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A GENERALIZATION OF THE PASCAL MATRIX AND ITS PROPERTIES

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Abstract. In this paper we introduce a generalization of the Pascal matrix and show it satisfies numerous properties. In particular, we firstly investigate various factorizations of such matrix. Explicit formula for the inverse of the generalized Pascal matrix is derived. In addition, explicit representations for the powers of the generalized Pascal matrix are derived for integer, rational and irrational exponents. Finally, we employ the formula for the power of the generalized Pascal matrix to find the inverse of the linear combination of the identity matrix and the generalized Pascal matrix.

Keywords: Pascal matrix; Binomial coefficient; Matrix inverse; Hadamard product.

1. Introduction

The matrices which contain the binomial coefficients as its elements has been a mine of topics for researchers. The most famous such matrix is the Pascal matrix, which has been playing a central role in matrix theory and combinatorics. The Pascal matrix $\mathcal{P}_n$ of order n is defined as

$$(\mathcal{P}_n)_{i,j} = \begin{cases} \binom{i-1}{j-1}, & i \geqslant j \\ 0, & \text{otherwise} \end{cases}, \quad 1 \leqslant i, j \leqslant n.$$

Since then, many generalizations of the Pascal matrix has been gaining a wide interest. The generalized Pascal matrix was defined in [2] as

$$(\mathcal{P}_n[x])_{i,j} = \begin{cases} x^{i-j}\binom{i-1}{j-1}, & i \geqslant j \\ 0, & \text{otherwise} \end{cases}, \quad 1 \leqslant i, j \leqslant n.$$

The properties of the generalized Pascal matrix were investigated by Call and Velleman [2] and Zhang [18]. For the sake of simplicity, in what follows we call $\mathcal{P}_n[x]$ simply the Pascal matrix.

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The extended generalized Pascal matrix $\Phi_n[x, y]$ was investigated by Zhang and Liu [19], and by Zhang and Wang [20], and is defined by

$$(\Phi_n[x, y])_{i,j} = \begin{cases} x^{i-j} y^{i+j-2} \binom{i-1}{j-1}, & i \geqslant j \\ 0, & \text{otherwise} \end{cases}, \quad 1 \leqslant i, j \leqslant n.$$

Since then, many generalizations of the Pascal matrix have been introduced. Zhao and Wang [22] introduced the concept of the Pascal functional matrix. This concept was further developed in papers [13, 16, 17]. In papers [8, 9], the authors generalized the Pascal matrix via the matrices filled with the symmetric polynomials.

Authors in [3, 6, 7, 15] investigated relations between the Pascal matrix and other special types of matrices. The Pascal matrix was also used for deriving combinatorial identities, as it was demonstrated in papers [5, 10, 12, 21].

The problem of representing the inverse of the Pascal matrix and the inverse of linear combinations of the identity and the Pascal matrix is well studied in the literature. For example, motivated by a problem from statistics, in [1] it is shown how to invert $I - \lambda \mathcal{P}_n[a]$. In particular, the inverse is the matrix with its main diagonal replaced by $1/(1 - \lambda)$ and its mth lower sub-diagonal multiplied by the constant $Li_{-m}(\lambda)$, where $Li_{-m}(\lambda)$ is the polylogarithm function..

Moreover, the matrix $(I_n + P_n)^{-1}$ is the Hadamard product $P_n \circ \Delta_n$, where Δ_n is the $n \times n$ lower triangular matrix containing the Euler polynomials, and the Hadamard product $A \circ B$ of two matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ is the matrix obtained by entry-wise multiplication of matrices A and B: $(A \circ B)_{i,j} = a_{i,j} b_{i,j}$. These representations might be useful in applications in control engineering, where it is needed to calculate the determinant and adjoint polynomials of the matrix $(\lambda I - A)^{-1}$.

The above results on Pascal matrices were extended in the case of the general Catalan matrices in paper [11].

