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DISCRETE LOCATION PROBLEM ON ARBITRARY SURFACE IN R^3*

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Abstract. We consider a discrete location problem in which locations of suppliers as well as locations of existing customers belong to arbitrary surface S in $\mathcal{R}^3$ and the distance between locations is the length of the shortest arc between all arcs connecting them. The lengths of trajectories that connect certain locations are calculated using coefficients of the first fundamental form of the surface S. The established discrete location problem on the surface is implemented and visualized in the programming package *Mathematica*.

Key words: Discrete location problem, First fundamental form, *Mathematica*.

1. Introduction

In the general case, the task of location problem is to define positions of some new facilities from the actual space in which some other relevant objects (points) are already placed. New facilities are centers that provide services and called *suppliers*; existing objects are the service users or clients, and called *customers*. Location problems occur frequently in real life. Many systems in the public and private sectors are characterized by facilities that provide homogeneous services at their locations to a given set of fixed points or customers. Examples of such facilities include warehouse location, positioning a computer and communication units, locating hospitals, police stations, locating fire stations in a city, locating base stations in wireless networks.

Different classifications of the location problems are known. The classification scheme from [12] assumes five positions in the order *Pos*1/*Pos*2/*Pos*3/*Pos*4/*Pos*5, where the meaning of each position is described as follows [12]:

*Pos*1 The number and type of new facilities to deploy.

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*Pos*2 Type of the location model with respect to the decision space. This information should at least be sensible to discrete, continuous and network location models. Continuous location models assume that the new location can be placed anywhere in some specified feasible region which often coincides with the complete plane; discrete models choose an optimal location from a discrete set of offered points, while in the network models new facilities can be placed either only on the nodes or on nodes and edges of the network.

*Pos*3 Particular information about the location model, such as information about the feasible solutions, capacity restrictions, etc.

*Pos*4 Relations between new and existing facilities. These relations may be expressed by a distance function or by accompanied costs.

*Pos*5 Definition of the distance dependent objective function.

We pay attention to selection of the distance function in the location problem. The distance between two points is the length of the shortest path connecting them. The metric by which the distance between two points is measured may be different in various instances [3]. In the calculating of distance between two points, the most common distance metrics in a continuous space are those known as the class of l_p distance metrics, primarily rectangular (l_1), Euclidean (l_2) and Chebyshev (l_∞) metric. Detailed explanation of various metrics one can find in Dictionary of distances [6]. Many factors affect on the process of metrics choosing. The most important factor is the nature of the problem. For example, if it is possible to move rectilinearly between two points, the distance between them is exactly given by the Euclidean (or straight-line distance) metric. On the other hand, in the cities where streets intersect under the right angle mainly, the distance between two points will be the best approximated using the rectangular metric (also known as the Manhattan, "city block" distance, the right-angle distance metric or taxicab distance). Choice of the metric is fundamental in the geometry construction. Taxicab geometry, considered by Hermann Minkowski in the 19th century, is a form of geometry in which the usual metric of Euclidean geometry is replaced by the rectangular metric in which the distance between two points is the sum of the absolute differences of their corresponding coordinates. Measures of distances in chess are a characteristic example. The distance between squares on the chessboard for rooks is measured in Manhattan distance; kings and queens use Chebyshev distance, and bishops use the Manhattan distance.

On the other hand, the earth's surface is approximately planar only on small dimensions. For this reason, it is reasonable to use spherical distances to solve the facility problems. Fundamental results regarding to the spherical distance location problem are established in Drezner and Wesolowsky (1978) [7], Aly, Kay and Litwhiler (1979) [1] and Drezner (1985), where the authors considered a modification of the Weber problem which consists of locating a new facility on a sphere, so that the weighted sum of distances to given demand points is minimized. A review of the spherical location problem is given in Wesolwsky (1983) [17] and Plastria (1995) [16].

A couple of variants and extensions of continuous location problems have been investigated in literature. Let us mention main between them. More complex problems include the placement of multiple facilities. Problems with barriers are the subject in [5, 11, 13, 14]. The location of undesirable (obnoxious) facilities requires to maximize minimum distances (see, e.g., [2, 8, 9, 15]. Location models with both desirable and undesirable facilities have been analyzed in [4].

