

FEJÉR INEQUALITY FOR DOUBLE INTEGRALS *

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Abstract. Fejér inequality for double integrals is established. Two mappings in connection with Fejér inequality for double integrals and some inequalities for Lipschitzian mappings in two variables are also established.

1. Introduction

Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following two inequalities hold

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq \int_a^b f(x) p(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b p(x) dx,$$

where $p : [a, b] \rightarrow \mathbf{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$. This inequality is known as the *Fejér inequality* for convex mappings (see [1,2,5,6]).

Let us consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in \mathbf{R}^2 with $a < b$ and $c < d$. Recall that the mapping $f : \Delta \rightarrow \mathbf{R}$ is convex in Δ if

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w),$$

for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

Dragomir [3] introduced a new concept of convexity, which is called the co-ordinated convex function, as follows: A function $f : \Delta \rightarrow \mathbf{R}$ which is convex in Δ is called *co-ordinated convex* on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbf{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbf{R}$, $f_x(v) = f(x, v)$, are convex for all $y \in [c, d]$ and $x \in [a, b]$.

An inequality of Hadamard's type for co-ordinated convex mappings on a rectangle from the plane \mathbf{R}^2 was established by Dragomir in [3], as follows:

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Theorem 1.1. Suppose that $f : \Delta \rightarrow \mathbf{R}$ is co-ordinated convex on Δ . Then

$$(1.2) \quad \begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \end{aligned}$$

The above inequalities are sharp.

If $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ is a co-ordinated convex function, then the following mapping on $[0, 1]^2$ can be defined

$$H(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dy dx.$$

The mapping H has the following properties (see [4]):

- (i) H is co-ordinated convex and monotonic non-decreasing on $[0, 1]^2$.
- (ii) we have the following bounds for H :

$$\begin{aligned} \sup_{t \in [0, 1]^2} H(t, s) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy = H(1, 1), \\ \inf_{t \in [0, 1]^2} H(t, s) &= f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = H(0, 0). \end{aligned}$$

Recently, Hwang, Tseng, and Yang [4] established a monotonic non-decreasing mapping connected with the Hadamard's inequality for co-ordinated convex functions in a rectangle from the plane, as follows:

Theorem 1.2. Suppose that $f : \Delta \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$ is co-ordinated convex on $\Delta := [a, b] \times [c, d]$ and the mapping $F : [0, 1]^2 \rightarrow \mathbf{R}$ is defined by

$$\begin{aligned} F(t, s) &= \frac{1}{4(b-a)(d-c)} \times \int_a^b \int_c^d \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)c + \left(\frac{1-s}{2}\right)y\right) \right. \\ &\quad + f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)d + \left(\frac{1-s}{2}\right)y\right) \\ &\quad + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)c + \left(\frac{1-s}{2}\right)y\right) \\ &\quad \left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)d + \left(\frac{1-s}{2}\right)y\right) \right] dy dx. \end{aligned}$$

Then

- (i) *The mapping F is a co-ordinated convex on $[0, 1]^2$.*
- (ii) *The mapping F is a co-ordinated monotonic nondecreasing on $[0, 1]^2$.*
- (iii) *We have the bounds*

$$\inf_{t \in [0,1]^2} F(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx = F(0, 0)$$

$$\sup_{t \in [0,1]^2} F(t, s) = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} = F(1, 1).$$

Also, some Hadamard's type inequalities for Lipschitzian mappings in two variables were given in [4] in the following way:

Theorem 1.3. *Let $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ satisfy Lipschitzian conditions, that is, let for (t_1, s_1) and $(t_2, s_2) \in [a, b] \times [c, d]$ we have*

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_1 |t_1 - t_2| + L_2 |s_1 - s_2|$$

where L_1 and L_2 are positive constants. Then

$$|F(t_1, s_1) - F(t_2, s_2)| \leq \frac{L_1 |t_1 - t_2| (b-a) + L_2 |s_1 - s_2| (d-c)}{4}$$

and

$$|H(t_1, s_1) - H(t_2, s_2)| \leq \frac{L_1 |t_1 - t_2| (b-a) + L_2 |s_1 - s_2| (d-c)}{4}.$$

In this paper Fejér type inequalities for co-ordinated convex functions on a rectangle from the plane are established. Two mappings associated with these inequalities and some of their properties, and refinements are given.

