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APPROXIMATE EVALUATION OF INTEGRALS OF ANALYTIC FUNCTIONS

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Abstract. A class of interpolatory rules of degree of precision nine has been constructed for the approximate evaluation of contour integrals of analytic functions along directed line segments in the complex plane. An optimal formula and some other specific formulas of degree nine belonging to this class have been determined.

1. Introduction

Birkoff and Young [1], Tosić [2], Acharya and Nayak [3] have considered the problem of numerical approximation of contour integral given by

$$I(f) = \int_L f(z)\,dz \tag{1.1}$$

where f is an analytic function in the disk $\Omega = \{z \,:\, |z - z_0| \le r,\ r > |h|\}$ and L is a directed line segment from the point $z_0 - h$ to the point $z_0 + h$. The rules formulated by Birkhoff and Young, Tosić, Acharya and Nayak (Ref [1] to [3]) are respectively the following

$$\begin{aligned} R_{BY}(f) &= \frac{h}{15}\big[24f(z_0) + 4\{f(z_0+h) + f(z_0-h)\} \\ &\qquad - \{f(z_0+ih) + f(z_0-ih)\}\big], \end{aligned} \tag{1.2}$$

$$\begin{aligned} R_T(f) &= \frac{16h}{15} f(z_0) + \frac{h}{6}\left(7/5 + \sqrt{7/3}\right)\{f(z_0+kh) + f(z_0-kh)\} \\ &\qquad + \frac{h}{6}\left(7/5 - \sqrt{7/3}\right)\{f(z_0+ikh) + f(z_0-ikh)\} \end{aligned} \tag{1.3}$$

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and

$$
\begin{aligned}
R_{AN}(f) = \frac{h}{315} & [448 f(z_0) + 113\{f(z_0+h) + f(z_0-h)\} \\
& -22\{f(z_0+ih) + f(z_0-ih)\} - 11h\ \{f'(z_0+h) - f'(z_0-h)\} \\
& +4ih\ \{f'(z_0+ih) - f'(z_0-ih)\}],
\end{aligned} \tag{1.4}
$$

where $i = \sqrt{-1}$, $k = (3/7)^{0.25}$ and prime " $'$ " denotes differentiation with respect to z. The rules $R_{BY}(f)$, $R_T(f)$ and $R_{AN}(f)$ have degree of precision five, seven and nine, respectively, and the first two rules are derivative free requiring five function evaluations. The rule $R_T(f)$ is the maximum accuracy formula in the class of seventh degree rules. In addition to these Lether [4], Acharya and Mohapatra [5] have constructed fifth degree rules for the numerical evaluation of the integral $I(f)$ which require function evaluations less than or equal to five.

The object of this paper to construct a class of ninth degree interpolatory rules meant for the approximate numerical evaluation of the contour integral $I(f)$ given by equation (1.1) with nodes at z_0, $z_0 \pm sh$, $z_0 \pm ish$ where s is a parameter in $(0,1]$ and finally derive the maximal accuracy formula and some more particular formulas of this class.

2. Formulation of the General Rule and Its Error

Let the proposed rule for the numerical approximation of the integral $I(f)$ be

$$
\begin{aligned}
R(f;\ s) = & Af(z) + B\{f(z_1) + f(z_3)\} + C\{f(z_2) + f(z_4)\} \\
& +Dhs\{f'(z_1) - f'(z_3)\} + Dhsi\{f'(z_2) - f'(z_4)\},
\end{aligned} \tag{2.1}
$$

where

$$
\begin{cases} z_j = z_0 + i^{j-1} sh, \quad j = 1(1)4, \\ 0 < s \le 1. \end{cases} \tag{2.2}
$$

