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A DIRECT PROOF OF THE CONNECTION OF TWO CONVERGENCE TESTS

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Abstract. This work is motivated by [1]. We give here a result about the connection of the CAUCHY's second test and RAABE-DUHAMEL's test. Our result improves the results from the cited paper which become particular cases of our studies.

1. Introduction

This paper is motivated by [1], where the connection between the following two convergence tests

$$\lim_{n\to\infty} \frac{\ln(1/a_n)}{\ln n} = \lambda_1 \qquad \text{(CAUCHY's second test)}$$

and

$$\lim_{n\to\infty} \left(n\left(1 - \frac{a_{n+1}}{a_n}\right)\right) = \lambda_2 \qquad \text{(RAABE-DUHAMEL's test)}$$

is studied. It is well-known that if the limit λ_2 exists, finite or not, then the limit λ_1 exists and $\lambda_1 = \lambda_2$. For proof and other comments see for example [3]. This can be obtained from the following inequality:

$$\begin{aligned} \underline{\lim}\left(n\left(1 - \frac{a_{n+1}}{a_n}\right)\right) \leqslant \underline{\lim}\frac{\ln(1/a_n)}{\ln n} \leqslant \overline{\lim}\frac{\ln(1/a_n)}{\ln n} \\ \leqslant \overline{\lim}\left(n\left(1 - \frac{a_{n+1}}{a_n}\right)\right) \end{aligned} \tag{1.1}$$

which is the main result from [1]. The proof of this inequality given in [1] is too technical, using also some elements of asymptotic theory, or other elements specific for the logarithm function. Here we give a direct proof of a result from which the inequality (1.1) follows as a particular case.

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2. The Results

We will use the following result:

Lemma 2.1. *For every real sequences* $(x_n)_{n\geqslant 1}$, $(y_n)_{n\geqslant 1}$, $y_n > 0$, *with the property that* $y_1 + y_2 + ... + y_n \to \infty$, *we have:*

$$\underline{\lim}\frac{x_n}{y_n} \leqslant \underline{\lim}\frac{x_1 + x_2 + ... + x_n}{y_1 + y_2 + ... + y_n} \leqslant \overline{\lim}\frac{x_1 + x_2 + ... + x_n}{y_1 + y_2 + ... + y_n} \leqslant \overline{\lim}\frac{x_n}{y_n}. \tag{2.1}$$

Proof. For each $a < \underline{\lim}\frac{x_n}{y_n}$, there exists $k \geqslant 1$ such that

$$\frac{x_n}{y_n} \geqslant a \quad , \quad \text{for all } n \geqslant k.$$

By adding the inequalities $ay_k \leqslant x_k$, $ay_{k+1} \leqslant x_{k+1}$, ..., $ay_n \leqslant x_n$, we obtain

$$x_1 + x_2 + ... + x_n \geqslant x_1 + x_2 + ... + x_{k-1} + a(y_k + y_{k+1} + ... + y_n),$$

then by dividing by $y_1 + y_2 + ... + y_n$, we deduce that

$$\frac{x_1 + x_2 + ... + x_n}{y_1 + y_2 + ... + y_n} \geqslant a + \frac{q}{y_1 + y_2 + ... + y_n}, \tag{2.2}$$

where

$$q = x_1 + x_2 + ... + x_{k-1} - a(y_1 + y_2 + ... + y_{k-1})$$

is a constant, nondepending on n. Consequently, by taking the inferior limit in (2.2) as $n \to \infty$ and using the fact that $q/(y_1 + y_2 + ... + y_n) \to 0$, we derive

$$\underline{\lim}\frac{x_1 + x_2 + ... + x_n}{y_1 + y_2 + ... + y_n} \geqslant a$$

and by de definition of the real number a arbitrary taken such that

$$a < \underline{\lim}\frac{x_n}{y_n},$$

the first inequality from (2.1) is proven. Finally, the last inequality from (2.1) follows for example, by replacing x_n by $-x_n$ in the first inequality of the relation (2.1). □

Now we will consider a series

$$\sum_{n=1}^{\infty} a_n$$

with positive terms, whose convergence we want to study. To do this, we will assume that the general term converges to zero, otherwise the series is divergent. Moreover, we will assume without loss of generality that

$$\frac{a_{n+1}}{a_n} \to 1,$$

otherwise the limit λ_2, if exists, is equal to $\pm\infty$.

Under these hypoteses, let us define

$$x_n = \ln a_n - \ln a_{n+1}, \quad y_n = \ln(n+1) - \ln n, \quad n \geqslant 1,$$

with $a_1 = 0$. We have:

$$\begin{aligned} \underline{\lim}\frac{x_n}{y_n} &= \underline{\lim}\frac{-\ln\frac{a_{n+1}}{a_n}}{\ln\left(1+\frac{1}{n}\right)} = \\ &= -\underline{\lim}\left(\frac{\ln\frac{a_{n+1}}{a_n}}{\frac{a_{n+1}}{a_n}-1}\cdot\left(\frac{a_{n+1}}{a_n}-1\right)\cdot\frac{n}{\ln\left(1+\frac{1}{n}\right)^n}\right) \\ &= \underline{\lim}\left(n\left(1-\frac{a_{n+1}}{a_n}\right)\right), \end{aligned} \tag{2.3}$$

where we used the usual limits

$$\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{\frac{a_{n+1}}{a_n}-1} = \lim_{x\to 1}\frac{\ln x}{x-1} = 1 \quad \text{and} \quad \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n = 1.$$

Further,

$$\underline{\lim}\frac{x_1+x_2+\ldots+x_{n-1}}{y_1+y_2+\ldots+y_{n-1}} = \underline{\lim}\frac{-\ln a_n}{\ln n} = \underline{\lim}\frac{\ln(1/a_n)}{\ln n}. \tag{2.4}$$

Now, by replacing (2.3) and (2.4) in the inequality (2.1), we obtain the conclusion (1.1).

At the end of the paper [1] it is mentioned the inequality

$$\underline{\lim}\frac{a_{n+1}}{a_n} \leqslant \underline{\lim}\sqrt[n]{a_n} \leqslant \overline{\lim}\sqrt[n]{a_n} \leqslant \overline{\lim}\frac{a_{n+1}}{a_n} \tag{2.5}$$

which is taken from [2, p. 277]. This easily follows from the Lemma 2.1 by taking here

$$x_n = \ln a_{n+1} - \ln a_n, \quad y_n = 1,$$

with $a_1 = 0$. If replace in (2.1), we obtain

$$\underline{\lim}\ln\frac{a_{n+1}}{a_n} \leqslant \underline{\lim}\frac{\ln a_n}{n} \leqslant \overline{\lim}\frac{\ln a_n}{n} \leqslant \overline{\lim}\ln\frac{a_{n+1}}{a_n}$$

or

$$\underline{\lim}\left(\ln\frac{a_{n+1}}{a_n}\right) \leqslant \underline{\lim}\left(\ln\sqrt[n]{a_n}\right) \leqslant \overline{\lim}\left(\ln\sqrt[n]{a_n}\right) \leqslant \overline{\lim}\left(\ln\frac{a_{n+1}}{a_n}\right)$$

and finally, (2.5) can be deduced by exponentiating the above inequality.

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