}$$

Proof. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$, We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$,

$$\begin{aligned} T_b(f)(x) &= \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] K(x, y) f(y) dy \\ &= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})] K(x, y) f(y) dy \\ &= \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \int_{R^n} K(x, y) f(y) dy \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_{\sigma^c} K(x, y) f(y) dy \\ &\quad + (-1)^m \int_{R^n} \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) K(x, y) f(y) dy \end{aligned}$$

and

$$\begin{aligned} A_{t_Q} T_b(f)(x) &= \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] K_t(x, y) f(y) dy \\ &= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})] K_t(x, y) f(y) dy \\ &= \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \int_{R^n} K_t(x, y) f(y) dy \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_{\sigma^c} K_t(x, y) f(y) dy \\ &\quad + (-1)^m \int_{R^n} \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) K_t(x, y) f(y) dy, \end{aligned}$$

then

$$|T_b(f)(x) - A_{t_Q} T_b(f)(x)| \leq \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \int_{R^n} K(x, y) f_1(y) dy \right|$$

$$\begin{aligned}
& + \left| \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_{\sigma^c} K(x, y) f_1(y) dy \right| \\
& + \left| \int_{R^n} \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) K(x, y) f_1(y) dy \right| \\
& + \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \int_{R^n} K_t(x, y) f_1(y) dy \right| \\
& + \left| \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_{\sigma^c} K_t(x, y) f_1(y) dy \right| \\
& + \left| \int_{R^n} \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) K_t(x, y) f_1(y) dy \right| \\
& + \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \int_{R^n} (K(x, y) - K_t(x, y)) f_2(y) dy \right| \\
& + \left| \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_{\sigma^c} (K(x, y) - K_t(x, y)) f_2(y) dy \right| \\
& + \left| \int_{R^n} \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) (K(x, y) - K_t(x, y)) f_2(y) dy \right| \\
& = I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x) + I_6(x) + I_7(x) + I_8(x) + I_9(x),
\end{aligned}$$

thus

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q |T_b(f)(x) - A_{t_Q} T_b(f)(x)| dx \\
& \leq \frac{1}{|Q|} \int_Q I_1(x) dx + \frac{1}{|Q|} \int_Q I_2(x) dx + \frac{1}{|Q|} \int_Q I_3(x) dx + \frac{1}{|Q|} \int_Q I_4(x) dx + \frac{1}{|Q|} \int_Q I_5(x) dx \\
& \quad + \frac{1}{|Q|} \int_Q I_6(x) dx + \frac{1}{|Q|} \int_Q I_7(x) dx + \frac{1}{|Q|} \int_Q I_8(x) dx + \frac{1}{|Q|} \int_Q I_9(x) dx \\
& = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9.
\end{aligned}$$

For I_1 , by the Hölder's inequality, and the L^s -boundedness of T , we get

$$I_1 \leq \frac{1}{|Q|} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \right| |T(f \chi_{2Q})(x)| dx$$

$$\begin{aligned} &\leq \left(\frac{1}{|Q|} \int_Q \prod_{j=1}^m |(b_j(x) - (b_j)_{2Q})|^{s'} dx \right)^{1/s'} \left(\frac{1}{|Q|} \int_Q |T(f\chi_{2Q})(x)|^s dx \right)^{1/s} \\ &\leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}). \end{aligned}$$

For I_2 , by the Minkowski's inequality, we get

$$\begin{aligned} I_2 &\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma| |T((b - (b)_{2Q})_{\sigma^c} f \chi_{2Q})(x)| dx \\ &\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|2Q|} \int_{2Q} |(b(x) - (b)_{2Q})_\sigma|^{s'} dx \right)^{1/s'} \left(\frac{1}{|Q|} \int_Q |T_{b_{\sigma^c}}(f)(x)|^s dx \right)^{1/s} \\ &\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} M_s(T_{b_{\sigma^c}}(f))(\tilde{x}). \end{aligned}$$

