FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 25 (2010), 1–5

CHARACTERIZATIONS OF RIGHT RING CONGRUENCES ON A SEMIRING *

Xiaofeng Zhu, Yuanlan Zhou, and Wenting Xiong

Abstract. This paper develops the relationship between ring congruences and normal subsemirings. As an application of this relationship, if right ring congruences on a semiring S exist, the structure of S is determined and the right ring congruences are characterized in terms of ring congruences and right zero band semiring congruences on S.

1. Introduction and Preliminaries

A *semiring* S is an algebraic structure $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations $+$ and $\cdot$ such that $(S, +)$ and $(S, \cdot)$ are semigroups, connected by ring-like distributivity (that is, $x(y+z) = xy+xz$, $(y+z)x = yx+zx$, for all x, y and z in S). Usually, we write $(S, +, \cdot)$ simply as S, and for any $x, y \in S$, write $x \cdot y$ simply as xy. An element a of a semiring S is called a *zero* if $a+x = x+a = x$ and $ax = xa = a$ for all $x \in S$. Clearly, the zero of a semiring S is unique if there exists. Usually, we denote it by 0. A non-empty subset K of a semiring S is a *subsemiring* if $a + b, ab \in K$ for every $a, b \in K$. A subsemiring K of S is an *ideal* if $as, sa \in K$ for any $a \in K$ and $s \in S$. A subsemiring K of S is called *reflexive* (*left-unitary*, *dense*) if its additive reduct $(S, +)$ is reflexive (unitary, dense). We denote the set of all additive idempotents of a semiring S by $E^+(S)$. Easily, we can prove that $E^+(S)$ is an ideal of $(S, \cdot)$. If C is a class of semirings and ϱ is a congruence on a semiring S then ϱ is called a $\mathscr{C}$-congruence if $S/\varrho \in \mathscr{C}$.

A semiring $S = (S, +, \cdot)$ is called an *idempotent semiring*, if $(\forall a \in S)\ a+a = a = a \cdot a$. An idempotent semiring S is called a *band semiring* ([6]), if S satisfies the conditions that for any $a, b \in S, a + ab + a = a, a + ba + a = a$. A band semiring S = (S,+, .) is called a T-band semiring ([6]), if the additive reduct (S, +) of S is a T-band, where T is, respectively, a *rectangular*, *right* (*left*) *zero*, *right* (*left*) *regular*, *regular*, *normal*, *commutative* band and so on. A semiring S is said to be a *right ring* ([2]), if S can be decomposed as a direct product of a ring R and a right zero band semiring E.

Received May 17, 2009.
2010 *Mathematics Subject Classification.* Primary 16Y60; Secondary 20M10
*The research is supported by the NSF of Jiangxi Province (0611051); the SF of Education Department of Jiangxi Province (GJJ09459) and the SF of Jiangxi Normal University (1905).

In Petrich and Reilly [5], the structure of a right group is characterized, and some properties are extended to a right ring in [2]. In [4], [7] and [9], some special semirings are studied. Based on those, the main purpose of this paper is to investigate the right ring congruences on a semiring S. A central notion considered is that of how the normal subsemiring associated with a group congruence on a semiring is related to right ring congruences on a semiring.

For other notations and terminology about semigroups and semirings not mentioned in this paper, the reader is referred to [1] and [3]. In this paper the phrases "right ring congruence", "ring congruence", and "right zero band ring congruence" will be denoted by *RRC*, *RC*, and *RZBC* respectively. When such a congruence is minimum, it will be denoted by *MRRC*, *MRC*, and *MRZBC* respectively.

2. Characterizations and structure

If σ is a *RC* on a semiring S, we define the set

$$N = \{a \in S \mid a\sigma \text{ is an additive zero of } S/\sigma\}.$$

Then N is a subsemiring of S and, moreover, N can be used in the following manner to generate σ. Define a relation σ_N on S by

$$\sigma_N = \{(a, b) \in S \times S \mid a + x, b + x \in N \text{ for some } x \in S\}. \tag{2.1}$$

It can be easily verified that $\sigma = \sigma_N$. The subset N which generates ring images of S, relative to the relation defined in equation (2.1), satisfies:

(2.2) N is an ideal of S;

(2.3) N is reflexive, i.e., if $a, b \in S$ and $a + b \in N$ then $b + a \in N$;

(2.4) N is left-unitary, i.e., $a, b \in S$ and $a, a + b \in N$ then $b \in N$;

(2.5) $a + b + c + d \in N \Leftrightarrow a + c + b + d \in N$;

(2.6) N is dense, i.e.,$(\forall s \in S)(\exists x, y \in S)\ s + x \in N \wedge y + s \in N$.

