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DIFFERENTIAL SUBORDINATION FOR MEROMORPHIC MULTIVALENT QUASI-CONVEX FUNCTIONS *

Maslina Darus and Imran Faisal

Abstract. An attempt has been made to introduce certain new classes of meromorphic multivalent quasi-convex functions and discuss its differential subordination properties in the punctured unit disk $\mathbb{U}$.

1. Introduction and preliminaries

Let $\mathcal{E}^{+}_{p,\alpha}$ denotes the family of all functions F, of the form

$$F(z) = \frac{1}{z^p} + \sum_{n=2}^{\infty} a_n z^{n-n/\alpha} \quad \alpha \in \mathbb{N} \setminus \{1\},\ p = 1, 2, ... \tag{1.1}$$

which are analytic in the punctured unit disk $\mathbb{U} = \{z : z \in C \ \ |z| < 1\}$.

Similarly $\mathcal{E}^{-}_{p,\alpha}$ denotes the family of all functions F, of the form

$$F(z) = \frac{1}{z^p} - \sum_{n=2}^{\infty} a_n z^{n-n/\alpha} \quad \alpha \in \mathbb{N} \setminus \{1\},\ p = 1, 2, ... \tag{1.2}$$

which are analytic in the punctured unit disk $\mathbb{U}$.

For two functions f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write $f(z) \prec g(z)$ or simply $f \prec g$ if there exists a Schwarz function w which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w| < 1$ such that $f(z) = g(w(z))\, z \in \mathbb{U}$.

Let $\phi : C^3 \times \mathbb{U} \rightarrow C$ and let h analytic in $\mathbb{U}$. Assume that p,ϕ are analytic and univalent in $\mathbb{U}$ and p satisfies the differential superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z). \tag{1.3}$$

Then p is called a solution of the differential superordination.

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An analytic function q is called a subordinant if $q \prec p$,for all p satisfying equation (1.3). A univalent function q such that $p \prec q$ for all subordinates p of equation (1.3) is said to be the best subordinant.

Let $\mathcal{E}_p^+$ be the class of all functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=2}^{\infty} a_n z^n \quad p = 1, 2, ...$$

which are analytic in the punctured unit disk $\mathbb{U}$.

Similarly $\mathcal{E}_p^-$ be the class of all functions of the form

$$f(z) = \frac{1}{z^p} - \sum_{n=2}^{\infty} a_n z^n \quad p = 1, 2, ...$$

which are analytic in the punctured unit disk $\mathbb{U}$.

A function $f \in \mathcal{E}_p^+(\mathcal{E}_p^-)$ is meromorphic multivalent starlike if $f(z) \neq 0$ and

$$-\operatorname{Re}(\frac{zf'(z)}{f(z)}) > 0, \quad z \in \mathbb{U}.$$

Similarly, a function f is meromorphic multivalent convex if $f'(z) \neq 0$ and

$$-\operatorname{Re}(1 + \frac{zf''(z)}{f'(z)}) > 0, \quad z \in \mathbb{U}.$$

Moreover, a function f is called meromorphic multivalent Quasi-convex function if there is a meromorphic multivalent convex function g such that

$$-\operatorname{Re}(\frac{(zf'(z))'}{g'(z)}) > 0 \quad z \in \mathbb{U}.$$

A function $F \in \mathcal{E}_{p,\alpha}^+(\mathcal{E}_{p,\alpha}^-)$ is meromorphic multivalent starlike if $F(z) \neq 0$ and

$$-\operatorname{Re}(\frac{zF'(z)}{F(z)}) > 0, \quad z \in \mathbb{U}.$$

Similar, $F \in \mathcal{E}_{p,\alpha}^+(\mathcal{E}_{p,\alpha}^-)$ is meromorphic multivalent convex if $F'(z) \neq 0$ and

$$-\operatorname{Re}(1 + \frac{zF''(z)}{F'(z)}) > 0, \quad z \in \mathbb{U}.$$

A function $f \in \mathcal{E}_{p,\alpha}^+(\mathcal{E}_{p,\alpha}^-)$ is called meromorphic multivalent Quasi-convex function if there exist a meromorphic multivalent convex function G such that $G'(z) \neq 0$ and

$$-\operatorname{Re}(\frac{(zF'(z))'}{G'(z)}) > 0 \quad z \in \mathbb{U}.$$

In the present paper, we establish some sufficient conditions for the functions belong to the classes $\mathcal{E}^{+}_{p,\alpha}$ and $\mathcal{E}^{-}_{p,\alpha}$ to satisfy

$$-\operatorname{Re}\left(\frac{(z^pF'(z))'}{G'(z)}\right) \prec q(z), \quad z \in \mathbb{U}$$

and q is the given univalent function in $\mathbb{U}$. Moreover, we give applications for these results in fractional calculus. In order to prove our subordination results, we need to the following lemmas in the sequel.