Take an s_0 such that $1 < s_0 < s < \infty$. Denote by $1/s_1 = 1/s_0 - 1/s$, then $1 < s_1 < \infty$. To estimate I_3 , we use Hölder's inequality and the L^{s_0} -boundedness of T .

$$\begin{aligned} I_3 &\leq \left(\frac{1}{|Q|} \int_Q |T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f \chi_{2Q})(x)|^{s_0} dx \right)^{1/s_0} \\ &\leq C \left(\frac{1}{|Q|} \int_Q |\prod_{j=1}^m (b_j(x) - (b_j)_{2Q})|^{s_0} |f(x) \chi_{2Q}(x)|^{s_0} dx \right)^{1/s_0} \\ &\leq C \left(\frac{1}{|2Q|} \int_{2Q} \prod_{j=1}^m |b_j(x) - (b_j)_{2Q}|^{s_1} dx \right)^{1/s_1} \left(\frac{1}{|2Q|} \int_{2Q} |f(x)|^s dx \right)^{1/s} \\ &\leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}). \end{aligned}$$

For I_4 , when $1/q + 1/q' = 1$, $1/s + 1/s' = 1$ and $q \leq s'$, $s < \infty$, by Hölder's inequality and (2) of Definition 1.2, we have

$$\begin{aligned} I_4 &\leq \frac{1}{|Q|} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \int_{R^n} K_t(x, y) f_1(y) dy \right| dx \\ &\leq \left(\frac{1}{|Q|} \int_Q \prod_{j=1}^m |b_j(x) - (b_j)_{2Q}|^{q'} dx \right)^{1/q'} \left(\frac{1}{|Q|} \int_Q \int_{R^n} |K_t(x, y)|^q |f_1(y)|^q dy dx \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
&\leq C\|\vec{b}\|_{BMO} \left[\frac{1}{|Q|} \int_Q \left(\int_{2Q} |K_t(x, y)|^{s'} dy \right)^{q/s'} \left(\int_{2Q} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
&\leq C\|\vec{b}\|_{BMO} \left[\frac{1}{|Q|} \int_Q |2Q|^{-q/s} \left(\int_{2Q} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
&\leq C\|\vec{b}\|_{BMO} \left(\frac{1}{|2Q|} \int_{2Q} |f(y)|^s dy \right)^{1/s} \\
&\leq C\|\vec{b}\|_{BMO} M_s(f)(\tilde{x}).
\end{aligned}$$

For I_5, I_6 when $1/s + 1/s_1 + 1/s_2 = 1$, we have

$$\begin{aligned}
I_5 &\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma| \int_{R^n} |(b(y) - (b)_{2Q})_{\sigma^c}| |K_t(x, y)| |f_1(y)| dy dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma|^{q'} dx \right)^{1/q'} \\
&\quad \times \left(\frac{1}{|Q|} \int_Q \int_{R^n} |(b(y) - (b)_{2Q})_{\sigma^c}|^q |K_t(x, y)|^q |f_1(y)|^q dy dx \right)^{1/q} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \\
&\quad \times \left[\frac{1}{|Q|} \int_Q \left(\int_{2Q} |f(y)|^s dy \right)^{q/s} \left(\int_{2Q} |(b(y) - (b)_{2Q})_{\sigma^c}|^{s_1} dy \right)^{q/s_1} \left(\int_{2Q} |K_t(x, y)|^{s_2} dy \right)^{q/s_2} dx \right]^{1/q} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \left(\int_{2Q} |(b(y) - (b)_{2Q})_{\sigma^c}|^{s_1} dy \right)^{1/s_1} |2Q|^{-1/s'_2} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \|\vec{b}_{\sigma^c}\|_{BMO} |2Q|^{1/s_1} |2Q|^{-1+1/s_2} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} \\
&\leq C\|\vec{b}\|_{BMO} M_s(f)(\tilde{x}).
\end{aligned}$$

$$I_6 \leq \frac{1}{|Q|} \int_Q \left(\int_{2Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right|^{s_1} dy \right)^{1/s_1} \left(\int_{2Q} |K_t(x, y)|^{s_2} dy \right)^{1/s_2}$$

$$\begin{aligned}
& \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} dx \\
& \leq C \|\vec{b}\|_{BMO} |2Q|^{1/s_1} |2Q|^{-1+1/s_2} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} \\
& \leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}).
\end{aligned}$$