A subsemiring of S satisfying (2.2)−(2.6) will be called a *normal subsemiring* of S. The following lemma relates ring congruences on S and related normal subsemirings.

Lemma 2.1. *The correspondence*

$$\varphi : \sigma \to N = \{a \in S \mid a\sigma \text{ is the additive zero of } S/\sigma\}$$

is a one-to-one correspondence from the set of the ring congruences onto the set of normal subsemirings of S.

Proof. First φ is evidently well defined. Now we prove φ is injective and surjective.

For any two ring congruences σ_1, σ_2 with $\varphi(\sigma_1) = \varphi(\sigma_2)$, if $(a, b) \in \sigma_1$, then there exists an element $c \in S$ such that $(a + c)\sigma_1 = (b + c)\sigma_1 = 0_{S/\sigma_1}$, which implies that

$a+c, b+c \in \varphi(\sigma_1) = \varphi(\sigma_2)$, then $(a+c)\sigma_2 = (b+c)\sigma_2 = 0_{S/\sigma_2}$, and therefore, $(a,b) \in \sigma_2$. This proves that $\sigma_1 \subseteq \sigma_2$. Similarly, $\sigma_2 \subseteq \sigma_1$. So φ is one-to-one.

For any normal subsemiring N of S, we can define a relation σ_N by means of (2.1). Obviously, σ_N is an equivalence relation. Let $(a,b) \in \sigma_N$ and $c \in S$. Then there exists $x \in S$ such that $a+x, b+x \in N$, for the element $a+c+x$ in S there exists $y \in S$ such that $a+c+x+y \in N$ since N is dense. Then $a+x+c+y \in N$ by (2.5), which implies $c+y \in N$. Therefore $b+c+x+y \in N$ means that $(a+c, b+c) \in \sigma_N$, and similarly, $(c+a, c+b) \in \sigma_N$. Further, $ac, bc, ac+xc = (a+x)c \in N, bc+xc = (b+x)c \in N$, since N is a ideal of S, so $(ac, bc) \in \sigma_N$, and similarly, $(ca, cb) \in \sigma_N$. Thus, we have proved σ_N is a congruence on the semiring S.

For each $a \in N$, we can easily verify that $a\sigma_N$ is the zero of S/σ_N, for each $b \in S$ there exists an element $x \in S$ such that $b+x \in N$ then $b\sigma_N + x\sigma_N = (b+x)\sigma_N$ which is the additive zero of S/σ_N. Then $(S,+)/\sigma_N$ is a group. For each $a, b \in S$ there exist $x, y \in S$ such that $a+x, b+y \in N$ and $a+x+b+y \in N$, then $a+b+x+y \in N$ by (2.5), and $y+a+b+x \in N$ since N is reflexive, then by the same reason, $b+a+x+y \in N$. Now we have found an element $x+y$ such that $a+b+x+y, b+a+x+y \in N$, which means that $(a+b, b+a) \in \sigma_N$, so σ_N is a ring congruence. Obviously, σ_N satisfies $\varphi(\sigma_N) = N$. So φ is onto. □

Now we generalize the preceding work to right ring congruences, the following theorem develops the sought for characterizations of a *RRC* on a semiring S. A *RRC* will now be classified as being trivial if it is a ring congruence or if it is a right zero band semiring congruence.