We give an extension of the planar location problem as well as the location problem on a sphere. Namely, instead of the plane or a sphere, arbitrary surface S in $\mathcal{R}^3$ is considered as the inhabitation for customers and suppliers. The trajectories that connect certain locations are arcs of curves on S. Instead of using the particular metric to calculate distances, we calculate lengths of curves from S connecting locations. In this way, we generalize results attained in the papers related with the spherical location problem, assuming that location of a new facility and locations of existing objects are on arbitrary surface S and that the distance between two points is the length of the shortest curve which connects these points. *Mathematica* computer program is used for calculations and as the useful teaching tool in visualization.

2. Solution and visualization of discrete location problem on the surface

We consider the discrete facility problem where the points, instead of being on a plane or on a sphere [7], are on arbitrary surface S in $\mathcal{R}^3$. Assume that $A_1, \dots, A_m$ are points on the surface S where some customers are located and let $B_1, \dots, B_r$ are potential locations on which is possible to place a new desired object (supplier). Suppose that the points A_i and B_k are connected by regular curves $C_{A_iB_k}$ which lie on the surface S. The sum of weighted distances from the potential location of the supplier B_k to the customers is equal to the sum

$$W_k = \sum_{i=1}^{m} w_i \cdot l_{ik},$$

where w_i is the weight associated with A_i and $l_{ik} = l_{A_iB_k}$ denotes the length of the arc $C_{A_iB_k} = C_{ik}$ connecting A_i and B_k.

Let us restate some known facts (see, for example [10]). Assume that the surface S is given by the parametric equation

$$S : r = r(u, v).$$

Then the equation of an arbitrary curve C on the surface S is

$$C : r(t) = r(u(t), v(t)).$$

The first fundamental form of the surface S is given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

where

$$E = r_u \cdot r_u,\ F = r_u \cdot r_v,\ G = r_v \cdot r_v$$

are the coefficients of the first fundamental form, r_u and r_v are the partial derivatives of the function $r(u, v)$ and the dot sign denotes the scalar product. The length of arc of the curve C on the surface S for $t \in (\alpha, \beta)$ is

$$s = \int_\alpha^\beta \|\dot{r}(u)\|\, du = \int_\alpha^\beta \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt.$$

Therefore, it is necessary to solve the discrete location problem on the surface S with given coordinates of customers $A_i,\ i = 1, \ldots, m$ and suppliers $B_k,\ k = 1, \ldots, r$, weighted coefficients $w_i,\ i = 1, \ldots, m$ and parametric equations of the arcs given by

$$C_{ik} : r_{ik} = r(u_{ik}(t), v_{ik}(t)),\ t \in (a_{ik}, b_{ik}),\ i = 1, \ldots, m,\ k = 1, \ldots, r.$$

The solution is obtained by the procedure which consists in several steps, as in the following.

Step 1. For each arc C_{ik} find α_{ik} and β_{ik} by solving the equations

$$r_{ik}(\alpha_{ik}) = A_i,\ r_{ik}(\beta_{ik}) = B_k,$$

where $[\alpha_{ik}, \beta_{ik}] \subseteq (a_{ik}, b_{ik})$.

Step 2. Find lengths of arcs C_{ik}

$$l_{ik} = \int_{\alpha_{ik}}^{\beta_{ik}} \|\dot{r}_{ik}(t)\|\, dt.$$

It is useful to benefit the next command in *Mathematica* for calculating these lengths:

```
arclengthprime[r_][t_] := Sqrt[Simplify[D[r[tt], tt].D[r[tt], tt]]] /. tt->t
leng[a_, b_][r_] := Abs[NIntegrate[arclengthprime[r][u], {u, a, b}]]
```

Step 3. Choose the weights w_i, $i = 1, \ldots, r$ and compute sums of weighted distances

$$W_k = \sum_{i=1}^{m} w_i \cdot l_{ik},\ k = 1, \ldots, r. \tag{2.1}$$

Step 4. Find the minimal sum of weighted distances

$$W_{k^*} = \min\{W_k |\ 1 \leq k \leq r\}. \tag{2.2}$$

Then the solution is the point B_{k^*}.

