FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 23 (2008), 49–62

DISCRETE AUTOREGRESSIVE STOPPING TIME MODEL*

Vladica Stojanović and Biljana Popović

Abstract. In the paper were described the new, original model used in modelling of time events in series with "low" frequency. As basic model's distributions we used distributions of discrete type, and the model called Discrete Autoregressive Stopping Time model (or D-AST model). The basic stochastic properties of D-AST model (moments, variance, correlation function) were described, too. The important segment of the paper is its statistical aspect, estimation of parameters and practical usage in the modelling of the real time series. In our case, we showed the usage of D-AST model in describing the intensity of changing dynamics of stock prices in some of the eminent Serbian companies, which are traded them on Belgrade Stock Exchange for quite a few years.

1. Introduction and Motivation

Indeterminacy in the market might be described with model in which price of some product is shown as stochastic process $S = (S_n)$, i.e., as family of random variables which depends on discrete time parameter $n \in D = \{0, 1, 2, \ldots\}$. The presumption that time moments n are discrete is based on the fact that in concrete exchange situation the price S is registered in different time intervals. Let us assume, as supplementary, that the probability space $(\Omega, \mathcal{F}, P)$ is expanded with filtration $F = (\mathcal{F}_n)$, i.e., family of σ-algebras which satisfies the condition

$$\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}.$$

In the basic interpretation, family $(\mathcal{F}_n)$ is the set of information about the market situation which is available to all its participants, up to the moment n. On this way, the evolution of the price is considered inside the filtrate space of probability $(\Omega, \mathcal{F}, F, P)$ which is named *stochastic basis.* For random variables S_n we assume that are $\mathcal{F}_n$ adaptive, i.e., value of price in the moment n depends on all available information described with $\mathcal{F}_n$.

Received November 7, 2007.
2000 *Mathematics Subject Classification.* Primary 62M10; Secondary 91B84, 91B72
*Partially supported by grants 144025 MSED Republic of Serbia

On the other side, some investigations of the real data recommend the statistical models which show the financial indexes dynamic like stochastic processes in continuous time. The evolution of financial indexes (price) can be represented with a piecewise constant functions, like in the figure 1, which have a constant value on the interval $\left[\tau_{k-1}, \tau_k\right)$, and then, in the (random) moment τ_k, the change comes.

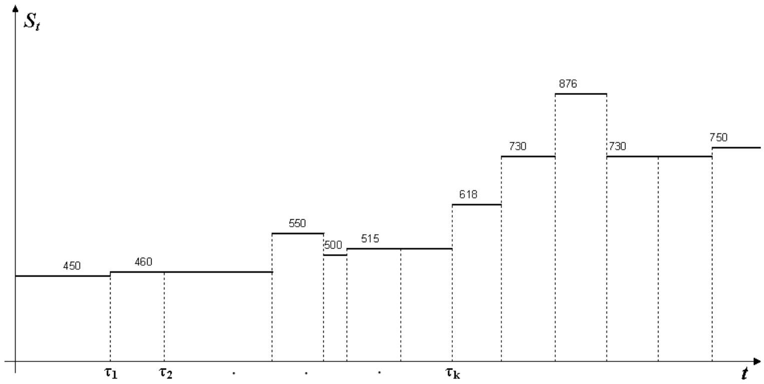

FIG. 1.1: Dynamic of the price of stocks of the company Hemofarm-Vršac. (Source: The Belgrade Stock Exchange)

On this way, natural expression for price dynamic is the model

$$S_t = S_0 + \sum_{k=1}^{\infty} \xi_k I(\tau_k \le t), \quad t \ge 0, \tag{1.1}$$

where S_0 is the initial value of the price in moment $t = 0$, and ξ_k are returns of price changes in the random moment τ_k. Here, the price is defined like stochastic process in continuous time, and the stochastic basis $\mathcal{B}$ must be extended with new, continuous-time basis $\widetilde{\mathcal{B}} = (\Omega, \mathcal{F}, \widetilde{F}, P)$, where $\widetilde{F} = \left(\widetilde{\mathcal{F}}_t\right)_{t\ge 0}$ and $\widetilde{\mathcal{F}}_t = \mathcal{F}_{[t]}$.