For I_7, I_8, I_9 , we have

$$\begin{aligned}
I_7 & \leq \left(\frac{1}{|Q|} \int_Q \prod_{j=1}^m |b_j(x) - (b_j)_{2Q}|^{q'} dx \right)^{1/q'} \\
& \quad \left(\frac{1}{|Q|} \int_Q \int_{R^n} |K(x, y) - K_t(x, y)|^q |f_2(y)|^q dy dx \right)^{1/q} \\
& \leq C \|\vec{b}\|_{BMO} \left[\frac{1}{|Q|} \int_Q \left(\int_{2Q^c} |K(x, y) - K_t(x, y)|^{s'} dy \right)^{q/s'} \left(\int_{2Q^c} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
& \leq C \|\vec{b}\|_{BMO} \left[\frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x, y) - K_t(x, y)|^{s'} dy \right)^{q/s'} \right. \\
& \quad \times \left. \left(\int_{Q(x_0, 2^{k+1}|x-x_0|)} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
& \leq C \|\vec{b}\|_{BMO} \left[\frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} C_k (2^k|x-x_0|)^{-nq/s} \left(\int_{Q(x_0, 2^{k+1}|x-x_0|)} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
& \leq C \|\vec{b}\|_{BMO} \left[\frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} C_k \left(\frac{1}{|Q(x_0, 2^{k+1}|x-x_0|)|} \int_{Q(x_0, 2^{k+1}|x-x_0|)} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
& \leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}) \sum_{k=1}^{\infty} C_k \\
& \leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}).
\end{aligned}$$

$$\begin{aligned}
I_8 &\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma| \int_{R^n} |(b(y) - (b)_{2Q})_{\sigma^c}| |K(x, y) - K_t(x, y)| |f_2(y)| dy dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma|^{q'} dx \right)^{1/q'} \\
&\quad \times \left(\frac{1}{|Q|} \int_Q \int_{R^n} |(b(y) - (b)_{2Q})_{\sigma^c}|^q |K(x, y) - K_t(x, y)|^q |f_2(y)|^q dy dx \right)^{1/q} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \left[\frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x, y) - K_t(x, y)|^{s_2} dy \right)^{q/s_2} \right. \\
&\quad \times \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |(b(y) - (b)_{2Q})_{\sigma^c}|^{s_1} dy \right)^{q/s_1} \\
&\quad \times \left. \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \left[\frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} C_k (2^k|x-x_0|)^{-nq/s'_2} \left(\int_{2^{k+1}Q} |(b(y) - (b)_{2Q})_{\sigma^c}|^{s_1} dy \right)^{q/s_1} \right. \\
&\quad \times \left. \left(\int_{2^{k+1}Q} |f(y)|^s dy \right)^{q/s} dx \right]^{1/q} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \left(\frac{1}{|Q|} \int_Q |x-x_0|^{-nq/s'_2} dx \right)^{1/q} \\
&\quad \times \sum_{k=1}^{\infty} C_k 2^{-kn/s'_2} |2^{k+1}Q|^{1/s_2} \|\vec{b}_{\sigma^c}\|_{BMO} M_s(f)(\tilde{x}) \\
&\leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}) \sum_{k=1}^{\infty} k C_k \\
&\leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}).
\end{aligned}$$