Lemma 1.1. *[11] Let q be convex univalent in the unit disk $\mathbb{U}$ and ψ and $\gamma \in C$ with*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\right) > 0.$$

If p is analytic in $\mathbb{U}$ and

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z),$$

then $p \prec q$ and q is the best dominant.

Lemma 1.2. *[10] Let q be univalent in the unit disk $\mathbb{U}$ and θ be analytic in a domain D containing $q(U)$. If $zq'(z)\theta(z)$ is starlike in $\mathbb{U}$ and*

$$zp'(z)\theta(p(z)) \prec zq(z)\theta(q(z))$$

then $p \prec q$ and q is the best dominant.

2. Subordination Theorems

In this section, we establish some sufficient conditions for subordination of analytic functions in the classes $\mathcal{E}^{+}_{p,\alpha}$ and $\mathcal{E}^{-}_{p,\alpha}$. Note also similar work has been seen for different subclasses done by other authors (see for example [4-7])

Theorem 2.1. *Let the function q be convex univalent in $\mathbb{U}$ such that $q'(z) \neq 0$ and*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\right) > 0, \quad \gamma \neq 0. \tag{2.1}$$

Suppose that $-\frac{(z^pF'(z))'}{G'(z)}$ is analytic in $\mathbb{U}$. If $F \in \mathcal{E}^{+}_{p,\alpha}$ satisfies the subordination

$$-\frac{(z^pF'(z))'}{G'(z)}\left(\psi + \gamma\left(\frac{z(z^pF'(z))''}{(z^pF'(z))'} - \frac{zG''(z)}{G'(z)}\right)\right) \prec \psi q(z) + \gamma z q'(z),$$

then

$$-\frac{(z^pF'(z))'}{G'(z)} \prec q(z),$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(z) = -\frac{(z^pF'(z))'}{G'(z)}, \quad z \in \mathbb{U}$$

It can easily observed that

$$\psi p(z) + \gamma z p'(z) = -\frac{(z^pF'(z))'}{G'(z)}(\psi + \gamma(\frac{z(z^pF'(z))''}{(z^pF'(z))'} - \frac{zG''(z)}{G'(z)}))$$
$$\prec \psi q(z) + \gamma z q'(z).$$

Then using the assumption the theorem the assertion of the theorem follows by an application of Lemma 1.1. □

Corollary 2.1. *Assume that eq. (2.1) holds. Let the function q be univalent in $\mathbb{U}$. Let $n = 1$, if q satisfies the subordination*

$$-\frac{(zF'(z))'}{G'(z)}(\psi + \gamma(\frac{z(zF'(z))''}{(zF'(z))'} - \frac{zG''(z)}{G'(z)})) \prec \psi q(z) + \gamma z q'(z),$$

then

$$-\frac{(zF'(z))'}{G'(z)} \prec q(z),$$

and q is the best dominant.

Theorem 2.2. *Let the function q be univalent in $\mathbb{U}$ such that $q \neq 0, z \in \mathbb{U}$ and $\frac{zq'(z)}{q(z)}$, is starlike univalent in $\mathbb{U}$. If $F \in \mathcal{E}^-_{p,\alpha}$ satisfies the subordination*

$$a(\frac{z(z^pF'(z))''}{(z^pF'(z))'} - \frac{zG''(z)}{G'(z)}) \prec a\frac{zq'(z)}{q(z)},$$

then

$$-\frac{(z^pF'(z))'}{G'(z)} \prec q(z),$$

and q is the best dominant.

Proof. Let the function ψ be defined by

$$\psi(z) = -\frac{(z^pF'(z))'}{G'(z)}, \quad z \in \mathbb{U}$$

By setting

$$\theta(\omega) = a/\omega, \ \omega \neq 0$$

it can be easily observed that θ is analytic in $C - \{0\}$. Then by simple computation we have

$$a\frac{z\psi'(z)}{\psi(z)} = a(\frac{z(z^pF'(z))''}{(z^pF'(z))'} - \frac{zG''(z)}{G'(z)})$$
$$\prec \psi q(z) + \gamma z q'(z).$$

Then using the assumption the theorem the assertion of the theorem follows by an application of Lemma 1.2. □

Corollary 2.2. *Assume that q is convex univalent in $\mathbb{U}$. Let $p = 1$, if $F \in \mathcal{E}^-_{p,\alpha}$ and*

$$a(\frac{z(zF'(z))''}{(zF'(z))'} - \frac{zG''(z)}{G'(z)}) \prec a\frac{zq'(z)}{q(z)},$$

then

$$-\frac{(zF'(z))'}{G'(z)} \prec q(z),$$

and q is the best dominant.