However, our point of view is to describe so called *discrete power of the events* which exists in the realizations of sequences $\xi = (\xi_k)$ and $\tau = (\tau_k)$. Such modelling needs some extended analysis on the stochastic properties of these two sequences. Especially important role in (1.1) plays the sequence $\tau = (\tau_k)_{k\ge 1}$ which describes the time changes of the values in the so called *stopping time* and is usually called *the stopping sequence*. Stopping sequence shows the property of irregularity in the dynamics of financial indexes, and it will be in the center of our attention. In order to do more precise analysis, we will use some of the most basic concepts which are in relation to this sequence, that is with which the stopping time is described.

Definition 1.1. Let us assume that the filtration $F = (\mathcal{F}_n)$ is given on the probability space $(\Omega, \mathcal{F}, P)$. The random variable $\tau = \tau(\omega)$, $\omega \in \Omega$, with set of values

$D = \{0, 1, 2, \ldots\}$ is called *Markov moment* with respect to the filter F if for every $n \in D$

$$(i) \qquad \big\{\omega \mid \tau(\omega) = n\big\} \in \mathcal{F}_n .$$

If, in addition,

$$(ii) \qquad P\big\{\omega \mid \tau(\omega) < \infty\big\} = 1$$

then τ is the Markov stopping time. $\triangle$

Markov moment is interpreted as the moment of time when some financial decision is brought up, for instance when buying or selling shares, or, in our case, the changing of value of the financial index. As, according to (i) and the definition of the filter F,

$$\big\{\omega \mid \tau(\omega) \leq n\big\} = \bigcup_{k=0}^{n} \big\{\omega \mid \tau(\omega) = k\big\} \in \mathcal{F}_n \,,$$

it will be

$$\big\{\omega \mid \tau(\omega) > n\big\} = \big\{\omega \mid \tau(\omega) \leq n\big\}^c \in \mathcal{F}_n .$$

With it we can point out the Markov property of $\tau(\omega)$, because the changing of the price values does not depend on the future prospects. On the other hand, the term stopping is directly connected with the premise (ii) which enables the realization of this values in the final time. Finally, we will supply the well known but necessary premises which enable us using the sequences of Markov stopping times, which we will use as the basic stochastic model in the work ahead.

Definition 1.2. Let $\tau = (\tau_k)$ be the stochastic sequence in which for every $k \geq 1$ the following conditions are satisfied:

(i) τ_k is Markov stopping time (in sense of the definition above);

(ii) with probability 1 is $\tau_k \leq \tau_{k+1}$ for all $k \in D$ and $\tau_k \to \infty$ when $k \to \infty$.

Then the sequence (τ_k) is called the Markov stopping time sequence, or just the stopping sequence. $\triangle$

2. D-AST Model. The Definition and the Basic Characteristics

In order to apply the idea of changing dynamics of the prices on some concrete data, above all on the time dynamic of changing the prices of the shares with which Serbian stock market trades, we will show the original model, based on the distribution of the random variables of the discrete type. The basic reason for introducing the changes like these lies in the fact that Serbian stock market has very slow tendency of changing the prices, which is in contradiction with the most of the developed foreign countries. Namely, it is very common that the prices of certain portfolios do not change during some long-lasting time period, so that their irregular dynamics can be interpreted with the number of days which pass between two successional changes of the price of shares.

On this way, the basic purpose of the model mentioned here is based upon the possibility of the stochastic modelling of time moments of price changes, which is usually shown by the stopping sequence (τ_k), but it is also based upon the description the intensity of those changes. Because of the emphasized low dynamics in price changing, distributions of the discrete type are extremely suitable stochastic apparatus which will, in this purpose, be used here. The basic idea for creating our model is based upon the autoregressive principle used in many stochastic models. Similar ideas in modelling changes of price in time domain we can find in the definition of so called autoregressive conditional duration (ACD) model, introduced by Engle and Russel [4], and innovated later by Bauwens and Giot [2], Meitz and Teräsvirta [6] or Stojanović [9]. In this purpose, we are introducing the next formal definition of stopping sequence, which is based on the recurrent, autoregressive dependence of its members, considered upon the prior realizations.

Definition 2.1. Let us assume that for every $k \in \mathbb{N}$ next conditions are fulfilled:

(*i*) (ε_k) is the sequence of the i.i.d. random variables with the set of values $D = \{0, 1, 2 \ldots\}$ so that $E(\varepsilon_k) = 1$, $\mathrm{Var}(\varepsilon_k) = \sigma^2$;

(*ii*) (λ_k) is the sequence of the independent random variables too, which is also independent of the sequence (ε_k), and which has the uniform distribution defined on the set $A = \{1, 2, \ldots, a\}$, $a \in \mathbb{N}$;

(*iii*) $\mathcal{F}_k = \mathcal{G}en\big\{(\varepsilon_j, \lambda_j) \mid j = 1, 2, \ldots, k\big\}$, $\mathcal{F}_0 = \varnothing$.