Corresponding implementation is given by the next *Mathematica* function.

```
discreteSurface[r_, lp_, lm_, lt_] :=
  Module[{d, n, i, j, s, p, a={}, b={}, pom1, pom2, ind=1, ras, ras2, sumdist={}},
      d = Length[lm]; n = Length[lp];
      For[i = 1, i <= n, i++,
         pom1 = {};
         For[j = 1, j <= d, j++,
             AppendTo[pom1, NSolve[lp[[i, 1]] == r[[i, j, 1]], t]]
         ];
         AppendTo[a, pom1];
      ];
      For[i = 1, i <= n, i++,
          pom2 = {};
          For[j = 1, j <= d, j++,
              AppendTo[pom2, NSolve[lm[[j, 1]] == r[[i, j, 1]], t]]
          ];
          AppendTo[b, pom2];
      ];
      For[i = 1, i <= d, i++,
          AppendTo[sumdist,
          Sum[lt[[k]]*leng[a[[k, i]], b[[k, i]]][r[[k, i]]], {k, 1, n}]];
      ];
      ras = sumdist[[1]];
      For[j = 2, j <= d, j++,
          ras2 = sumdist[[j]];
          If[ras > ras2, ind = j; ras = ras2];
      ];
      Return[lm[[ind]]];
  ];
```

Example 2.1. Let be given the locations of two points

$$A_1(\sqrt{2}/2, \sqrt{2}/2, 1/2),\ A_2(1, e, e)$$

on the surface $S : r(u, v) = (u, v, uv)$. Two locations are possible for new object (supplier):

$$B_1(0, 1, 0),\ B_2(1, 1, 1).$$

The weighted coefficients corresponding to A_1 and A_2 are equal to $w_1 = 3$ and $w_2 = 1$, respectively. Suppose that the arcs between two points A_i and B_k are given by the following equations:

$$\begin{aligned}
C_{11} &: r(t) = (\cos t, \sin t, \cos t \sin t),\\
C_{12} &: r(t) = (\cos t, \cos t, \cos^2 t),\\
C_{21} &: r(t) = (t, 1 + (e-1)t, t + (e-1)t^2),\\
C_{22} &: r(t) = (1, e^t, e^t).
\end{aligned}$$

Determine the coordinates of the new object which minimizes (2.1) and corresponds to the minimal sum of weighted distances (2.2). The next program in *Mathematica* gives us graphical presentation of the surface S with observing paths.

```
p = Plot3D[u*v, {u, -1, 2}, {v, -2, 4},
      MeshStyle-> GrayLevel[.4], Shading->False, PlotPoints->20,
      DisplayFunction->Identity
    ];

c11[u_] := {Cos[u], Sin[u], Cos[u]*Sin[u]}
q = ParametricPlot3D[
      Append[c11[u], {RGBColor[1,0,0], AbsoluteThickness[2]}]//Evaluate,
```

```
            {u, Pi/4, Pi/2},
            DisplayFunction -> Identity
    ];

c12[u_] := {Cos[u], Cos[u], Cos[u]^2}
r = ParametricPlot3D[ Append[c12[u], {RGBColor[0,1,0], AbsoluteThickness[2]}]//Evaluate,
                     {u, 0, Pi/4},
                     DisplayFunction -> Identity
    ];

e := 2.61
c21[t_] := {t, 1 + (e - 1)*t, t + (e - 1)*t^2}
s =ParametricPlot3D[ Append[c21[u], {RGBColor[0,0,1], AbsoluteThickness[2]}]//Evaluate,
                    {u, 0, 1},
                    DisplayFunction -> Identity
   ];

c22[t_] := {1, e^t, e^t}
f = ParametricPlot3D[ Append[c22[u], {RGBColor[0,1,1], AbsoluteThickness[2]}]//Evaluate,
                     {u, 0, 1},
                     DisplayFunction -> Identity
    ];
Show[  p, q, r, s, f,
         ViewPoint->{3,-1,4}, BoxRatios->{1,1,1},
         Boxed->False, Ticks->None, Axes->False, DisplayFunction->$DisplayFunction
    ]
```

Graphical illustration made using *Mathematica* is presented on Figure 1.