3. Applications of Fractional Integral Operator

In this section we introduce some applications of section (2) containing fractional integral operators. Assume that $f(z) = \sum_0^\infty \phi_n z^n$ and let us begin with the following definition. Note also similar work has been seen for different subclasses done by other authors (see for example [1, 2, 3, 8, 9]).

Definition 3.1. The fractional integral of order α is defined for a function f by,

$$I_z^\alpha f(z) = \frac{1}{\Gamma(\alpha)}\int_0^z f(z)(z-\zeta)^{\alpha-1}d\zeta, \quad 0 \le \alpha < 1$$

where, the function *f(z)* is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Note that $I_z^\alpha f(z) = f(z) \times z^{\alpha-1}/\Gamma(\alpha)$ for $z > 0$ and 0. Let

$$f(z) = \sum_0^\infty \phi_n z^{n-n/\beta+1-\alpha},$$

this implies that

$$I_z^\alpha f(z) = f(z) \times z^{\alpha-1}/\Gamma(\alpha) = z^{\alpha-1}/\Gamma(\alpha)\sum_0^\infty \phi_n z^{n-n/\beta+1-\alpha} \quad for z > 0$$

$$= \sum_o^\infty a_n z^{n-n/\beta}, \quad where \quad a_n = \phi_n/\Gamma(\alpha),$$

thus

$$1/z^p \pm I_z^\alpha f(z) \in \mathcal{E}^+_{p,\alpha}(\mathcal{E}^-_{p,\alpha})$$

Theorem 3.1. *Let the function q be convex univalent in $\mathbb{U}$ such that $q' \neq 0$ and*

$$\mathrm{Re}(1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}) > 0, \quad \gamma \neq 0. \tag{3.1}$$

Suppose that $-\frac{(z^p(1/z^p+I_z^\alpha f(z))')'}{(1/z^p+I_z^\alpha g(z))'}$ *is analytic in* $\mathbb{U}$. *If* $F \in \mathcal{E}_{p,\alpha}^{+}$ *satisfies the subordination*

$$-\frac{(z^p(1/z^p + I_z^\alpha f(z))')'}{(1/z^p + I_z^\alpha g(z))'}(\psi+\gamma(\frac{z(z^p(1/z^p + I_z^\alpha f(z))')''}{(z^p(1/z^p + I_z^\alpha f(z))')'}-\frac{z(1/z^p + I_z^\alpha g(z))''}{(1/z^p + I_z^\alpha g(z))'})) \prec \psi q(z)+\gamma z q'(z),$$

then

$$-\frac{(z^p(1/z^p + I_z^\alpha f(z))')'}{(1/z^p + I_z^\alpha g(z))'} \prec q(z),$$

and q *is the best dominant.*

Proof. Let the function p be defined by

$$p(z) = -\frac{(z^p(1/z^p + I_z^\alpha f(z))')'}{(1/z^p + I_z^\alpha g(z))'}, \quad z \in \mathbb{U}$$

It can easily observed that

$$\psi p(z)+\gamma z p'(z) = -\frac{(z^p(1/z^p + I_z^\alpha f(z))')'}{(1/z^p + I_z^\alpha g(z))'}(\psi+\gamma(\frac{z(z^p(1/z^p + I_z^\alpha f(z))')''}{(z^p(1/z^p + I_z^\alpha f(z))')'}-\frac{z(1/z^p + I_z^\alpha g(z))''}{(1/z^p + I_z^\alpha g(z))'}))$$

$$\prec \psi q(z) + \gamma z q'(z).$$

Then using the assumption the theorem the assertion of the theorem follows by an application of lemma 1.1. □

Theorem 3.2. *Let the assumptions of theorem 2.2. hold, then*

$$-\frac{(z^p(1/z^p - I_z^\alpha f(z))')'}{(1/z^p - I_z^\alpha g(z))'} \prec q(z), \quad z \in U$$

where $F(z) = (1/z^p - I_z^\alpha f(z)$, $G(z) = (1/z^p - I_z^\alpha g(z)$ *and* q *is the best dominant.*

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Maslina Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor
D. Ehsan, Malaysia
`maslina@ukm.my`

Imran Faisal
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor
D. Ehsan, Malaysia
`faisalmath@gmail.com`

(corresponding author: maslina@ukm.my)