2. Fejér Inequality for double integrals with two related mappings

In order to prove our main theorems, we need the following lemma:

Lemma 2.1. *Let $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ be a co-ordinated convex function and let*

$$a \leq y_1 \leq x_1 \leq x_2 \leq y_2 \leq b \text{ with } x_1 + x_2 = y_1 + y_2,$$

$$c \leq w_1 \leq v_1 \leq v_2 \leq w_2 \leq d \text{ with } v_1 + v_2 = w_1 + w_2.$$

Then, for the convex partial mappings $f_y : [a, b] \rightarrow \mathbf{R}$, $f_y(t) = f(t, y)$ and $f_x : [c, d] \rightarrow \mathbf{R}$, $f_x(s) = f(x, s)$, for all $y \in [c, d]$ and $x \in [a, b]$, respectively, the following holds:

$$(2.1) \quad f(x_1, s) + f(x_2, s) \leq f(y_1, s) + f(y_2, s), \quad \forall s \in [c, d],$$

and

$$(2.2) \quad f(t, v_1) + f(t, v_2) \leq f(t, w_1) + f(t, w_2), \quad \forall t \in [a, b],$$

Proof. Consider $f_y : [a, b] \rightarrow \mathbf{R}$, $f_y(t) = f(t, y)$. If $y_1 = y_2$ then we are done. Suppose $y_1 \neq y_2$ and write

$$x_1 = \frac{y_2 - x_1}{y_2 - y_1} y_1 + \frac{x_1 - y_1}{y_2 - y_1} y_2, \quad x_2 = \frac{y_2 - x_2}{y_2 - y_1} y_1 + \frac{x_2 - y_1}{y_2 - y_1} y_2,$$

since f is convex, we have

$$\begin{aligned} f(x_1, s) + f(x_2, s) &\leq \frac{y_2 - x_1}{y_2 - y_1} f(y_1, s) + \frac{x_1 - y_1}{y_2 - y_1} f(y_2, s) \\ &+ \frac{y_2 - x_2}{y_2 - y_1} f(y_1, s) + \frac{x_2 - y_1}{y_2 - y_1} f(y_2, s) \\ &= \frac{2y_2 - (x_1 + x_2)}{y_2 - y_1} f(y_1, s) + \frac{(x_1 + x_2) - 2y_1}{y_2 - y_1} f(y_2, s) \\ &= f(y_1, s) + f(y_2, s), \end{aligned}$$

what means that (2.1) holds. In a similar way we can show that (2.2) holds, which completes the proof. \square

Fejér-type inequality for co-ordinated convex mappings is established as follows:

Theorem 2.1. *Let $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ be a co-ordinated convex function. Then the following double inequality holds*

$$(2.3) \quad \begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{\int_a^b \int_c^d f(x, y) p(x, y) dy dx}{\int_a^b \int_c^d p(x, y) dy dx} \\ &\leq \frac{f(a, c) + f(c, d) + f(b, c) + f(b, d)}{4}, \end{aligned}$$

where $p : [a, b] \times [c, d] \rightarrow \mathbf{R}$ is positive, integrable, and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. The above inequalities are sharp.

Proof. Since p is positive, integrable, and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, by the inequalities (2.1) and (2.2) we have