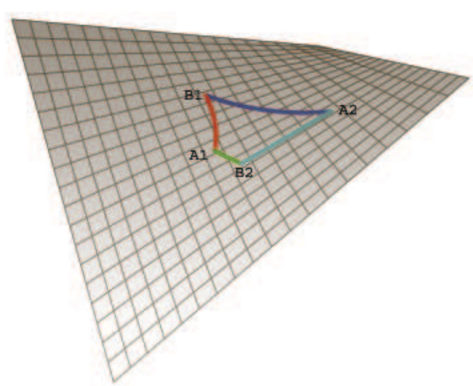

Figure 1. Distance as the length of curve

Let calculate the coefficients of the first fundamental form of the surface S:

$$r_u = (1, 0, v),\ r_v = (0, 1, u),\ \ E = 1 + v^2,\ \ F = u\,v,\ \ G = 1 + u^2.$$

Look at the each curve C particulary and compute its length l:

$$C_{11}:\begin{cases} u(t)=\cos t, & \dot u(t)=-\sin t\\ v(t)=\sin t, & \dot v(t)=\cos t,\end{cases}$$

$$l_{11}=\int_{\pi/4}^{\pi/2}\sqrt{E\dot u^2+2F\dot u\dot v+G\dot v^2}dt\approx 0.96$$

$$C_{12}:\begin{cases} u(t)=\cos t, & \dot u(t)=-\sin t\\ v(t)=\cos t, & \dot v(t)=-\sin t,\end{cases}$$

$$l_{12}=\int_{0}^{\pi/4}\sqrt{E\dot u^2+2F\dot u\dot v+G\dot v^2}dt\approx 0.65$$

$$C_{21}:\begin{cases} u(t)=t, & \dot u(t)=1\\ v(t)=1+(e-1)t, & \dot v(t)=e-1,\end{cases}$$

$$l_{21}=\int_{0}^{1}\sqrt{E\dot u^2+2F\dot u\dot v+G\dot v^2}dt\approx 3.42$$

$$C_{22}:\begin{cases} u(t)=1, & \dot u(t)=0\\ v(t)=e^t, & \dot v(t)=e^t,\end{cases}$$

$$l_{22}=\int_{0}^{1}\sqrt{E\dot u^2+2F\dot u\dot v+G\dot v^2}dt\approx 2.43.$$

Now we can calculate the sum of the weighted distances from the potential locations of the supplier to the customers.

$$W_1=w_1\cdot l_{11}+w_2\cdot l_{21}=3\cdot 0.96+1\cdot 3.42=6.3$$
$$W_2=w_1\cdot l_{12}+w_2\cdot l_{22}=3\cdot 0.65+1\cdot 2.43=4.38.$$

Therefore, the new object will be built on the location B_2, i.e. on the location whose coordinates are $(1,1,1)$. The sum of the weighted distances adjoint to this point is 4.38.

This model could be solved by the next *Mathematica* expression

```
e = Exp[1] // N;
discreteSurface[{{{Cos[t], Sin[t], Cos[t]*Sin[t]}, {Cos[t], Cos[t], Cos[t]^2}},
                {{t, 1+(e-1)*t, t+(e-1)*t^2}, {1, e^t, e^t}}},
                {{1.41/2, 1.41/2, 0.5}, {1, e, e}}, {{0, 1, 0}, {1, 1, 1}}, {3, 1}]
```

Example 2.2. Let $A_1(0,0,0)$ and $A_2(\pi,0,0)$ be the locations of two points on the surface defined as

$$S:\ z=\sin(x+\sin y).$$

There are two possible locations for the new object: $B_1(0,\pi,0)$ and $B_2(\pi,\pi,0)$. The weighted coefficients corresponding to A_1 and A_2 are $w_1=3$ and $w_2=1$, respectively. The arcs between the points A_i and B_k are parts of curves given by the next equations:

$$C_{11}: r(t)=(0,t,\sin(\sin t)),$$
$$C_{12}: r(t)=(t,t,\sin(t+\sin t)),$$
$$C_{21}: r(t)=(t+\pi,-t,\sin(t+\pi+\sin(-t))),$$
$$C_{22}: r(t)=(\pi,t,\sin(\pi+\sin t)).$$

Determine the coordinates of new object. Graphical illustration made using *Mathematica* is presented on Figure 2.