$$\begin{aligned}
I_9 &\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x,y) - K_t(x,y)|^{s_2} dy \right)^{1/s_2} \\
&\quad \times \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} \prod_{j=1}^m |b_j(y) - (b_j)_{2Q}|^{s_1} dy \right)^{1/s_1} \\
&\quad \times \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |f(y)|^s dy \right)^{1/s} dx \\
&\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} C_k (2^k|x-x_0|)^{-n/s'_2} \left(\int_{2^{k+1}Q} \prod_{j=1}^m |b_j(y) - (b_j)_{2Q}|^{s_1} dy \right)^{1/s_1} \\
&\quad \times \left(\int_{2^{k+1}Q} |f(y)|^s dy \right)^{1/s} dx \\
&\leq \frac{1}{|Q|} \int_Q |x-x_0|^{-n/s'_2} dx \sum_{k=1}^{\infty} C_k 2^{-kn/s'_2} |2^{k+1}Q|^{1/s_2} \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}) \\
&\leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}) \sum_{k=1}^{\infty} k C_k \\
&\leq C \|\vec{b}\|_{BMO} M_s(f)(\tilde{x}).
\end{aligned}$$

This completes the proof of the theorem .

Theorem 2.3 Let T be a singular integral operators with non-smooth kernels as in Definition 1.2. If the sequence $\{C_k\} \in l^1$ and $b_j \in Lip_{\beta}(R^n)$ for $j = 1, \dots, m$ with $0 < \beta < 1$, then for any $1 < s < \infty$, there exists a constant $C > 0$ such that for all smooth functions f with compact support, and any $\tilde{x} \in R^n$,

$$M_A^\#(T_b(f))(\tilde{x}) \leq C \|\vec{b}\|_{Lip_{\beta}} M_{m\beta,s}(f)(\tilde{x}), \text{a.e.}$$

Proof. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$, We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$, same as in Theorem 2.2, we write

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q |T_b(f)(x) - A_{t_Q} T_b(f)(x)| dx \\
&\leq \frac{1}{|Q|} \int_Q I_1(x) dx + \frac{1}{|Q|} \int_Q I_2(x) dx + \frac{1}{|Q|} \int_Q I_3(x) dx + \frac{1}{|Q|} \int_Q I_4(x) dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{|Q|} \int_Q I_5(x) dx + \frac{1}{|Q|} \int_Q I_6(x) dx + \frac{1}{|Q|} \int_Q I_7(x) dx \\
& + \frac{1}{|Q|} \int_Q I_8(x) dx + \frac{1}{|Q|} \int_Q I_9(x) dx \\
= & J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9.
\end{aligned}$$

Hölder's inequality gives that

$$\begin{aligned}
J_1 & \leq C \|\vec{b}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \left(\frac{1}{|Q|} \int_Q |T(f\chi_{2Q})(x)|^s dx \right)^{1/s} \\
& \leq C \|\vec{b}\|_{Lip_\beta} \left(\frac{1}{|Q|^{1-m\beta s/n}} \int_{2Q} |f(y)|^s dy \right)^{1/s} \\
& \leq C \|\vec{b}\|_{Lip_\beta} M_{m\beta,s}(f)(\tilde{x}).
\end{aligned}$$

For J_2 , let η, η' be the integers such that $\eta + \eta' = m$, $0 \leq \eta < m$, $0 < \eta' \leq m$. By using the Hölder's inequality, the weighted boundedness of T on L^r we get

$$\begin{aligned}
J_2 & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \left(\int_Q |(b(x) - b_{2Q})_\sigma|^{s'} dx \right)^{1/s'} \left(\int_Q |T((b - b_{2Q})_{\sigma^c} f \chi_{2Q})(x)|^s dx \right)^{1/s} \\
& \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \left(\int_Q |(b(x) - b_{2Q})_\sigma|^{s'} dx \right)^{1/s'} \left(\int_{2Q} |(b(y) - b_{2Q})_{\sigma^c} f(y)|^s dy \right)^{1/s} \\
& \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \|\vec{b}_\sigma\|_{Lip_\beta} |Q|^{\frac{\eta\beta}{n}} |Q|^{1/s'} \|\vec{b}_{\sigma^c}\|_{Lip_\beta} |Q|^{\frac{\eta'\beta}{n}} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} \\
& \leq C \|\vec{b}\|_{Lip_\beta} M_{m\beta,s}(f)(\tilde{x}).
\end{aligned}$$