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x, y) dy dx &= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) p(x, y) dy dx \\ &+ \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) p(a+b-x, c+d-y) dy dx \\ &= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] p(x, y) dy dx \end{aligned}$$

$$\begin{aligned}
& \leq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x, y) + f(a+b-x, c+d-y)] p(x, y) dy dx \\
& = \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x, y) p(x, y) dy dx + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(x, y) p(x, y) dy dx \\
& = \int_a^b \int_c^d f(x, y) p(x, y) dy dx
\end{aligned}$$

and

$$\begin{aligned}
& \frac{f(a, c) + f(c, d) + f(b, c) + f(b, d)}{4} \int_a^b \int_c^d p(x, y) dy dx \\
& = \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[\frac{f(a, c) + f(c, d) + f(b, c) + f(b, d)}{4} \right] p(x, y) dy dx \\
& + \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[\frac{f(a, c) + f(c, d) + f(b, c) + f(b, d)}{4} \right] p(a+b-x, c+d-y) dy dx \\
& = \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[\frac{f(a, c) + f(c, d) + f(b, c) + f(b, d)}{2} \right] p(x, y) dy dx \\
& \geq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x, y) + f(a+b-x, c+d-y)] p(x, y) dy dx \\
& = \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x, y) p(x, y) dy dx + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(x, y) p(x, y) dy dx \\
& = \int_a^b \int_c^d f(x, y) p(x, y) dy dx
\end{aligned}$$

which completes the proof.

In (2.3), take $p(x, y) = 1$. Then the inequality (2.3) is reduced to the double inequality (1.2), and therefore, if we choose $f(x, y) = xy$ in (2.3), then we have the

equality, which shows that (2.3) is sharp. \square

The next theorem introduces a mapping associated with the first inequality in (2.3) and describes some of its properties.

Theorem 2.2. *Let $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ be a co-ordinated convex function, and consider the function $p : [a, b] \times [c, d] \rightarrow \mathbf{R}$ which is positive, integrable, and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. Let S be a function defined on $[0, 1]^2$ by*

$$S(t, s) = \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) p(x, y) dy dx$$

Then S is a co-ordinated convex function on $[0, 1]^2$, non-decreasing on $[0, 1]^2$. Moreover,

$$\inf_{(t,s) \in [0,1]^2} S(t, s) = S(0, 0) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x, y) dy dx$$

and

$$\sup_{(t,s) \in [0,1]^2} S(t, s) = S(1, 1) = \int_a^b \int_c^d f(x, y) p(x, y) dy dx$$

Proof. Fix $s \in [0, 1]$. Then for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} S(\alpha t_1 + \beta t_2, s) &= \int_a^b \int_c^d f\left((\alpha t_1 + \beta t_2)x + [1 - (\alpha t_1 + \beta t_2)]\frac{a+b}{2}, \right. \\ &\quad \left. sy + (1-s)\frac{c+d}{2}\right) p(x, y) dy dx \\ &= \int_a^b \int_c^d f\left(\alpha\left(t_1x + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2x + (1-t_2)\frac{a+b}{2}\right), \right. \\ &\quad \left. sy + (1-s)\frac{c+d}{2}\right) p(x, y) dy dx \\ &\leq \alpha \int_a^b \int_c^d f\left(t_1x + (1-t_1)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) p(x, y) dy dx \\ &\quad + \beta \int_a^b \int_c^d f\left(t_2x + (1-t_2)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) p(x, y) dy dx \\ &= \alpha S(t_1, s) + \beta S(t_2, s). \end{aligned}$$

If $t \in [0, 1]$ is fixed, then for all $s_1, s_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we also have:

$$S(t, \alpha s_1 + \beta s_2) \leq \alpha S(t, s_1) + \beta S(t, s_2),$$

which shows that S is co-ordinated convex on $[0, 1]^2$.

To show that S is non-decreasing on $[0, 1]^2$, fix $s \in [0, 1]$. If $0 \leq t_1 \leq t_2 \leq 1$ with $x \in [a, \frac{a+b}{2}]$, then

$$\begin{aligned} t_2 x + (1 - t_2) \frac{a+b}{2} &\leq t_1 x + (1 - t_1) \frac{a+b}{2} \leq t_1 (a + b - x) + (1 - t_1) \frac{a+b}{2} \\ &\leq t_2 (a + b - x) + (1 - t_2) \frac{a+b}{2}, \end{aligned}$$

where,

$$\begin{aligned} \left[t_1 x + (1 - t_1) \frac{a+b}{2} \right] + \left[t_1 (a + b - x) + (1 - t_1) \frac{a+b}{2} \right] &= \\ \left[t_2 x + (1 - t_2) \frac{a+b}{2} \right] + \left[t_2 (a + b - x) + (1 - t_2) \frac{a+b}{2} \right], \end{aligned}$$