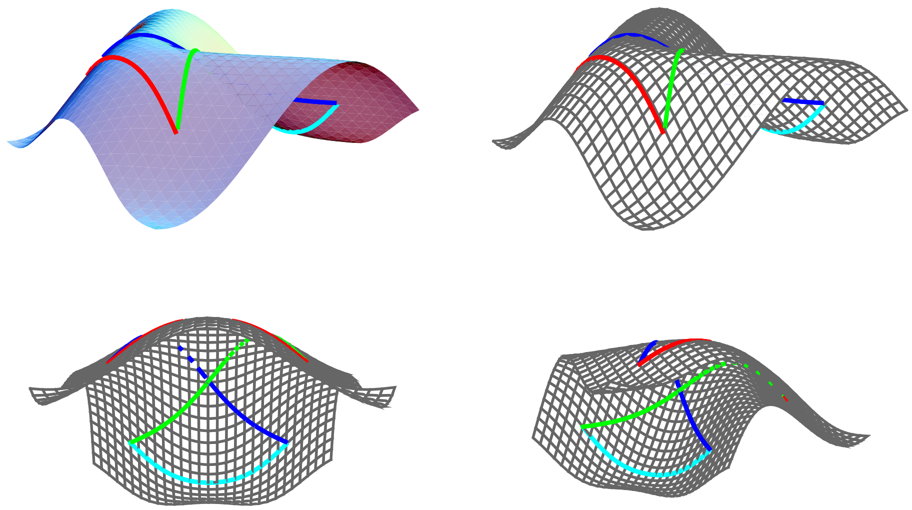

Figure 2. Distance is computed as the length of arc

Corresponding lengths of arcs are:

$$l_{11} = \int_0^{\pi} \sqrt{1 + \cos^2(\sin t) \cdot \cos^2 t}\, dt \approx 3.68$$
$$l_{12} = \int_0^{\pi} \sqrt{2 + \cos^2(t + \sin t) \cdot (\cos t + 1)^2}\, dt \approx 5.08$$
$$l_{21} = \int_{-\pi}^{0} \sqrt{2 + \cos^2(t + \pi + \sin(-t)) \cdot (\cos(-t) - 1)^2}\, dt \approx 5.08$$
$$l_{22} = \int_0^{\pi} \sqrt{1 + \cos^2(\pi + \sin t) \cdot \cos^2 t}\, dt \approx 3.68.$$

The sum of weighted distances are:

$$W_1 = w_1 \cdot l_{11} + w_2 \cdot l_{21} = 3 \cdot 3.68 + 1 \cdot 5.08 = 16.12$$
$$W_2 = w_1 \cdot l_{12} + w_2 \cdot l_{22} = 3 \cdot 5.08 + 1 \cdot 3.68 = 18.92.$$

Therefore, the new object will be built on the location $B_1(0, \pi, 0)$. The sum of weighted distances for that point is equal to 16.12.

This model could be solved by the next *Mathematica* command

```
discreteSurface[{{{0, t, Sin[Sin[t]]}, {t, t, Sin[t + Sin[t]]}},
                 {{t+Pi, -t, Sin[t+Pi+Sin[-t]]}, {Pi, t, Sin[Pi+Sin[t]]}}},
                 {{0, 0, 0}, {Pi, 0, 0}}, {{0, Pi, 0}, {Pi, Pi, 0}}, {3, 1}]
```

3. Conclusion

It seems interesting and reasonable to investigate the location problem and its various extensions in a more general non-convex case, where the shortest length of arc is used as distance instead of a particular metrics. Also, a lot of real-word problems are non-convex.

We consider the discrete location problem where the points, instead of being on a plane or on a sphere, lie on arbitrary surface S. arbitrary surface S in $\mathcal{R}^3$ is used instead of the plane or a sphere. The trajectories that connect certain locations are arcs of curves on S. Lengths of these curves are calculated as a generalization of the usage of a particular metric in distances calculation.

Visual and interactive representation of the location problem is useful in pedagogical purposes. Visualization and interactive tools help students to quickly obtain an intuitive feel of the location problem as well as to learn the topic more rapidly and with lesser pain.

We use the programming package *Mathematica* as a powerful visualization tool and a powerful platform for performing calculations.

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