In this case, the sequence of random variables (τ_k) represents *the discrete autoregressive stopping time model* (*D-AST model*) if it satisfies the recurrent relation

$$\tau_k = \tau_{k-1} + \lambda_k\, \varepsilon_k, \quad k \geq 1 \quad (\tau_0 = 0). \quad \triangle \tag{2.1}$$

As we can see, the stated definition proposes stopping sequence (τ_k) in the form of random walk sequence with discrete distribution which depends on the distribution of the two sequences. The first of these, sequence (ε_k) with the normalized distribution has the role of noise. Its values affects the changes of the next realization of stopping sequence. Yet, mean value of these changes is equivalent to the unit of time in which we observe the dynamics of change of the value of the regarded financial sequence. That fact is, like we will see, rarely convenience in practice modelling.

On the other hand, the sequence (λ_k) implies the addition changes in values of (τ_k), and have, as for the distinction to the normalized sequence (ε_k), the more emphatic and bigger fluctuation. This sequence we interpret like the *intensity of a reaction* of D-AST model related to its previous realization, and we emphasize the important role of the parameter a which is the parameter of the distribution of the sequence (λ_k). As it is in the average $\varepsilon_k \sim 1$, it means that the value of parameter a, in free interpretation, will show the *upper limit of intensity of fluctuations* in the stopping sequence. In this way, its role, altogether with parameter σ^2, the variance of the noise (ε_k), is very important in the description of the irregularity in dynamics of the viewed financial sequence. If it is, for example, $\lambda_k \equiv \varepsilon_k \equiv 1$, the dynamics of observed sequence is *quite regular*, i.e., the changes of its values will happen in every moment in which we observe its behavior. This situation can be described by means of the parameters themselves, because in that case we have $(a, \sigma^2) = (1, 0)$.

This fact can be, in some way, used in an investigation of the grade of regularity in behavior of some empiric financial sequence. The possible hypotheses about it, that the uniform distribution of the sequence (λ_k) is concentrated in the point $x = 1$, that means that $a = 1$, corresponds to the fact that the dynamics of the considered financial sequence has no emphasized irregularity, that means that every time interval $\big[\tau_{k-1}, \tau_k\big)$ which satisfies the inequality $\tau_k - \tau_{k-1} > 1$ is the consequence of the fluctuation inside the values of the sequence (ε_k). Of course, a and σ^2 are the unknown parameters of D-AST model, which are necessary to be estimated upon the realization $\tau_1, \ldots, \tau_N$, so, we shall pay the great attention on the problem further on. First of all, we shall appoint the basic stochastic properties of the sequence (τ_k).

Theorem 2.1. *The sequence of random variables* (τ_k)*, defined by* (2.1)*, represent the stopping time sequence in the sense of the definition* 1.2*. Thereat, the basic stochastic properties of this sequence are*

(*i*) $E(\tau_k) = \dfrac{k}{2}\,(a+1)$;

(*ii*) $\operatorname{Var}(\tau_k) = \dfrac{k}{12}\,(a+1)\big[2(\sigma^2+1)(2a+1) - 3(a+1)\big]$;

(*iii*) $\operatorname{Corr}(\tau_k, \tau_{k+h}) = \sqrt{\dfrac{k}{k+h}}$, $h \geq 0$.

Proof. It is obvious that the condition (*i*) from the Definition 1.2 is fulfilled as well as $\tau_k \leq \tau_{k+1}$, $k = 0, 1, 2, \ldots$, which is valid almost sure.

Let us notice that

$$\big\{\tau_k \not\to \infty, k \to \infty\big\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{\lambda_n \varepsilon_n = 0\}.$$

According to the well known property of the continuity of probability, we have

$$P\big\{\tau_k \not\to \infty, k \to \infty\big\} = \lim_{k\to\infty} \left(\lim_{m\to\infty} \prod_{n=k}^{k+m} P\big\{\lambda_n \varepsilon_n = 0\big\} \right) = 0.$$

This completes the definition of the sequence (τ_k). Its stochastic properties which are discussed in this theorem (mean, variance and correlate function) can be easily verified by the standard computation. □

