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Ser. Math. Inform. 26 (2011), 9–15

THE UNIQUENESS THEOREM FOR MEROMORPHIC FUNCTIONS SHARING A SET*

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Abstract. The study about the field of uniqueness of meromorphic function is important and interesting for the researchers. In this paper, we prove the uniqueness theorem of nonconstant meromorphic functions which share a common set and obtain some uniqueness results.

1. Introduction

Let $f(z)$ and $g(z)$ are nonconstant meromorphic functions in the complex $\mathbb{C}$. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$ and $m(r, f)$ and so on, see [1-3]. In particular, $B(f)$ denotes the family of all meromorphic functions $a(z)$ such that $T(r, a(z)) = S(r, f)$, where $r \longrightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure. So we define

$$E_f(B) = \bigcup_{a\in B}\{f(z) - a = 0, counting - multiplicities\},$$

$$\overline{E}_f(B) = \bigcup_{a\in B}\{f(z) - a = 0, ignoring - multiplicities\}.$$

The f and g share a set B CM, resp. IM, provided that $E_f(B) = E_g(B)$, resp. $\overline{E}_f(B) = \overline{E}_g(B)$

P. Li and C. C. Yang [4] proved the following theorem,

Theorem A [4]. Let f be a non-constant entire function and a_1, a_2 be two distinct complex numbers. If f and f' share the set $\{a_1, a_2\}$ CM, then f takes one of the following conclusions:

Received March 29, 2011.
2010 *Mathematics Subject Classification.* Primary 30D35

*The author was supported in part by the Scientific Research Foundation from Yunnan Province Education Committee (2010Y167), and the Quality Engineering Construction of Honghe University (ZDKC 1111).

$(i) f = f'$.

$(ii) f + f' = a_1 + a_2$.

$(iii) f = c_1 e^{cz} + c_2 e^{-cz}$, with $a_1 + a_2 = 0$, where c, c_1, c_2 are non-zero constants which satisfy $c^2 \neq 1$ and $c_1 c_2 = \frac{1}{4} a_1^2 (1 - \frac{1}{c_1})$.

J. Heittokangas and R. Korhonen [5] obtained the following,

Theorem B [5]. Let f be a transcendental meromorphic function of finite order, let $c \in \mathbb{C} \backslash \{0\}$, and let $a_1, a_2, a_3 \in B(f) \bigcup \{\infty\}$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM, and a_3 IM, then $f(z) \equiv f(z+c)$.

Kai Liu[6] got the result as following,

Theorem C [6]. Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C} \backslash \{0\}$,and let $a(z) \in B(f)$ be a non-vanishing periodic entire function with period c. If $f(z)$ and $f(z+c)$ share the set $\{a(z), -a(z)\}$ CM, then $f(z)$ must take one of the following conclusions:

$(i) f(z) \equiv f(z+c)$.

$(ii) f(z) + f(z+c) = 0$.

$(iii) f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, where $\frac{h_1(z+c)}{h_1(z)} = -e^{\gamma}$, $\frac{h_2(z+c)}{h_2(z)} = e^{\gamma}$, $h_1(z) h_2(z) = a(z)^2 (1 - e^{-2\gamma})$ and γ is a polynomial.

It is natural to ask what will be happen when f is a transcendental meromorphic function in theorem C. In this paper, we obtain the following results.

Theorem 1. Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \backslash \{0\}$, and let $a(z) \in B(f)$ be a non-vanishing periodic meromorphic function with period c. If $f(z)$ and $f(z+c)$ share the set $\{a(z), -a(z)\}$ CM, and $\overline{N}(r, f^2(z+c) - a^2) = \overline{N}(r, f^2(c) - a^2) = S(r, f)$, then $f(z)$ must take one of the following conclusions:

$(i) f(z) \equiv f(z+c)$.

$(ii) f(z) + f(z+c) = 0$.

$(iii) f(z) = \frac{1}{2}(\psi_1 + \psi_2)$, where $\frac{\psi_1(z+c)}{\psi_1(z)} = -\varphi(z)$, $\frac{\psi_2(z+c)}{\psi_2(z)} = \varphi(z)$, $\psi_1 \psi_2 = a(z)^2 (1 - \varphi(z)^{-2})$ and $\varphi(z)$ is a meromorphic function.