By inequality (2.1) and the assumptions that p is positive, integrable, and symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned} S(t_1, s) &= \int_a^b \int_c^d f\left(t_1 x + (1 - t_1) \frac{a+b}{2}, sy + (1 - s) \frac{c+d}{2}\right) p(x, y) dy dx \\ &= \int_a^{\frac{a+b}{2}} \int_c^d f\left(t_1 x + (1 - t_1) \frac{a+b}{2}, sy + (1 - s) \frac{c+d}{2}\right) p(x, y) dy dx \\ &\quad + \int_a^{\frac{a+b}{2}} \int_c^d \left[f\left(t_1 (a + b - x) + (1 - t_1) \frac{a+b}{2}, sy + (1 - s) \frac{c+d}{2}\right) \right. \\ &\quad \left. p(a + b - x, y) \right] dy dx \\ &= \int_a^{\frac{a+b}{2}} \int_c^d \left[f\left(t_1 x + (1 - t_1) \frac{a+b}{2}, sy + (1 - s) \frac{c+d}{2}\right) \right. \\ &\quad \left. + f\left(t_1 (a + b - x) + (1 - t_1) \frac{a+b}{2}, sy + (1 - s) \frac{c+d}{2}\right) \right] p(x, y) dy dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_a^{\frac{a+b}{2}} \int_c^d \left[f\left(t_2x + (1-t_2)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \right. \\
&\quad \left. + f\left(t_2(a+b-x) + (1-t_2)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \right] p(x, y) dy dx \\
&= \int_a^{\frac{a+b}{2}} \int_c^d f\left(t_2x + (1-t_2)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) p(x, y) dy dx \\
&\quad + \int_a^{\frac{a+b}{2}} \int_c^d \left[f\left(t_2(a+b-x) + (1-t_2)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \right. \\
&\quad \left. p(a+b-x, y) \right] dy dx \\
&= \int_a^b \int_c^d f\left(t_2x + (1-t_2)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) p(x, y) dy dx \\
&= S(t_2, s).
\end{aligned}$$

This shows that $S(t, \circ)$ is co-ordinated non-decreasing for all $t \in [0, 1]$. If $t \in [0, 1]$ is fixed, then for all $s \in [0, 1]$ we also have that $S(\circ, s)$ is co-ordinated non-decreasing for all $s \in [0, 1]$. Thus S is co-ordinated monotonic non-decreasing on $[0, 1]^2$. Therefore,

$$\begin{aligned}
S(t, s) &\geq S(0, s) \geq S(0, 0) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x, y) dy dx, \\
S(t, s) &\leq S(t, 1) \leq S(1, 1) = \int_a^b \int_c^d f(x, y) p(x, y) dy dx,
\end{aligned}$$

which completes the proof. \square

Now, a mapping associated with the second inequality in (2.3) and some of its properties are considered.

Theorem 2.3. *Let $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ be a co-ordinated convex function, and consider the function $p : [a, b] \times [c, d] \rightarrow \mathbf{R}$ which is positive, integrable, and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. Let A be a function defined on $[0, 1]^2$ by*

$$\begin{aligned}
A(t, s) &= \frac{1}{4} \int_a^b \int_c^d \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \\
&\quad \left. + f\left(\frac{1+t}{2}a + \frac{1-t}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \right]
\end{aligned}$$

$$\begin{aligned}
& +f\left(\frac{1+t}{2}b+\frac{1-t}{2}x, \frac{1+s}{2}c+\frac{1-s}{2}y\right)p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
& +f\left(\frac{1+t}{2}b+\frac{1-t}{2}x, \frac{1+s}{2}d+\frac{1-s}{2}y\right)p\left(\frac{x+b}{2}, \frac{y+d}{2}\right)\Big] dydx
\end{aligned}$$