The goal of the present paper is to introduce a generalization of the Pascal matrix and to extend earlier results on Pascal matrix to this generalization. The results are presented in the following section. First, we examine some factorizations of the generalized Pascal matrix. After that, we find explicit formulae for the inverse, as well as for the power of the generalized Pascal matrix for integer and rational exponent. Later, we find the power of the generalized Pascal matrix for the irrational exponent. Finally, we apply the formula for the power of the generalized Pascal matrix to find the explicit representation of the inverse of linear combination of the identity and generalized Pascal matrix.

2. The generalized Pascal matrix

Definition 2.1. The generalized Pascal matrix $\mathcal{P}_n[r; x]$ of order n, is defined by

$$(\mathcal{P}_n[r; x])_{i,j} = x^{i-j} \binom{i + r - 2}{i - j}$$

for all $1 \leqslant i, j \leqslant n$, $r \in \mathbb{Z}$ and $x \in \mathbb{R}$.

In the case $r = 1$, the generalized Pascal matrix reduces to the Pascal matrix, so we have the relation $\mathcal{P}_n[1; x] = \mathcal{P}_n[x]$.

Example 2.1. The generalized Pascal matrix $\mathcal{P}_n[r; x]$ of order 5 is equal to

$$\mathcal{P}_5[r;x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \binom{r}{1}x & 1 & 0 & 0 & 0 \\ \binom{r+1}{2}x^2 & \binom{r+1}{1}x & 1 & 0 & 0 \\ \binom{r+2}{3}x^3 & \binom{r+2}{2}x^2 & \binom{r+2}{1}x & 1 & 0 \\ \binom{r+3}{4}x^4 & \binom{r+3}{3}x^3 & \binom{r+3}{2}x^2 & \binom{r+3}{1}x & 1 \end{bmatrix}$$

Let us define matrices

$$\mathcal{P}_n^{(k)}[r;x] = I_k \oplus \mathcal{P}_{n-k}[r;x] = \begin{bmatrix} I_k & 0 \\ 0 & \mathcal{P}_{n-k}[r;x] \end{bmatrix},$$

for each $0 \leqslant k < n$. Recall that $A \oplus B$ denotes the direct sum of matrices A and B. Next, define matrix $\mathcal{S}_n[r; x]$ element-wise as

$$(\mathcal{S}_n[r;x])_{i,j} = \begin{cases} x^{i-j}\binom{i+r-2}{i-j}, & j = 1 \\ x^{i-j}, & i \geqslant j \\ 0, & \text{otherwise} \end{cases}$$

and let $\mathcal{S}_n^{(k)}[r; x] = I_k \oplus \mathcal{S}_{n-k}[r; x]$, for each $0 \leqslant k < n$. Then we have the following results.

Lemma 2.1. *For natural n, integer r and real x we have*

$$\mathcal{P}_n[r;x] = \mathcal{S}_n[r;x]\mathcal{P}_n^{(1)}[r;x]. \tag{2.1}$$

Proof. The proof goes straightforward. □

After recursively applying (2.1), we obtain the factorization of the generalized Pascal matrix.

Theorem 2.1. *We have the following factorization of the generalized Pascal matrix $\mathcal{P}_n[r; x]$*

$$\mathcal{P}_n[r;x] = \mathcal{S}_n^{(0)}[r;x]\mathcal{S}_n^{(1)}[r;x]\cdots\mathcal{S}_n^{(n-1)}[r;x]. \tag{2.2}$$