Theorem 2. Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \backslash \{0\}$, and let $a(z) \in B(f)$ be a non-vanishing periodic meromorphic function with period c. If $f(z)$ and $f(z+c)$ share the sets $\{a(z), -a(z)\}$, $\{0\}$ CM, and $\overline{N}(r, f^2(z+c) - a^2) = \overline{N}(r, f^2(z) - a^2) = S(r, f)$, then $f(z) = \pm f(z+c)$.

Theorem 3. Let f be a transcendental meromorphic function of finite order, and let a be a non-zero finite constant. If $f(z)$ and $\Delta_c f = f(z+c) - f(z)$ share the set $\{a, -a\}$ CM, and $\overline{N}(r, (\Delta_c f)^2 - a^2) = \overline{N}(r, f^2 - a^2) = S(r, f)$, then $f(z+c) = 2 f(z)$.

Theorem 4. There exists a set B with two elements such that if f is a transcendental meromorphic function of finite order with at most finitely many zeros and $E_{f(z)}(B) = E_{f(z+c)}(B)$, and $\overline{N}(r, f^2(z+c) - a^2) = \overline{N}(r, f^2(z) - a^2) = S(r, f)$, then $f(z+c) = \pm f(z)$.

2. Several lemmas

In this section, we give several lemmas to prove the above theorems.

Lemma 1 [1]. Let f be a transcendental meromorphic function, $P_k(f)$ denote a polynomial in f of degree k, and $a_i, i = 1, 2, ..., n$, denote finite distinct constants in $\mathbb{C}$. Let

$$g = \frac{P_k(f)f'}{(f - a_1)...(f - a_n)}.$$

If $k < n$, then $m(r, g) = S(r, f)$.

Lemma 2 [7]. Let f be a non-constant meromorphic function, $c \in \mathbb{C}, \delta < 1$, and $\varepsilon > 0$. Then

$$m(r, \frac{f(z + c)}{f(z)}) = o(\frac{T(r + |c|, f)^{1+\varepsilon}}{r^\delta})$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 3 [8]. Let f be a non-constant meromorphic function of finite order, $c \in \mathbb{C}, \delta < 1$. Then

$$m(r, \frac{f(z + c)}{f(z)}) = o(\frac{T(r, f)}{r^\delta}) = S(r, f).$$

where $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 4 [8]. Let f be a non-constant meromorphic function of finite order, and let $c \in \mathbb{C}, n \in \mathbb{N}$. Then for any small periodic function $a(z) \in B(f)$ with period c,

$$m(r, \frac{\Delta_c^n f}{f(z) - a(z)}) = S(r, f).$$

Lemma 5. Let f be a non-constant meromorphic function of finite order, and let $a(z) \in B(f)$. If f and $\Delta_c f$ share the set $\{a, -a\}$ CM, and $\overline{N}(r, (\Delta_c f)^2 - a^2) = \overline{N}(r, f^2 - a^2) = S(r, f)$, then

$$(\Delta_c f - a)(\Delta_c f + a) = (f - a)(f + a)\varphi^2(z), \tag{2.1}$$

where $\varphi^2(z)$ is a meromorphic function such that $T(r, \varphi^2(z)) = S(r, f)$.

Proof. Let $g = \Delta_c f$. Since f and g are meromorphic functions and share the set $\{a, -a\}$ CM, there exists an meromorphic function $\varphi^2(z)$ such that

$$(g - a)(g + a) = (f - a)(f + a)\varphi^2(z). \tag{2.2}$$

So we get

$$\varphi^2(z) = \frac{g^2 - a^2}{f^2 - a^2}. \tag{2.3}$$

By the second fundamental theorem, we have

$$
\begin{aligned}
T(r,f) &\leq \overline{N}(r,\frac{1}{f-a}) + \overline{N}(r,\frac{1}{f+a}) + \overline{N}(r,f) + S(r,f)\\
&\leq \overline{N}(r,\frac{1}{g-a}) + \overline{N}(r,\frac{1}{g+a}) + \overline{N}(r,g) + S(r,f)\\
&\leq 3T(r,g) + S(r,f)
\end{aligned}
$$

On the other hand, the fact that f is an meromorphic function of finite order, we have $T(r,g) \leq 3T(r,f) + S(r,f)$. Therefore, $S(r,f) = S(r,g)$.