Then, A is co-ordinated convex function on $[0, 1]^2$, symmetric about $(\frac{1}{2}, \frac{1}{2})$, non-decreasing on $[0, 1]^2$, and

$$\begin{aligned}
\inf_{(t,s) \in [0,1]^2} A(t, s) = A(0, 0) &= \int_a^b \int_c^d f(x, y) p(x, y) dy dx \\
\sup_{(t,s) \in [0,1]^2} A(t, s) = A(1, 1) &= \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \int_a^b \int_c^d p(x, y) dy dx.
\end{aligned}$$

Proof. Fix $s \in [0, 1]$. Then for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned}
A(\alpha t_1 + \beta t_2, s) &= \\
&= \frac{1}{4} \int_a^b \int_c^d \left[f\left(\frac{1+\alpha t_1 + \beta t_2}{2}a + \frac{1-(\alpha t_1 + \beta t_2)}{2}, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \\
&\quad + f\left(\frac{1+\alpha t_1 + \beta t_2}{2}a + \frac{1-(\alpha t_1 + \beta t_2)}{2}, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\
&\quad + f\left(\frac{1+\alpha t_1 + \beta t_2}{2}b + \frac{1-(\alpha t_1 + \beta t_2)}{2}, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
&\quad \left. + f\left(\frac{1+\alpha t_1 + \beta t_2}{2}b + \frac{1-(\alpha t_1 + \beta t_2)}{2}, \frac{1+s}{2}d + \frac{1-s}{2}y\right)\right] p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) dy dx \\
&= \frac{1}{4} \int_a^b \int_c^d \left[f\left(\alpha \frac{(1+t_1)a + (1-t_1)x}{2} + \beta \frac{(1+t_2)a + (1-t_2)x}{2}, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \\
&\quad + f\left(\alpha \frac{(1+t_1)a + (1-t_1)x}{2} + \beta \frac{(1+t_2)a + (1-t_2)x}{2}, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\
&\quad + f\left(\alpha \frac{(1+t_1)b + (1-t_1)x}{2} + \beta \frac{(1+t_2)b + (1-t_2)x}{2}, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
&\quad \left. + f\left(\alpha \frac{(1+t_1)b + (1-t_1)x}{2} + \beta \frac{(1+t_2)b + (1-t_2)x}{2}, \frac{1+s}{2}d + \frac{1-s}{2}y\right)\right] p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) dy dx \\
&\leq \frac{1}{4} \int_c^d \left\{ \alpha \int_a^b \left[f\left(\frac{(1+t_1)}{2}a + \frac{(1-t_1)}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \right. \\
&\quad + f\left(\frac{(1+t_1)}{2}a + \frac{(1-t_1)}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\
&\quad + f\left(\frac{(1+t_1)}{2}b + \frac{(1-t_1)}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
&\quad \left. \left. + f\left(\frac{(1+t_1)}{2}b + \frac{(1-t_1)}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+d}{2}\right)\right] dx \right\}
\end{aligned}$$

$$\begin{aligned}
& + \beta \int_a^b \left[f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \\
& + f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\
& + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
& \left. + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) \right] dy \\
& = \alpha A(t_1, s) + \beta A(t_2, s)
\end{aligned}$$

If $t \in [0, 1]$ is fixed, then for all $s_1, s_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we also have:

$$A(t, \alpha s_1 + \beta s_2) \leq \alpha A(t, s_1) + \beta A(t, s_2),$$

which shows that A is co-ordinated convex on $[0, 1]^2$. To show that A is non-decreasing on $[0, 1]^2$, fix $s \in [0, 1]$. Let $0 \leq t_1 \leq t_2 \leq 1$. Since

$$\begin{aligned}
& \int_a^b \int_c^d \left[f\left(\left(\frac{1+t_1}{2}\right)b + \left(\frac{1-t_1}{2}\right)x, \left(\frac{1+s}{2}\right)c + \left(\frac{1-s}{2}\right)y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \right] dy dx \\
& = \int_a^b \int_c^d \left[f\left(\left(\frac{1+t_1}{2}\right)b + \left(\frac{1-t_1}{2}\right)(b+a-x), \left(\frac{1+s}{2}\right)c + \left(\frac{1-s}{2}\right)y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \right] dy dx
\end{aligned}$$