Example 2.2. Setting $n = 5$ in Lemma 2.1 and Theorem 2.1, we get

$$\begin{aligned} \mathcal{P}_5[r;x] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \binom{r}{1}x & 1 & 0 & 0 & 0 \\ \binom{r+1}{2}x^2 & \binom{r+1}{1}x & 1 & 0 & 0 \\ \binom{r+2}{3}x^3 & \binom{r+2}{2}x^2 & \binom{r+2}{1}x & 1 & 0 \\ \binom{r+3}{4}x^4 & \binom{r+3}{3}x^3 & \binom{r+3}{2}x^2 & \binom{r+3}{1}x & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \binom{r}{1}x & 1 & 0 & 0 & 0 \\ \binom{r+1}{2}x^2 & x & 1 & 0 & 0 \\ \binom{r+2}{3}x^3 & x^2 & x & 1 & 0 \\ \binom{r+3}{4}x^4 & x^3 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \binom{r}{1}x & 1 & 0 & 0 \\ 0 & \binom{r+1}{2}x^2 & \binom{r+1}{1}x & 1 & 0 \\ 0 & \binom{r+2}{3}x^3 & \binom{r+2}{2}x^2 & \binom{r+2}{1}x & 1 \end{bmatrix} \end{aligned}$$

and

$$\mathcal{P}_5[r;x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \binom{r}{1}x & 1 & 0 & 0 & 0 \\ \binom{r+1}{2}x^2 & x & 1 & 0 & 0 \\ \binom{r+2}{3}x^3 & x^2 & x & 1 & 0 \\ \binom{r+3}{4}x^4 & x^3 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \binom{r}{1}x & 1 & 0 & 0 \\ 0 & \binom{r+1}{2}x^2 & x & 1 & 0 \\ 0 & \binom{r+2}{3}x^3 & x^2 & x & 1 \end{bmatrix} \times$$
$$\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \binom{r}{1}x & 1 & 0 \\ 0 & 0 & \binom{r+1}{2}x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \binom{r}{1}x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \mathcal{S}_5^{(0)}[r;x]\mathcal{S}_5^{(1)}[r;x]\mathcal{S}_5^{(2)}[r;x]\mathcal{S}_5^{(3)}[r;x]\mathcal{S}_5^{(4)}[r;x].$$

By setting $r = 1$ in Theorem 2.1, we regain the well-known factorization of the Pascal matrix (for details consult [18]):

Example 2.3.

$$\mathcal{P}_5[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ x^2 & 2x & 1 & 0 & 0 \\ x^3 & 3x^2 & 3x & 1 & 0 \\ x^4 & 4x^3 & 6x^2 & 4x & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ x^2 & x & 1 & 0 & 0 \\ x^3 & x^2 & x & 1 & 0 \\ x^4 & x^3 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ 0 & x^2 & x & 1 & 0 \\ 0 & x^3 & x^2 & x & 1 \end{bmatrix} \times$$
$$\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x & 1 & 0 \\ 0 & 0 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we investigate the properties of the power of the Pascal matrix.

Theorem 2.2. *The inverse of the generalized Pascal matrix is given by*

$$\mathcal{P}_n[r;x]^{-1} = \mathcal{P}_n[r;-x]. \tag{2.3}$$

for $n \in \mathbb{N}$, $r \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Proof. We show that $\mathcal{P}_n[r;x]\mathcal{P}_n[r;-x] = I_n$, where I_n is the identity matrix of order n. It is clear that $(\mathcal{P}_n[r;x]\mathcal{P}_n[r;-x])_{i,i} = 1$, for all $1 \leqslant i \leqslant n$, as well as

$(\mathcal{P}_n[r;x]\mathcal{P}_n[r;-x])_{i,j} = 0$ for $i < j$. Thus, suppose that $i > j$. Later it follows that

$$\begin{aligned}(\mathcal{P}_n[r;x]\mathcal{P}_n[r;-x])_{i,j} &= \sum_{k=j}^{i} x^{i-k}\binom{i+r-2}{i-k}(-x)^{k-j}\binom{k+r-2}{k-j}\\ &= x^{i-j}\sum_{k=0}^{i-j}(-1)^k\binom{i+r-2}{i-j-k}\binom{k+j+r-2}{k}.\end{aligned}$$