Lemma 2.2. *If there exists a nontrivial right ring congruence ϱ on a semiring S then there exist a nontrivial ring congruence σ on S and a nontrivial right zero band semiring congruence π on S.*

Proof. Assume ϱ is a nontrivial *RRC* on S and that $S/\varrho = R \times E$ where R is a ring and E is a right zero band semiring. Note that $E \cong E(S/\varrho)$. For each e in E, define L_e to be the preimage of $R \times \{e\}$, and define K to be the kernel of ϱ, i.e., the union of all the ϱ-classes meeting E. Evidently $S = \bigcup\{L_e : e \in E\}$, with the union disjoint. If $s \in L_e, t \in L_f$ and $e, f \in E$, then

$$(s+t)\varrho^\natural = (s\varrho^\natural) + (t\varrho^\natural) \in (R \times \{e\}) + (R \times \{f\}).$$

But $(R \times \{e\}) + (R \times \{f\}) \subseteq R \times \{f\}$ and therefore $s+t \in L_f$, i.e., L_f is a left ideal of $(S,+)$. Now we show that K satisfies (2.2). Obviously, K is the preimage of $\{0_R\} \times E$, for each $a, b \in K$, there exist $e, f \in E$ such that $(a+b)\varrho = (a+b)\varrho^\natural = a\varrho^\natural + b\varrho^\natural = (0_R, e) + (0_R, f) = (0_R, f) \in \{0_R\} \times E$ which means that $a+b \in K$. If $k \in K, s \in S$ then exists $g \in R, c, d \in E$ such that $(ks)\varrho = k\varrho^\natural \cdot s\varrho^\natural = (0_R, c) \cdot (g, d) = (0_R, cd) \in \{0_R\} \times E$ then $ks \in K$, similarly, $sk \in K$. So K is a ideal of S. We can also easily verify that K satisfies (2.2)–(2.6). So K is a dense normal subsemiring of S. Let $\sigma = \sigma_K$ as defined by (2.1). Now we define a relation π on S by

$$\pi = \{(a,b) \in S \times S \mid a, b \in L_e \text{ for some } e \in E\}$$

then we can easily verify that π is a right zero band semiring congruence. □

Thus, a *RRC* ϱ induces a *RC* σ_K, where K is the kernel of ϱ, and a *RZBC* π, where the π-classes are left ideal of $(S,+)$. The exact relationship between ϱ, σ_K and π is exhibited in the following theorem.

Theorem 2.1. *Let ϱ be a right ring congruence on a semiring S. Then S/ϱ admits a direct product decomposition $S/\varrho = R \times E$ where R is a ring and E is a right zero band semiring. Let $K = \{a \in S \mid a\rho \in E(S/\varrho)\}$, and for each $e \in E$, let L_e the preimage of $R \times \{e\}$, and denote by π the right zero band semiring congruence induced on S by the $\{L_e : e \in E\}$. Then:*

(1) $\varrho = \sigma_K \cap \pi$.

(2) $S/\varrho \cong S/\sigma \times S/\pi$.

Proof. (1) If $a\,\varrho\,b$, then $a\varrho^\natural = (r,e) = b\varrho^\natural$ for some (r,e) in $R \times E$. There exists an element x in S such that $x\varrho\natural = (-r,e)$, and therefore $(x+a)\varrho^\natural = (-r,e)+(r,e) = (0_R,e)$, and similarly $(x+b)\varrho^\natural = (0_R,e)$, then $x+a, x+b \in K$ and therefore $a\,\sigma_K\,b$. Since $a\varrho^\natural, b\varrho^\natural \in R \times \{e\}$, $a\pi b$ and thus $\varrho \subseteq \sigma_K \cap \pi$.

Conversely, $(a,b) \in \sigma_K \cap \pi$ implies that there exists e in E such that $a\varrho^\natural, b\varrho^\natural \in R\times\{e\}$, and that there exists x in S such that $x+a, x+b \in K$. Thus $(x+a)\varrho^\natural = (0_R,e) = (x+b)\varrho^\natural$. If $x \in L_f$, then there exist $x' \in S$ such that $(x'+x)\varrho^\natural = (0_R,f)$,then

$$a\varrho^\natural = (0_R,f) + a\varrho^\natural = (x'+x+a)\varrho^\natural = (x'+x+b)\varrho^\natural = (0_R,f) + b\varrho^\natural = b\varrho^\natural,$$

i.e., $a\varrho b$ and $\sigma_K \cap \pi \subseteq \rho$.