we have

$$\begin{aligned}
A(t_1, s) &= \frac{1}{4} \int_a^b \int_c^d \left[f\left(\left(\frac{1+t_1}{2}\right)a + \left(\frac{1-t_1}{2}\right)x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \\
&\quad + f\left(\left(\frac{1+t_1}{2}\right)a + \left(\frac{1-t_1}{2}\right)x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\
&\quad + f\left(\left(\frac{1+t_1}{2}\right)b + \left(\frac{1-t_1}{2}\right)(b+a-x), \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
&\quad \left. + f\left(\left(\frac{1+t_1}{2}\right)b + \left(\frac{1-t_1}{2}\right)(b+a-x), \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) \right] dy dx.
\end{aligned}$$

Also, since

$$\begin{aligned}
\left(\frac{1+t_2}{2}\right)a + \left(\frac{1-t_2}{2}\right)x &\leq \left(\frac{1+t_1}{2}\right)a + \left(\frac{1-t_1}{2}\right)x \leq \left(\frac{1+t_1}{2}\right)b + \left(\frac{1-t_1}{2}\right)(b+a-x) \\
&\leq \left(\frac{1+t_2}{2}\right)b + \left(\frac{1-t_2}{2}\right)(b+a-x); \\
\left[\left(\frac{1+t_1}{2}\right)a + \left(\frac{1-t_1}{2}\right)x\right] + \left[\left(\frac{1+t_1}{2}\right)b + \left(\frac{1-t_1}{2}\right)(b+a-x)\right] \\
&= \left[\left(\frac{1+t_2}{2}\right)a + \left(\frac{1-t_2}{2}\right)x\right] + \left[\left(\frac{1+t_2}{2}\right)b + \left(\frac{1-t_2}{2}\right)(b+a-x)\right]
\end{aligned}$$

and f is co-ordinated convex on $[a, b] \times [c, d]$, by Lemma 2.1, we have

$$\begin{aligned}
A(t_1, s) &\leq \frac{1}{4} \int_a^b \int_c^d \left[f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \\
&\quad + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}(b+a-x), \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
&\quad + f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\
&\quad \left. + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}(b+a-x), \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) \right] dy dx \\
&= \frac{1}{4} \int_a^b \int_c^d \left[f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right. \\
&\quad + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\
&\quad + f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\
&\quad \left. + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) \right] dy dx \\
&= A(t_1, s).
\end{aligned}$$

This shows that $A(t, \circ)$ is co-ordinated non-decreasing for all $t \in [0, 1]$. If $t \in [0, 1]$ is fixed, then for all $s \in [0, 1]$, we also have $A(\circ, s)$ is co-ordinated non-decreasing for all $s \in [0, 1]$. Thus A is co-ordinated monotonic non-decreasing on $[0, 1]^2$. Now,

$$\begin{aligned}
A(t, s) &\geq A(0, s) \geq A(0, 0) \\
&= \frac{1}{4} \int_a^b \int_c^d \left[f\left(\frac{a+x}{2}, \frac{c+y}{2}\right) p\left(\frac{a+x}{2}, \frac{c+y}{2}\right) + f\left(\frac{a+x}{2}, \frac{d+y}{2}\right) p\left(\frac{a+x}{2}, \frac{d+y}{2}\right) \right. \\
&\quad \left. + f\left(\frac{b+x}{2}, \frac{c+y}{2}\right) p\left(\frac{b+x}{2}, \frac{c+y}{2}\right) + f\left(\frac{b+x}{2}, \frac{d+y}{2}\right) p\left(\frac{b+x}{2}, \frac{d+y}{2}\right) \right] dy dx \\
&= \int_a^b \int_c^d f(x, y) p(x, y) dy dx
\end{aligned}$$

$$\begin{aligned}
A(t, s) &\leq A(t, 1) \leq A(1, 1) \\
&= \frac{1}{4} \int_a^b \int_c^d \left[f(a, c) p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) + f(a, d) p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \right. \\
&\quad \left. + f(b, c) p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) + f(b, d) p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) \right] dy dx \\
&= \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \int_a^b \int_c^d p(x, y) dy dx
\end{aligned}$$

and thus the proof is completed. \square

3. Lipschitzian mappings

In this section Fejér type inequalities for Lipschitzian mappings in two variables are established.