By employing the properties for the Pochhammer function $(a)_k = a(a+1)\cdots(a+k-1)$, we obtain

$$\begin{aligned}(\mathcal{P}_n[r;x]\mathcal{P}_n[r;-x])_{i,j} &= x^{i-j}\sum_{k=0}^{i-j}(-1)^k\frac{(r+j+k-1)_{i-j-k}}{(i-j-k)!}\frac{(j+r-1)_k}{k!}\\ &= x^{i-j}\sum_{k=0}^{i-j}(-1)^k\frac{(j+r-1)_{i-j}}{(i-j-k)!\,k!}.\end{aligned}$$

After multiplying each summand by factor $1 = (i-j)!/(i-j)!$, we get the following

$$\begin{aligned}(\mathcal{P}_n[r;x]\mathcal{P}_n[r;-x])_{i,j} &= x^{i-j}\frac{(j+r-1)_{i-j}}{(i-j)!}\sum_{k=0}^{i-j}(-1)^k\frac{(i-j)!}{(i-j-k)!\,k!}\\ &= x^{i-j}\binom{i+r-2}{i-j}\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k}.\end{aligned}$$

Finally, after applying the binomial theorem, we finish this part of the proof, since

$$(\mathcal{P}_n[r;x]\mathcal{P}_n[r;-x])_{i,j} = x^{i-j}\binom{i+r-2}{i-j}(1-1)^{i-j} = 0.$$

Similarly, one can obtain $\mathcal{P}_n[r;-x]\mathcal{P}_n[r;x] = I_n$, and the proof is therefore finished. □

Setting $r = 1$ in Eq. (2.3), we regain the well-known formula for the inverse of the Pascal matrix (see [2, 18]).

Corollary 2.1. *For natural n and real x,*

$$\mathcal{P}_n[x]^{-1} = \mathcal{P}_n[-x].$$

Theorem 2.3. *For arbitrary dimension n, integers r and k, and real x, the following relation between powers of the generalized Pascal matrices is valid*

$$\mathcal{P}_n[r;x]^k = \mathcal{P}_n[r;kx]. \tag{2.4}$$

Proof. At first, let $k \geqslant 0$. We employ the principle of the mathematical induction. The basic case $\mathcal{P}_n[r;x]^0 = \mathcal{P}_n[r;0] = I_n$ goes trivially. After applying the inductive hypothesis

$$\mathcal{P}_n[r;x]^{k+1} = \mathcal{P}_n[r;x]\mathcal{P}_n[r;kx]$$

we get

$$\begin{aligned}\left(\mathcal{P}_n[r;x]^{k+1}\right)_{i,j} &= x^{i-j}\sum_{l=0}^{i-j} k^l \binom{i+r-2}{i-j-l}\binom{l+j+r-2}{l} \\ &= x^{i-j}\sum_{l=0}^{i-j} k^l \frac{(r+j+l-1)_{i-j-l}}{(i-j-l)!}\frac{(j+r-1)_l}{l!}\frac{(i-j)!}{(i-j)!} \\ &= x^{i-j}\binom{i+r-2}{i-j}\sum_{l=0}^{i-j}\binom{i-j}{l}k^l.\end{aligned}$$

Again, by applying the binomial theorem, we prove the inductive step

$$\left(\mathcal{P}_n[r;x]^{k+1}\right)_{i,j} = x^{i-j}\binom{i+r-2}{i-j}(k+1)^{i-j}.$$

Since $\mathcal{P}_n[r;x]^{-1} = \mathcal{P}_n[r;-x]$, a similar induction shows that Eq. (2.4) holds in the case $k < 0$. This completes the proof. □

By putting $r = 1$ in Eq. (2.4), we regain the well-known formula for the power of the Pascal matrix (see [2, 15]).