For J_3 , by Hölder's inequality, we have

$$\begin{aligned}
J_3 & \leq \frac{1}{|Q|} \left(\int_Q |T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f \chi_{2Q})(x)|^s dx \right)^{1/s} |Q|^{1-1/s} \\
& \leq \frac{C}{|Q|} |Q|^{1-1/s} \left(\int_{2Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) f(y) \right|^s dy \right)^{1/s} \\
& \leq \frac{C}{|Q|} |Q|^{1-1/s} \|\vec{b}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s}
\end{aligned}$$

$$\leq C\|\vec{b}\|_{Lip_\beta} M_{m\beta,s}(f)(\tilde{x}).$$

For J_4 , J_5 and J_6 , by Hölder's inequality and (2) of Definition 1.2, we have

$$\begin{aligned}
J_4 &\leq \frac{1}{|Q|} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \int_{R^n} K_t(x,y) f \chi_{2Q}(y) dy \right| dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \frac{1}{|Q|} \int_Q \left(\int_{2Q} |K_t(x,y)|^{s'} dy \right)^{1/s'} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \frac{1}{|Q|} \int_Q |2Q|^{-1/s} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} M_{m\beta,s}(f)(\tilde{x}). \\
J_5 &\leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma| \int_{R^n} |(b(y) - (b)_{2Q})_{\sigma^c}| |K_t(x,y)| |f_1(y)| dy dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{Lip_\beta} \|\vec{b}_{\sigma^c}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \frac{1}{|Q|} \int_Q \int_{R^n} |K_t(x,y)| |f_1(y)| dy dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \frac{1}{|Q|} \int_Q \left(\int_{2Q} |K_t(x,y)|^{s'} dy \right)^{1/s'} \left(\int_{2Q} |f(y)|^s dy \right)^{1/s} dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} M_{m\beta,s}(f)(\tilde{x}). \\
J_6 &\leq C\|\vec{b}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \frac{1}{|Q|} \int_Q \int_{R^n} |K_t(x,y)| |f_1(y)| dy dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} M_{m\beta,s}(f)(\tilde{x}).
\end{aligned}$$

For J_7 , J_8 and J_9 , let η, η' be the integers such that $\eta + \eta' = m$, $0 \leq \eta < m$, $0 < \eta' \leq m$, we have

$$\begin{aligned}
J_7 &\leq \frac{1}{|Q|} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \right| \int_{R^n} |K(x,y) - K_t(x,y)| |f_2(y)| dy dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} |Q|^{\frac{m\beta}{n}} \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x,y) - K_t(x,y)|^{s'} dy \right)^{1/s'} \\
&\quad \times \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |f(y)|^s dy \right)^{1/s} dx
\end{aligned}$$

$$\begin{aligned}
&\leq C\|\vec{b}\|_{Lip_\beta}|Q|^{\frac{m\beta}{n}}\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}C_k(2^k|x-x_0|)^{-n/s}\left(\int_{Q(x_0,2^{k+1}|x-x_0|)}|f(y)|^sdy\right)^{1/s}dx \\
&\leq C\|\vec{b}\|_{Lip_\beta}\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}C_k|2^{k+1}Q|^{\frac{m\beta}{n}} \\
&\quad \times\left(\frac{1}{|Q(x_0,2^{k+1}|x-x_0|)|}\int_{Q(x_0,2^{k+1}|x-x_0|)}|f(y)|^sdy\right)^{1/s}dx \\
&\leq C\|\vec{b}\|_{Lip_\beta}M_{m\beta,s}(f)(\tilde{x})\sum_{k=1}^{\infty}C_k \\
&\leq C\|\vec{b}\|_{Lip_\beta}M_{m\beta,s}(f)(\tilde{x}).
\end{aligned}$$