It is noteworthy that the rule $R(f;s)$ is exact if the integrand $f(z)$ is an odd monomial $(z-z_0)^{2m-1}$, $m = 1, 2, \ldots$ Making $R(f;s)$ exact for even degree monomials $(z-z_0)^{2m}$, $m = 0, 1, 2, 3$ and 4, leads to the following system of equations:

$$
\begin{cases}
A + 2B + 2C = 2h, \\
B - C + 2(D - E) = \dfrac{h}{3s^2}, \\
B + C + 4(D + E) = \dfrac{h}{5s^4}, \\
B - C + 6(D - E) = \dfrac{h}{7s^6}, \\
B + C + 8(D + E) = \dfrac{h}{9s^8}.
\end{cases} \tag{2.3}
$$

solving the equations (2.3) we get

$$\left\{\begin{array}{lcl} A &=& 2h\left\{1-\dfrac{2}{5s^4}+\dfrac{1}{9s^8}\right\}, \\ B &=& h\left\{\dfrac{1}{4s^2}+\dfrac{1}{5s^4}-\dfrac{1}{28s^6}-\dfrac{1}{18s^8}\right\}, \\ C &=& h\left\{-\dfrac{1}{4s^2}+\dfrac{1}{5s^4}+\dfrac{1}{28s^6}-\dfrac{1}{18s^8}\right\}, \\ D &=& h\left\{-\dfrac{1}{24s^2}-\dfrac{1}{40s^4}+\dfrac{1}{56s^6}+\dfrac{1}{72s^8}\right\}, \\ E &=& h\left\{\dfrac{1}{24s^2}-\dfrac{1}{40s^4}-\dfrac{1}{56s^6}+\dfrac{1}{72s^8}\right\}. \end{array}\right. \tag{2.4}$$

It is evident that the degree of precision of the rule $R(f;s)$ is nine for every value of $s \in (0,1]$.

As f is analytic in the disk Ω, $f(z)$ is expansible in Taylor series for $|z-z_0| \le r$ and

$$f(z)=\sum_{n=0}^{+\infty} a_n(z-z_0)^n, \qquad a_n=\frac{1}{n!}f^{(n)}(z_0). \tag{2.5}$$

If $E(f;s)$ denotes the error in the approximation $R(f;s)$ of $I(f)$, i.e.,

$$E(f;s)=I(f)-R(f;s), \tag{2.6}$$

then using equation (2.5) in equation (2.6) we get after simplification

$$E(f)=h^{11}a_{10}u(s)+h^{13}a_{12}v(s)+h^{15}a_{14}w(s)+\cdots, \tag{2.7}$$

where

$$\left\{\begin{array}{l} u(s)=2\left(\dfrac{1}{11}-\dfrac{2s^4}{7}+\dfrac{s^8}{3}\right), \\ v(s)=2\left(\dfrac{1}{13}-\dfrac{2s^4}{9}+\dfrac{s^8}{5}\right), \\ w(s)=2\left(\dfrac{1}{15}-\dfrac{2s^4}{11}+\dfrac{s^8}{7}\right). \end{array}\right. \tag{2.8}$$

It is pertinent to note that the function $u(s)$ which is the associated with the coefficient a_{10} in the expression for the truncation error $E(f;s)$ is nonzero for all $s \in (0,1]$, the range of the parameter s. In view of this and the discussions made above we have the following theorem.

Theorem 2.1. *The rule $R(f;\ s)$, given by equations* (2.2) *and* (2.5), *meant for the numerical evaluation of the contour integral $I(f)$ of an analytic function f has degree of precision nine and the truncation error $E(f;s)$ associated with $R(f;s)$ satisfies $E(f)=O\left(h^{11}\right)$ for all $s \in (0,1]$.*

3. Some Particular Cases

Some particular cases of the general rule $R(f;s)$ have been considered below:

(i) Setting $s = 1$ leads to the rule $R_{AN}(f)$ constructed by Acharya and Nayak [3].