Corollary 2.2. *For natural n, integer k and real x,*

$$\mathcal{P}_n[x]^k = \mathcal{P}_n[kx].$$

Theorem 2.3 is easily expanded to rational exponents. But, if k is irrational, then does $\mathcal{P}_n[r;kx]$ still represent a matrix which deserves to be regarded as $\mathcal{P}_n[r;x]^k$? If we recall how irrational exponents for real numbers work, we will see that for $a > 0$ the expression a^x is defined to be e^{xl}, where $l = \ln a$. By analogy, if $\mathcal{P}_n[r;kx]$ is to be regarded as $\mathcal{P}_n[r;x]^k$, we might expect that there exist matrix L such that $\mathcal{P}_n[r;x] = e^{xL}$. It will be in our attention to find such matrix.

Matrix exponentials are defined by simply plugging matrices into the usual Maclaurin series for the exponential function. In other words, for any square matrix A, the exponential of A is defined to be the matrix

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \ldots + \frac{A^k}{k!} + \ldots$$

(consult [14] for details).

The following well-known theorem will be very useful for the rest of the work.

Theorem 2.4. [2, 14] *Let A be any square matrix. Then*

- *For any numbers s and t,* $e^{(s+t)A} = e^{sA}e^{tA}$.
- *eA is invertible, and* $(e^A)^{-1} = e^{-A}$.
- $\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A$, *where* $\frac{d}{dt}e^{tA}$ *is the matrix resulting from taking the derivative with respect to t of each entry of* e^{tA}.

Definition 2.2. The matrix $L_n[r]$ of order n, is element-wise equal to

$$(L_n[r])_{i,j} = \begin{cases} j + r - 1, & i = j + 1 \\ 0, & \text{otherwise} \end{cases},$$

for all $1 \leqslant i, j \leqslant n$ and $r \in \mathbb{Z}$.

Our attention is to prove that $\mathcal{P}_n[r, x] = e^{xL_n[r]}$. To that effort, we prove the following auxiliary result.

Lemma 2.2. *For every nonnegative integer k, the entries of the matrix* $L_n[r]^k$ *are given by*

$$(L_n[r]^k)_{i,j} = \begin{cases} (j + r - 1)_k, & i = j + k \\ 0, & \textit{otherwise} \end{cases}.$$

Proof. We employ the principle of the mathematical induction. The basic case follows straightforward. Let us assume the inductive hypothesis on $L_n[r]^{k+1} = L_n[r]L_n[r]^k$. It is not hard to conclude that $(L_n[r]^{k+1})_{i,j} = 0$ for $i \neq j + k + 1$, while in the case $i = j + k + 1$ we have

$$(L_n[r]^{k+1})_{i,j} = (i + r - 2)(j + r - 1)_k = (j + k + r - 1)(j + r - 1)_k = (j + r - 1)_{k+1},$$

and the proof is therefore finished. □

Theorem 2.5. *For* $n \in \mathbb{N}$, $r \in \mathbb{Z}$ *and* $x \in \mathbb{R}$, *we have*

$$\mathcal{P}_n[r; x] = e^{xL_n[r]}.$$

Proof. Suppose there is a matrix $L'_n[r]$ such that $\mathcal{P}_n[r; x] = e^{xL'_n[r]}$. Then

$$\frac{d}{dx}\mathcal{P}_n[r; x] = L'_n[r]e^{xL'_n[r]} = L'_n[r]\mathcal{P}_n[x],$$

so

$$\frac{d}{dx}\mathcal{P}_n[r; x]\,|_{x=0} = L'_n[r]\mathcal{P}_n[r; 0] = L'_n[r]I_n = L'_n[r].$$