$$\begin{aligned}
J_8 &\leqslant \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-(b)_{2Q})_\sigma| \\
&\quad \times\int_{R^n}|(b(y)-(b)_{2Q})_{\sigma^c}||K(x,y)-K_t(x,y)||f_2(y)|dydx \\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{Lip_\beta}|x-x_0|^{\eta\beta}\|\vec{b}_{\sigma^c}\|_{Lip_\beta} \\
&\quad \times\frac{1}{|Q|}\int_Q\int_{R^n}|y-x_0|^{\eta'\beta}|K(x,y)-K_t(x,y)||f_2(y)|dydx \\
&\leq C\|\vec{b}\|_{Lip_\beta}\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\left(\int_{2^k|x-x_0|\leq|x-y|<2^{k+1}|x-x_0|}|K(x,y)-K_t(x,y)|^{s'}dy\right)^{1/s'} \\
&\quad \times(2^{k+1}|x-x_0|)^{m\beta}\left(\int_{2^k|x-x_0|\leq|x-y|<2^{k+1}|x-x_0|}|f(y)|^sdy\right)^{1/s}dx \\
&\leq vC\|\vec{b}\|_{Lip_\beta}\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}C_k|2^{k+1}Q|^{\frac{m\beta}{n}} \\
&\quad \times\left(\frac{1}{|Q(x_0,2^{k+1}|x-x_0|)|}\int_{Q(x_0,2^{k+1}|x-x_0|)}|f(y)|^sdy\right)^{1/s}dx \\
&\leq C\|\vec{b}\|_{Lip_\beta}M_{m\beta,s}(f)(\tilde{x}).
\end{aligned}$$

$$\begin{aligned}
J_9 &\leq C\|\vec{b}\|_{Lip_\beta} \frac{1}{|Q|} \int_Q \int_{R^n} |y - x_0|^{m\beta} |K(x, y) - K_t(x, y)| |f_2(y)| dy dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |K(x, y) - K_t(x, y)|^{s'} dy \right)^{1/s'} \\
&\quad \times (2^{k+1}|x - x_0|)^{m\beta} \left(\int_{2^k|x-x_0| \leq |x-y| < 2^{k+1}|x-x_0|} |f(y)|^s dy \right)^{1/s} dx \\
&\leq C\|\vec{b}\|_{Lip_\beta} M_{m\beta, s}(f)(\tilde{x}).
\end{aligned}$$

This completes the proof of the theorem .

3. Applications

3.1. Boundedness on Lebesgue Spaces

Lemma 3.1(see [3][10]) For any $\gamma > 0$, there exists a constant $C > 0$ independent of γ such that

$$|\{x \in R^n : M(f)(x) > D\lambda, M_A^\#(f)(x) \leq \gamma\lambda\}| \leq C\gamma|\{x \in R^n : M(f)(x) > \lambda\}|$$

for $\lambda > 0$, where D is a fixed constant which only depends on n . So that

$$\|M(f)\|_{L^p} \leq C\|M_A^\#(f)\|_{L^p}$$

for every $f \in L^p(R^n)$, $1 < p < \infty$.

Lemma 3.2(see [6]) The fractional maximal operator $M_{\beta, r}$ is bounded from $L^p(R^n)$ to $L^p(R^n)$, where $0 < \beta < n$, $r < p < n/\beta$, and $1/q = 1/p - \beta/n$.

Theorem 3.1 Let T be a singular integral operator with non-smooth kernel as in Definition 1.2. If the sequence $\{kC_k\} \in l^1$ and $b_j \in BMO(R^n)$ for $j = 1, \dots, m$, then T_b is bounded on $L^p(R^n)$ for any $1 < p < \infty$, that is

$$\|T_b(f)\|_{L^p} \leq C\|\vec{b}\|_{BMO}\|f\|_{L^p},$$

where $C > 0$ is independent of f .

Proof. Choose $1 < s < p$ in Theorem 2.2 and by using Lemma 3.1, We first consider the case $m = 1$, we have

$$\begin{aligned}
\|T_b(f)\|_{L^p} &\leq \|M(T_b(f))\|_{L^p} \leq C\|M_A^\#(T_b(f))\|_{L^p} \\
&\leq C\|b\|_{BMO} (\|M_s(f) + M_s(T(f))\|_{L^p}) \leq C\|b\|_{BMO} (\|f\|_{L^p} + \|T(f)\|_{L^p}) \\
&\leq C\|b\|_{BMO}\|f\|_{L^p}.
\end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 3.1 by induction. This finishes the proof.