Theorem 3.1. *Let $f : \Delta^2 \rightarrow \mathbf{R}$ satisfy Lipschitzian conditions, that is, let for (t_1, s_1) and $(t_2, s_2) \in \Delta^2$, we have*

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_1 |t_1 - t_2| + L_2 |s_1 - s_2|$$

where L_1, L_2 are positive constants.

If there exists $m > 0$, such that $|p| \leq m$, for all $(x, y) \in [a, b] \times [c, d]$, then

$$(3.1) \quad |A(t_1, s_1) - A(t_2, s_2)| \leq \frac{mL_1}{4} (d - c) (b - a)^2 |t_1 - t_2| + \frac{mL_2}{4} (b - a) (d - c)^2 |s_1 - s_2|$$

for all $(t, s) \neq (0, 0)$, $\forall (t, s) \in \Delta^2$. Also,

$$(3.2) \quad |S(t_1, s_1) - S(t_2, s_2)| \leq \frac{m}{4} (L_1 |t_1 - t_2| (b - a) + L_2 |s_1 - s_2| (d - c))$$

Proof. For (t_1, s_1) and $(t_2, s_2) \in \Delta^2$, we have

$$\begin{aligned} & |A(t_1, s_1) - A(t_2, s_2)| \\ & \leq \frac{1}{4} \iint_{a \ c}^{b \ d} \left[\left| f\left(\frac{1+t_1}{2}a + \frac{1-t_1}{2}x, \frac{1+s_1}{2}c + \frac{1-s_1}{2}y\right) - f\left(\frac{1+t_2}{2}a + \frac{1-t_2}{2}x, \frac{1+s_2}{2}c + \frac{1-s_2}{2}y\right) \right| \right. \\ & \quad \times \left| p\left(\frac{a+x}{2}, \frac{c+y}{2}\right) \right| \\ & \quad + \left| f\left(\frac{1+t_1}{2}a + \frac{1-t_1}{2}x, \frac{1+s_1}{2}d + \frac{1-s_1}{2}y\right) - f\left(\frac{1+t_2}{2}a + \frac{1-t_2}{2}x, \frac{1+s_2}{2}d + \frac{1-s_2}{2}y\right) \right| \\ & \quad \times \left| p\left(\frac{a+x}{2}, \frac{d+y}{2}\right) \right| \\ & \quad + \left| f\left(\frac{1+t_1}{2}b + \frac{1-t_1}{2}x, \frac{1+s_1}{2}c + \frac{1-s_1}{2}y\right) - f\left(\frac{1+t_2}{2}b + \frac{1-t_2}{2}x, \frac{1+s_2}{2}c + \frac{1-s_2}{2}y\right) \right| \\ & \quad \times \left| p\left(\frac{b+x}{2}, \frac{c+y}{2}\right) \right| \\ & \quad + \left| f\left(\frac{1+t_1}{2}b + \frac{1-t_1}{2}x, \frac{1+s_1}{2}d + \frac{1-s_1}{2}y\right) - f\left(\frac{1+t_2}{2}b + \frac{1-t_2}{2}x, \frac{1+s_2}{2}d + \frac{1-s_2}{2}y\right) \right| \\ & \quad \times \left| p\left(\frac{b+x}{2}, \frac{d+y}{2}\right) \right| \right] dy dx \end{aligned}$$