Thus, there is at most one matrix $L'_n[r]$ such that $\mathcal{P}_n[r; x] = e^{xL'_n[r]}$. By calculating $\frac{d}{dx}\mathcal{P}_n[r; x]\,|_{x=0}$, it is not hard to conclude that $L'_n[r] = L_n[r]$, where $L_n[r]$ is the matrix

from Definition 2.2. By employing the result from Lemma 2.2, we conclude that $L_n[r]^k = 0$ for $k \geqslant n$, so we have

$$e^{xL_n[r]} = \sum_{k=0}^{n-1} \frac{x^k}{k!} L_n[r]^k.$$

We see that $(e^{xL_n[r]})_{i,j} = 0$ for $i < j$, as well that $(e^{xL_n[r]})_{i,i} = 1$. Now, suppose that $i > j$ and let $k = i - j$. In this case we can employ the result from Lemma 2.2, and obtain

$$(e^{xL_n[r]})_{i,j} = \frac{x^k}{k!}(L_n[r]^k)_{i,j} = x^k \frac{(j+r-1)_k}{k!} = x^k \binom{i+r-2}{k} = (\mathcal{P}_n[r;x])_{i,j}.$$

In this way, the proof is completed. □

Example 2.4. $\frac{d}{dx}\mathcal{P}_5[r;x]$ is the matrix resulting from taking the derivative with respect to x of each entry of $\mathcal{P}_5[r;x]$

$$\frac{d}{dx}\mathcal{P}_5[r;x] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \binom{r}{1} & 0 & 0 & 0 & 0 \\ (r+1)\binom{r}{1}x & \binom{r+1}{1} & 0 & 0 & 0 \\ (r+2)\binom{r+1}{2}x^2 & (r+2)\binom{r+1}{1}x & \binom{r+2}{1} & 0 & 0 \\ (r+3)\binom{r+2}{3}x^3 & (r+3)\binom{r+2}{2}x^2 & (r+3)\binom{r+2}{1}x & \binom{r+3}{1} & 0 \end{bmatrix}$$

Thus we have

$$L_5[r] = \frac{d}{dx}\mathcal{P}_5[r;x]\,|_{x=0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ r & 0 & 0 & 0 & 0 \\ 0 & r+1 & 0 & 0 & 0 \\ 0 & 0 & r+2 & 0 & 0 \\ 0 & 0 & 0 & r+3 & 0 \end{bmatrix}$$

and

$$L_5[r]^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ r(r+1) & 0 & 0 & 0 & 0 \\ 0 & (r+1)(r+2) & 0 & 0 & 0 \\ 0 & 0 & (r+2)(r+3) & 0 & 0 \end{bmatrix},$$

$$L_5[r]^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ r(r+1)(r+2) & 0 & 0 & 0 & 0 \\ 0 & (r+1)(r+2)(r+3) & 0 & 0 & 0 \end{bmatrix},$$

$$L_5[r]^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ r(r+1)(r+2)(r+3) & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Corollary 2.3. *For natural n and real x and y, the following relation holds*

$$\mathcal{P}_n[r;x+y] = \mathcal{P}_n[r;x]\mathcal{P}_n[r;y]. \tag{2.5}$$

Proof. By applying Theorem 2.5, we get $\mathcal{P}_n[r;x+y] = e^{x+y}L_n[r] = e^{xL_n[r]}e^{yL_n[r]} = \mathcal{P}_n[r;x]\mathcal{P}_n[r;y]$. This completes the proof. □

Setting $r = 1$ in Eq. (2.5), we regain the well-known identity for the Pascal matrix [2, 18].

$$\mathcal{P}_n[x+y] = \mathcal{P}_n[x]\mathcal{P}_n[y].$$

At the end of this section, we make use of the formula (2.4) to find the explicit inverse of the matrix $I_n - \lambda\mathcal{P}_n[r;x]$.

Theorem 2.6. *The inverse $\mathcal{Q}_n[r;x]$ of the matrix $I_n - \lambda\mathcal{P}_n[r;x]$ is defined for all numbers $|\lambda| < 1$. The entries of $\mathcal{Q}_n[r;x]$ are*