Theorem 3.2 Let T be a singular integral operators with non-smooth kernels as in Definition 1.2 and the sequence $\{C_k\} \in l^1$. If $0 < \beta < 1$, $\vec{b} = (b_1, \dots, b_m)$ with $b_j \in Lip_\beta(R^n)$ for $1 \leq j \leq m$, then T_b is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/m\beta$ and $1/p - 1/q = m\beta/n$, that is

$$\|T_b(f)\|_{L^q} \leq C\|\vec{b}\|_{Lip_\beta}\|f\|_{L^p},$$

where $C > 0$ is independent of f .

Proof. By using Lemma 3.2 and the boundedness of T , we have

$$\begin{aligned} \|T_b(f)\|_{L^q} &\leq \|M(T_b(f))\|_{L^q} \leq C\|M_A^\#(T_b(f))\|_{L^q} \\ &\leq C\|\vec{b}\|_{Lip_\beta}\|M_{m\beta,s}(f)\|_{L^q} \\ &\leq C\|f\|_{L^p}. \end{aligned}$$

This complete the proof.

3.2. Boundedness on Morrey Type Spaces

The estimates for sharp maximal functions can be applied to obtain the boundedness properties not only on Lebesgue spaces, but also on Morrey type spaces.

Morrey spaces have been great value through the years in studying the local behavior of solutions to second elliptic partial differential equations. For the boundedness of classical singular integral operators on Morrey spaces, referring to [1][5].

Definition 3.1 A function $f \in L_{loc}^p(R^n)$ is said belong to the classical Morrey space $M_p^q(R^n)$, $1 \leq p \leq q < \infty$, if

$$\|f\|_{M_p^q} = \sup_{B \subset R^n} |B|^{\frac{1}{q} - \frac{1}{p}} \left(\int_B |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Definition 3.2 For a general positive function φ on $R^n \times R^+$, the generalized Moerry space $L^{p,\varphi}$ with $1 \leq p < \infty$ is defined as follows:

$$L^{p,\varphi}(R^n) = \{f \in L_{loc}^p(R^n), \|f\|_{L^{p,\varphi}} < +\infty\},$$

where

$$\|f\|_{L^{p,\varphi}} = \sup_{x \in R^n, r > 0} \left(\frac{1}{\varphi(x, r)} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p}.$$

Lemma 3.3(see [8]) Let φ_1 be a positive function on $R^n \times R^+$ and there exists a C_0 satisfying $0 < C_0 < 2^n$ such that

$$\varphi(x, 2r) \leq C_0\varphi(x, r) \text{ for all } x \in R^n, r > 0. \quad (3.1)$$

If $1 < p < \infty$, then

$$\|Mf\|_{L^{p,\varphi}} \leq \|f\|_{L^{p,\varphi}} \text{ and } \|Mf\|_{L^{p,\varphi}} \leq \|f^\#\|_{L^{p,\varphi}},$$

where C is independent of f .

Lemma 3.4(see [9]) Let φ_1 be a positive function on $R^n \times R^+$. Suppose $0 < \alpha < n$, $1 < l < p_1 < n/\alpha$, $1/p_2 = 1/p_1 - \alpha/n$ and $\varphi_2^{1/p_2} = \varphi_1^{1/p_1}$. If there exists $0 < C_1 < 2^{np_1/p_2}$ such that (3.1) holds for φ_1 and C_1 , then

$$\|M_{\alpha,l}f\|_{L^{p_2,\varphi_2}} \leq C\|f\|_{L^{p_1,\varphi_1}},$$

where C is independent of f .

Theorem 3.3 Let T be a singular integral operators with non-smooth kernels as in Definition 1.2 and the sequence $\{C_k\} \in l^1$. If φ is a positive function on $R^n \times R^+$ such that (3.1) holds and $1 < p < \infty$, then T is bounded on $L^{p,\varphi}(R^n)$.