$$\begin{aligned}
&\leq \frac{m}{4} \int_a^b \int_c^d \left[L_1 \left| \left(\frac{t_1 - t_2}{2} \right) a + \left(\frac{t_2 - t_1}{2} \right) x \right| + L_2 \left| \left(\frac{s_1 - s_2}{2} \right) c + \left(\frac{s_2 - s_1}{2} \right) y \right| \right. \\
&\quad \left. + L_1 \left| \left(\frac{t_1 - t_2}{2} \right) a + \left(\frac{t_2 - t_1}{2} \right) x \right| + L_2 \left| \left(\frac{s_1 - s_2}{2} \right) d + \left(\frac{s_2 - s_1}{2} \right) y \right| \right. \\
&\quad \left. + L_1 \left| \left(\frac{t_1 - t_2}{2} \right) b + \left(\frac{t_2 - t_1}{2} \right) x \right| + L_2 \left| \left(\frac{s_1 - s_2}{2} \right) c + \left(\frac{s_2 - s_1}{2} \right) y \right| \right. \\
&\quad \left. + L_1 \left| \left(\frac{t_1 - t_2}{2} \right) b + \left(\frac{t_2 - t_1}{2} \right) x \right| + L_2 \left| \left(\frac{s_1 - s_2}{2} \right) d + \left(\frac{s_2 - s_1}{2} \right) y \right| \right] \\
&= \frac{mL_1}{2} (d - c) \int_a^b \left[\left| \left(\frac{t_1 - t_2}{2} \right) a + \left(\frac{t_2 - t_1}{2} \right) x \right| + \left| \left(\frac{t_1 - t_2}{2} \right) b + \left(\frac{t_2 - t_1}{2} \right) x \right| \right] dx \\
&\quad + \frac{mL_2}{2} (b - a) \int_a^b \left[\left| \left(\frac{s_1 - s_2}{2} \right) c + \left(\frac{s_2 - s_1}{2} \right) y \right| + \left| \left(\frac{s_1 - s_2}{2} \right) d + \left(\frac{s_2 - s_1}{2} \right) y \right| \right] dy \\
&= \frac{mL_1}{4} (d - c) (b - a)^2 |t_1 - t_2| + \frac{mL_2}{4} (b - a) (d - c)^2 |s_1 - s_2|.
\end{aligned}$$

Also,

$$\begin{aligned}
&|S(t_1, s_1) - S(t_2, s_2)| \\
&\leq \int_a^b \int_c^d \left[\left| f \left(t_1 x + (1 - t_1) \frac{a+b}{2}, s_1 y + (1 - s_1) \frac{c+d}{2} \right) \right. \right. \\
&\quad \left. \left. - f \left(t_2 x + (1 - t_2) \frac{a+b}{2}, s_2 y + (1 - s_2) \frac{c+d}{2} \right) \right| |p(x, y)| \right] dx dy \\
&\leq m \int_a^b \int_c^d \left[L_1 \left| (t_1 - t_2) \left(x - \frac{a+b}{2} \right) \right| + L_2 \left| (s_1 - s_2) \left(y - \frac{c+d}{2} \right) \right| \right] dy dx \\
&= \frac{mL_1 |t_1 - t_2|}{(b - a)} \int_a^b \left| x - \frac{a+b}{2} \right| dx + \frac{mL_2 |s_1 - s_2|}{(d - c)} \int_c^d \left| y - \frac{c+d}{2} \right| dy \\
&= \frac{m}{4} (L_1 |t_1 - t_2| (b - a) + L_2 |s_1 - s_2| (d - c))
\end{aligned}$$

which completes the proof. \square

Remark 3.1. If we take $t_1 = s_1 = 0$ and $t_2 = s_2 = 1$ in Theorem 3.1, then (3.1) and (3.2) can be reduced to

$$\begin{aligned}
(3.3) \quad &\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \int_a^b \int_c^d p(x, y) dy dx - \int_a^b \int_c^d p(x, y) f(x, y) dy dx \right| \\
&\leq m (b - a) (d - c) \left[\frac{L_1}{4} (b - a) + \frac{L_2}{4} (d - c) \right],
\end{aligned}$$

and

$$(3.4) \quad \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x, y) dy dx - \int_a^b \int_c^d p(x, y) f(x, y) dy dx \right| \\ \leq \frac{m}{4} (L_1(b-a)^2 + L_2(d-c)^2),$$

respectively. The inequalities (3.3) and (3.4) are the Fejér type inequalities for Lipschitzian mapping in two variables.

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