$$(\mathcal{Q}_n[r;x])_{i,i} = \frac{1}{1-\lambda}$$

on the main diagonal and

$$(\mathcal{Q}_n[r;x])_{i,j} = (\mathcal{P}_n[r;x])_{i,j}\, Li_{j-i}(\lambda)$$

for $i > j$, where $Li_n(z)$ is the polylogarithm function

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

Proof. It is known that if $\|\cdot\|$ is a matrix norm and if $\|A\| < 1$, $A \in \mathbb{R}^{n\times n}$, then $I - A$ is invertible and $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$ (see, for example [4]). For any $|\lambda| < 1$, we can express the inverse of the matrix $I_n - \lambda\mathcal{P}_n[r;x]$ as the following infinite matrix sum

$$(I_n - \lambda\mathcal{P}_n[r;x])^{-1} = \sum_{k=0}^{\infty} \lambda^k \mathcal{P}_n[r;x]^k.$$

In view of (2.4), we obtain

$$(\mathcal{Q}_n[r;x])_{i,j} = \sum_{k=0}^{\infty} \lambda^k (\mathcal{P}_n[r;kx])_{i,j} = \binom{i+r-2}{i-j} x^{i-j} \sum_{k=0}^{\infty} \lambda^k k^{i-j}.$$

Finally, we get

$$(\mathcal{Q}_n[r;x])_{i,j} = (\mathcal{P}_n[r;x])_{i,j} \left(\sum_{k=0}^{\infty} \lambda^k k^{i-j} \right)$$

and the proof is finished after separate analysis of cases $i = j$ and $i > j$. □

Example 2.5. Setting $n = 5$ in Theorem 2.6, we obtain

$$Q_5[r;x] = \begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 \\ \binom{r}{1}\frac{x\lambda}{(\lambda-1)^2} & \frac{1}{1-\lambda} & 0 & 0 & 0 \\ \binom{r+1}{2}\frac{-x^2\lambda(\lambda+1)}{(\lambda-1)^3} & \binom{r+1}{1}\frac{x\lambda}{(\lambda-1)^2} & \frac{1}{1-\lambda} & 0 & 0 \\ \binom{r+2}{3}\frac{x^3\lambda\left(\lambda^2+4\lambda+1\right)}{(\lambda-1)^4} & \binom{r+2}{2}\frac{-x^2\lambda(\lambda+1)}{(\lambda-1)^3} & \binom{r+2}{1}\frac{x\lambda}{(\lambda-1)^2} & \frac{1}{1-\lambda} & 0 \\ \binom{r+3}{4}\frac{-x^4\lambda\left(\lambda^3+11\lambda^2+11\lambda+1\right)}{(\lambda-1)^5} & \binom{r+3}{3}\frac{x^3\lambda\left(\lambda^2+4\lambda+1\right)}{(\lambda-1)^4} & \binom{r+3}{2}\frac{-x^2\lambda(\lambda+1)}{(\lambda-1)^3} & \binom{r+3}{1}\frac{x\lambda}{(\lambda-1)^2} & \frac{1}{1-\lambda} \end{bmatrix}.$$

Setting $r = 1$ in Theorem 2.6, we obtain the well-known result for the inverse of the matrix $I_n - \lambda\mathcal{P}_n[x]$ (see [1]).

Now we find the matrix $\Delta_n[\lambda]$ satisfying $(I_n - \lambda\mathcal{P}_n[x])^{-1} = \mathcal{P}_n[x] \circ \Delta_n[\lambda]$.

Theorem 2.7. *For the parameter λ satisfying $|\lambda| < 1$, the inverse $(I_n - \lambda\mathcal{P}_n[x])^{-1}$ can be expressed as*

$$(I_n - \lambda\mathcal{P}_n[x])^{-1} = \mathcal{P}_n[x] \circ \Delta_n[\lambda],$$

where

$$(\Delta_n)_{i,j}[\lambda] = \begin{cases} Li_{j-i}(\lambda), & i \geqslant j \\ 0, & i < j \end{cases}$$

and $Li_n(z)$ is the polylogarithm function.

Proof. The proof goes directly from Theorem 2.6. □

Setting $r = 1$ in Theorem 2.7, we anticipate the result from [15] for the Pascal matrix.

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