(ii) The function $u(s)$ associated with the leading coefficient in the error expression $E(f;s)$ given by equations (2.8) and (2.9) is positive for all real values of the parameter s and $u(s)$ attains its minimum value for $s = (3/7)^{0.25}$. Hence for $s = (3/7)^{0.25}$ the general rule $R(f;s)$ yields the following maximum accuracy formula

$$\begin{aligned} R_{MAF}(f) = \frac{544h}{405} f(z_0) &+ \frac{h}{6}\left(133/135 + \sqrt{7/3}\right)\{f(\varsigma_1) + f(\varsigma_3)\} \\ &+ \frac{h}{6}\left(133/135 - \sqrt{7/3}\right)\{f(\varsigma_2) + f(\varsigma_4)\} \\ &+ \frac{7h^2 s}{405}\{f'(\varsigma_1) - f'(\varsigma_3) + if'(\varsigma_2) - if'(\varsigma_4)\}, \end{aligned} \tag{3.1}$$

where $\varsigma_j = z_0 + (i)^{j-1} h(3/7)^{0.25}$, $j = 1(1)4$. It is noteworthy that the $R_{MAF}(f)$ is a modification of the formula $R_T(f)$ given by equation (1.3) due to Tosić [2].

(iii) The general rule $R(f;s)$ requires nine function evaluations. To make $R(f;s)$ computationally more efficient, number of function evaluation needs to be reduced. This is achieved by equating the coefficient D (in equation (2.4)) to zero. Which leads to the following cubic equation

$$105t^3 + 63t^2 - 45t - 35 = 0$$

where $t = s^2$. The equation (**??**) has the root

$$t = 0.698045648995087341,$$

by applying the Newton Raphson method and using D-arithmetic. Thus for

$$s = 0.835491262069859929 \quad (= s^*),$$

the general rule $R(f;s)$ boils down to the complexity reduced ninth degree rule $R_{CR}(f)$ in which the coefficient $D = 0$.

(iv) By taking two different values of s, i.e., s_1 and s_2 the following extrapolated approximation $\bar{R}(f;s_1,s_2)$ from $R(f;s_1)$ and $R(f;s_2)$ to $I(f)$ has degree of precision eleven. Thus,

$$\bar{R}(f;s_1;s_2) = \frac{R(f;s_1)\,u(s_2) - R(f;s_2)\,u(s_1)}{u(s_2) - u(s_1)} \tag{3.2}$$

and

$$I\,(f) - R\,(f; s_1, s_2) = h^{13} a_{12} X\,(s_1, s_2) + \cdots ,$$

where

$$X\,(s_1, s_2) = \frac{v\,(s_1)\,u\,(s_2) - v\,(s_2)\,u\,(s_1)}{u\,(s_2) - v\,(s_1)}$$

and the functions $u(s)$ and $v(s)$ have been prescribed in equation (2.8).

4. Numerical Tests

For the purpose of numerical verification of the rules constructed and discussed the following integral is considered

$$I = \int_{-i}^{i} e^z\,dz \approx 1.682941969\,i.$$

This value is exact upto nine decimal places. The integral I has been computed for a sequence of values of the parameter s and the results have been presented in the following table:

s	Approximate value of I/i
0.5	1.682942013
0.6	1.682942001
0.7	1.682942180
0.8	1.682941986
0.9	1.682941995
1.0	1.682942045
$(3/7)^{1/4}$	1.682941969
$\sqrt{0.6}$	1.682941988
s^*	1.682941982

The approximation of the integral I by the rules $R_{BY}\,(f)$ and $R_T\,(f)$ (given by equations (1.2) and (1.3)) are $1.682417145\,i$ and $1.682940726\,i$, respectively. As expected the maximum accuracy formula $R_{MAF}\,(f)$ given by Eq. (3.1) is most accurate in the class of ninth degree rules generated by $R\,(f; s)$, $s \in (0, 1]$.

For the numerical test of integration method $\bar{R}\,(f; s_1, s_2)$ given by equation (3.2) the parametric value $s_1 = 0.5$ and $s_2 = 1.0$ have been set and the following approximation of the integral I has been obtained

$$\bar{R}\,(f; 0.5,\, 1.0) = 1.682941969\,i.$$

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