From (2.3) and by the second fundamental theorem, we obtain

$$
\begin{aligned}
T(r,\varphi^2(z)) &\leq \overline{N}(r,\varphi^2(z)) + \overline{N}(r,\frac{1}{\varphi^2(z)}) + \overline{N}(r,\frac{1}{\varphi^2(z)-1}) + S(r,\varphi^2(z))\\
&\leq 2\overline{N}(r,g^2-a^2) + 2\overline{N}(r,f^2-a^2) + S(r,\varphi^2(z))\\
&\leq S(r,f).
\end{aligned}
$$

Hence $T(r,\varphi^2(z)) = S(r,f)$.

3. Proof of the Theorems

3.1. Proof of Theorem 1

Proof. Recall that the idea of the proof is similar to the proof of [6]. Since $f(z)$ be a transcendental meromorphic function of finite order and f and g share the set $\{a,-a\}$ CM, there exists an meromorphic function $\varphi^2(z)$ such that

$$(f(z+c)-a(z))(f(z+c)+a(z)) = (f(z)-a(z))(f(z)+a(z))\varphi^2(z), \tag{3.1}$$

Since $T(r,\varphi^2(z)) = S(r,f)$.

Case 1. If $\varphi^2(z) = 1$, from (3.1), we get $f(z) = f(z+c)$ or $f(z) + f(z+c) \equiv 0$.

Case 2. If $\varphi^2(z) \neq 1$, let $\psi_1 := f(z) - \varphi^{-1}(z)f(z+c)$ and $\psi_2 := f(z) + \varphi^{-1}(z)f(z+c)$. Then

$$f(z) = \frac{1}{2}(\psi_1+\psi_2), f(z+c) = \frac{1}{2}\varphi(z)(\psi_2-\psi_1). \tag{3.2}$$

From (3.1), we have

$$\psi_1\psi_2 = a(z)^2(1-\varphi(z)^{-2}), \tag{3.3}$$

which means that

$$N(r, \frac{1}{\psi_i}) = S(r, f), N(r, \psi_i) = S(r, f), i = 1, 2. \tag{3.4}$$

From the expressions of ψ_1 and ψ_2, we get $T(r, \psi_i) \leq 2T(r, f) + S(r, f)$, so that $S(r, \psi_i) = o(T(r, f)), i = 1, 2$.

Let $\alpha := \frac{\psi_1(z+c)}{\psi_1(z)}, \beta := \frac{\psi_2(z+c)}{\psi_2(z)}$. From (3.4) and by Lemma 3, we have

$$\begin{aligned} T(r, \alpha) &= m(r, \alpha) + N(r, \tfrac{1}{\psi_1}) + N(r, \psi_1) && (3.5)\\ &= S(r, f), T(r, \beta) && (3.6)\\ &= m(r, \beta) + N(r, \tfrac{1}{\psi_2}) + N(r, \psi_2) && (3.7)\\ &= S(r, f). && (3.8) \end{aligned}$$

From (3.2), we get

$$\varphi(z)\psi_2(z) - \varphi(z)\psi_1(z) = \psi_1(z + c) + \psi_2(z + c). \tag{3.9}$$

From the definition of α and β, we conclude that

$$(\varphi(z) + \alpha)\psi_1 = (\varphi(z) - \beta)\psi_2. \tag{3.10}$$

From (3.3) and (3.10), it follows that

$$(\varphi(z) + \alpha)\psi_1^2 - (\varphi(z) - \beta)a(z)^2(1 - \varphi(z)^{-2}) = 0. \tag{3.11}$$

By (3.8), (3.11) and Lemma 5 we get $\alpha = -\varphi(z)$ and $\beta = \varphi(z)$. Otherwise, we get $T(r, \psi_1) = S(r, f)$. Combining (3.2) and (3.3), we conclude that $T(r, f) = S(r, f)$, which is impossible. Thus, we have completed the proof of Theorem 1.