Proof. For $1 < p < \infty$, there exist an s such that $1 < s < p$. It follows from Theorem 2.1 that

$$\begin{aligned} \|Tf\|_{L^{p,\varphi}} &\leq \|M(Tf)\|_{L^{p,\varphi}} \leq C\|M_A^\#(Tf)\|_{L^{p,\varphi}} \\ &\leq C\|M_s(f)\|_{L^{p,\varphi}} = C\|M(|f|^s)\|_{L^{p/s,\varphi}}^{1/s} \\ &\leq C\||f|^s\|_{L^{p/s,\varphi}}^{1/s} = C\|f\|_{L^{p,\varphi}}. \end{aligned}$$

This completes the proof of the theorem

Theorem 3.4 Let T be a singular integral operators with non-smooth kernels as in Definition 1.2 and the sequence $\{kC_k\} \in l^1$. If φ is a positive function on $R^n \times R^+$ such that (3.1) holds, $1 < p < \infty$ and $b_j \in BMO(R^n)$ for $j = 1, \dots, m$, then T_b is bounded on $L^{p,\varphi}$.

Proof. Taking an s such that $1 < s < p$. By Theorem 2.2 and Theorem 3.3, when $m = 1$, we have

$$\begin{aligned} \|T_b(f)\|_{L^{p,\varphi}} &\leq \|M(T_b(f))\|_{L^{p,\varphi}} \leq C\|M_A^\#(T_b(f))\|_{L^{p,\varphi}} \\ &\leq C\|b\|_{BMO} (\|M_s(Tf)\|_{L^{p,\varphi}} + \|M_s(f)\|_{L^{p,\varphi}}) \\ &\leq C\|b\|_{BMO} \left(\|M(|Tf|^s)\|_{L^{p/s,\varphi}}^{1/s} + \|M(|f|^s)\|_{L^{p/s,\varphi}}^{1/s} \right) \\ &\leq C\|b\|_{BMO} \left(\||Tf|^s\|_{L^{p/s,\varphi}}^{1/s} + \||f|^s\|_{L^{p/s,\varphi}}^{1/s} \right) \\ &\leq C\|b\|_{BMO} (\|Tf\|_{L^{p,\varphi}} + \|f\|_{L^{p,\varphi}}) \\ &\leq C\|b\|_{BMO}\|f\|_{L^{p,\varphi}}. \end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 3.4 by induction. This finishes the proof.

Theorem 3.5 Let T be a singular integral operators with non-smooth kernels as in Definition 1.2 and the sequence $\{C_k\} \in l^1$. If φ_1 is a positive function on $R^n \times R^+$ and $0 < \beta < n$, $b_j \in Lip_\beta(R^n)$ for $j = 1, \dots, m$, $1 < p_1 < n/m\beta$,

$1/p_2 = 1/p_1 - m\beta/n$ and $\varphi_2^{1/p_2} = \varphi_1^{1/p_1}$, and there exists $0 < C_1 < 2^{np_1/p_2}$ such that (3.1) holds for φ_1 and C_1 , then T_b is bounded from $L^{p_1, \varphi_1}(R^n)$ into $L^{p_2, \varphi_2}(R^n)$.

Proof. For $1 < p_1 < n/m\beta$, there exist an s such that $1 < s < p_1$. It follows from Theorem 2.3, Lemma 3.4 and Theorem 3.3 that

$$\begin{aligned} \|T_b(f)\|_{L^{p_2, \varphi_2}} &\leq \|M(T_b(f))\|_{L^{p_2, \varphi_2}} \leq C\|M_A^\#(T_b(f))\|_{L^{p_2, \varphi_2}} \\ &\leq C\|\vec{b}\|_{Lip_\beta} (\|M_{m\beta, s}(f)\|_{L^{p_2, \varphi_2}}) \\ &\leq C\|\vec{b}\|_{Lip_\beta} \|f\|_{L^{p_1, \varphi_1}}. \end{aligned}$$

This finishes the proof.

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