(2) From part (1), $\varrho = \sigma_K \cap \pi$, we can define a map θ from S/ϱ to $S/\sigma_K \times S/\pi$ by $(s\varrho)\theta = (s\sigma_K, s\pi)$. Clearly, θ is well defined. For $s,t \in S$

$$(s\varrho + t\varrho)\theta = ((s+t)\varrho)\theta = ((s+t)\sigma_K, (s+t)\pi) = (s\sigma_K, s\pi) + (t\sigma_K, t\pi) = (s\varrho)\theta + (t\varrho)\theta$$

$$(s\varrho \cdot t\varrho)\theta = (st\varrho)\theta = (s\sigma_K \cdot t\sigma_K, s\pi \cdot t\pi) = (s\sigma_K, s\pi)\cdot(t\sigma_K, t\pi) = (s\varrho)\theta \cdot (t\varrho)\theta$$

and hence θ is a homomorphism. For $(s\sigma_K, t\pi) \in S/\sigma \times S/\pi$, let $t \in L_e$ and $k \in K \cap L_e$. Thus $((s+k)\varrho)\theta = ((s+t)\sigma_K, (s+k)\pi) = (s\sigma_K + k\sigma_K, k\pi) = (s\sigma_K, t\pi)$, which means that θ is onto. If $(s\varrho)\theta = (t\varrho)\theta$ then $s\sigma_K t$ and $s\pi t$, and therefore $s\,\varrho\,t$, θ is one-to-one on S/ϱ. □

A sufficient condition for the existence of a minimum right ring congruence is given in the next theorem.

Theorem 2.2. *Let S be a semiring having nontrivial right ring congruences. If S has a minimum normal subsemiring K, then S has a (unique) minimum right ring congruence.*

Proof. We denote the set of left ideal partitions of $(S,+)$ by $\{P_i \mid i \in I\}$, each P_i induces a *RZBC*, say π_i. Define a congruence π on S by $\pi = \bigcap\{\pi_i : i \in I\}$. If $a,b \in S$, then $(a+b)\pi_i = b\pi_i$ and $(a+a)\pi_i = (a\cdot a)\pi_i = a\pi_i$, for each $i \in I$, implies that $(a+b)\pi = b\pi$ and $(a+a)\pi = (a\cdot a)\pi = a\pi$, which means that π is the *MRZBC* on S. The *MRC* on S is evidently σ_K. The *MRRC* on S is therefore $\sigma_K \cap \pi$. □

The following corollary is a direct consequence of Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let S be a semiring. Then the following are equivalent:*

(1) *ϱ is the mimimum right ring congruence on S.*

(2) *$\varrho = \sigma \cap \pi$, where σ is the minimum ring congruence on S and π is the minimum right zero band semiring congruence on S.*

REFERENCES

1. J. S. Golan: *The Theory of Semirings and Their Applications*, Kluwer Academic Publishers, 1999.
2. Y. Q. Guo, K. P. Shum, and M. K. Sen: *The semigroup structure of left Clifford semirings*, Acta Mathematica Sinica, English Series 19 (4) (2003), 783–792.
3. J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford: Clarendon Press, 1995.
4. F. Pastijn and Y. Q. Guo, *Semirings which are unions of rings*, Science in China (series A) 45 (2) (2002), 172–195.
5. M. Petrich and N. R. Reilly, *Completely regular semigruops*, A Wiley-Interscience Publication, 1999.
6. M. K. Sen, Y. Q. Guo, and K. P. Shum, *A class of idempotent semirings*, Semigroup Forum 60 (2000), 351-367.
7. J. Zeleznikow, *Regular semirings*, Semigroup Forum 23 (1981), 119–136.
8. Y. L. Zhou, The PhD thesis, Zhongshan University, 2006.
9. Y. L. Zhou, Y. Q. Guo, and Z. P., Wang *The semigroup structure of orthorings*, Acta Mathematica Scientia, 2008.

Xiaofeng Zhu, Yuanlan Zhou, Wenting Xiong
Department of Mathematics, Jiangxi Normal University,
Nanchang, Jiangxi, 330022, P. R. China
Correspondence: ylzhou185@163.com