3.2. Proof of Theorem 2

Proof. It suffices to consider the case (iii) in Theorem 1. We assume that $f(z_0) = 0$. Since $f(z)$ and $f(z + c)$ share 0 CM, then $\psi_1(z_0) + \psi_2(z_0) = 0$ and $\psi_1(z_0 + c) + \psi_2(z_0 + c) = 0$. Hence

$$\frac{\psi_1(z_0 + c)}{\psi_1(z_0)} \cdot \frac{\psi_2(z_0)}{\psi_2(z_0 + c)} = 1. \tag{3.12}$$

From $\frac{\psi_1(z+c)}{\psi_1(z)} = -\varphi(z)$, $\frac{\psi_2(z+c)}{\psi_2(z)} = \varphi(z)$, we obtain

$$\frac{\psi_1(z_0 + c)}{\psi_1(z_0)} \cdot \frac{\psi_2(z_0)}{\psi_2(z_0 + c)} = -1 \tag{3.13}$$

a contradiction. Hence 0 must be the Picard exceptional value of $f(z)$ and $f(z+c)$, which implies that $\psi_1(z)+\psi_2(z)\neq 0$. Combining this with $\psi_1\psi_2 = a(z)^2(1-\varphi(z)^{-2})$, we get the following equation

$$\psi_1(z)+\psi_2(z) = \frac{a(z)^2(1-\varphi(z)^{-2})+\psi_1^2}{\psi_1} = 2f(z), \tag{3.14}$$

From (3.3), (3.14) and by Lemma 5, we have

$$N(r,\psi_1^2) = S(r,f), N(r,\frac{1}{\psi_1^2}) = S(r,f), N(r,\frac{1}{a(z)^2(1-\varphi(z)^{-2})+\psi_1^2}) = S(r,f).$$

Applying the second main theorem for three small target functions, we get

$$\begin{aligned} T(r,f)+S(r,f) &= T(r,\psi_1^2) \leq N(r,\psi_1^2) + N(r,\tfrac{1}{\psi_1^2}) \\ &+N(r,\tfrac{1}{a(z)^2(1-\varphi(z)^{-2})+\psi_1^2}) + S(r,\psi_1) \\ &= S(r,f), \end{aligned}$$

which is a contradiction. So we can remove the case (iii) to get $f(z) = \pm f(z+c)$.

3.3. Proof of Theorem 3

Proof. From Lemma 5, we must have $T(r,\varphi^2(z)) = S(r,f)$. If $\varphi^2(z) = 1$, thus $f(z+c) = 2f(z)$. If $\varphi^2(z)\neq 1$, using a method similar to the proof of Theorem 1, we easily get $\frac{\psi_1(z+c)}{\psi_1(z)} = 1-\varphi(z)$, $\frac{\psi_2(z+c)}{\psi_2(z)} = 1+\varphi(z)$, and $\psi_1(z)\psi_2(z) = a(z)^2(1-\varphi(z)^{-2})$. Then we obtain

$$\psi_1(z+c)\psi_2(z+c) = \psi_1(z)\psi_2(z)(1-\varphi(z))(1+\varphi(z)) = a(z)^2(1-\varphi(z+c)^{-2}).$$

Thus, by computing, we can get

$$\varphi^2(z)+\varphi^{-2}(z)-\varphi^{-2}(z+c) = 1.$$

From the above equation and [3, Theorem 1.56], we get $\varphi^2(z) = 1$, which is a contradiction to our assumption. That implies $f(z+c) = 2f(z)$. Thus, we have completed the proof of Theorem 3.

3.4. Proof of Theorem 4

Proof. Assume that $B = \{a,-a\}, a\in\mathbb{C}\backslash\{0\}$. From (3.3) and Lemma 5, we have $N(r,\psi_1)+N(r,\frac{1}{\psi_1}) = S(r,f)$. since $2f(z) = \psi_1+\psi_2$ and $\psi_1\psi_2 = a(z)^2(1-\varphi(z)^{-2})$, we get

$$\frac{a(z)^2(1-\varphi(z)^{-2})+\psi_1^2}{\psi_1} = 2f(z).$$

Since f has finitely many zeros, then $N(r, \frac{1}{a(z)^2(1-\varphi(z)^{-2})+\psi_1^2}) = S(r, \psi_1)$. By the second main theorem for three small target functions, we obtain

$$T(r, \psi_1) \leq N(r, \psi_1) + N(r, \frac{1}{\psi_1}) + N(r, \frac{1}{a(z)^2(1-\varphi(z)^{-2})+\psi_1^2}) + S(r, \psi_1) \leq S(r, \psi_1)$$

a contradiction. So we can remove the case (iii) of Theorem 1. Thus, we have completed the proof of Theorem 4.

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