According to the above mentioned, it is clear that (τ_k) is the nonstationary sequence and that it can produce difficulties in practical applications. That is why we shall consider the sequence of increments of this stopping sequence

$$X_k = \tau_k - \tau_{k-1} = \lambda_k \varepsilon_k, \quad k = 1, 2, \dots$$

only. It is the i.i.d. sequence of random variables, and its distribution is enough for the full description of the stopping sequence. In the simple way we can appoint basic stochastic properties of the sequence (X_k), mean value

$$E(X_k) = E(\lambda_k) = \frac{a+1}{2} \tag{2.2}$$

and the variance

$$\begin{aligned} \mathrm{Var}(X_k) &= E(\lambda_k^2)E(\varepsilon_k^2) - \big[E(\lambda_k)\big]^2 \\ &= \frac{a+1}{12}\big[2(\sigma^2+1)(2a+1) - 3(a+1)\big]. \end{aligned} \tag{2.3}$$

Further on, we shall use the notation $\mathrm{Var}(X_k) = V$.

In some practical applications (see for instance section 4.) we will assume that the sequence (ε_k) has normalized Poisson's distribution, defined as

$$P\big\{\varepsilon_k = m\big\} = \frac{1}{e \cdot m!}, \quad m, k \in D. \tag{2.4}$$

Then $\sigma^2 = 1$ and the distribution of the sequences (X_k), according to the definition of D-AST model, can be described as

$$P\{X_k = m\} = \begin{cases} 1/e\,, & m = 0, \\ 1/(ae) \displaystyle\sum_{d \in D(m)} 1/d!\,, & m \geq 1, \end{cases}$$

where $D(m)$ is the set of divisors d of the number m with the obvious property $m/d \leq a$.

3. Parameters Estimation

The basic problem in the estimation and later formation of the D-AST model over the empirical data, is the estimation of the unknown parameter (a, σ^2) based on the realization $\tau_1, \dots, \tau_N$ of the stopping sequence. As this sequence is exchanged by the stationary sequence of the increments $X_k = \tau_k - \tau_{k-1}$, the simplest possibility of evaluating the unknown parameters give us the equations (2.2) and (2.3), which express the mean and the variance of the sequence (X_k). Using the method of moments, where we use the notation

$$\bar{X}_N = \frac{1}{N}\sum_{k=1}^{N} X_k, \quad \bar{D}_N = \frac{1}{N}\sum_{k=1}^{N}\big(X_k - \bar{X}_N\big)^2,$$

we get the estimates

$$\text{(3.1)} \qquad \begin{cases} \widetilde{a} = 2\bar{X}_N - 1 \\ \widetilde{\sigma}^2 = \left[12\bar{D}_N + 3(\widetilde{a}+1)^2\right]\left[2(\widetilde{a}+1)(2\widetilde{a}+1)\right]^{-1} - 1, \end{cases}$$

for which the "good" stochastic properties are shown in the next proposition.

Theorem 3.1. *Estimate $\widetilde{a}$ is unbiased, strictly consistent and asymptotically normal estimate of the parameter a. Estimate $\widetilde{\sigma}^2$ is strictly consistent and asymptotically normal estimate of the parameter σ^2.*

Proof. It is easy to notice that $\widetilde{a}$ is unbiased and strictly consistent estimate of the parameter a, because it is the linear function of the unbiased and strictly consistent estimate $\bar{X}_N$. Likewise, based on the asymptotic normality of the empirical mean value, that is, convergency

$$\sqrt{N}\Big(\bar{X}_N - \frac{a+1}{2}\Big) \xrightarrow{d} \mathcal{N}(0, V_1),$$

where $V_1 = \lim\limits_{N\to\infty} \mathrm{Var}(\sqrt{N}\bar{X}_N) = V$, according to (3.1), we get

$$\text{(3.2)} \qquad \sqrt{N}\big(\widetilde{a} - a\big) \xrightarrow{d} \mathcal{N}(0, V_2),$$

where $V_2 = 4\,V$.

Similarly, estimate $\widetilde{\sigma}^2$, as the continuous function $\psi : \mathbb{R}^2 \to \mathbb{R}$ of the strictly consistent estimate $(\widetilde{a}, \bar{D}_N)$, is strictly consistent. Furthermore, the asymptotic normality of $\widetilde{\sigma}^2$ is based on the asymptotic normality of the $\widetilde{a}$, that is, on the convergency (3.2), as well as the convergency,

$$\sqrt{N}\Big(\bar{D}_N - \sigma^2\Big) \xrightarrow{d} \mathcal{N}(0, M_4 - \sigma^4)$$

where $M_4 = E(X_k^4)$. Then, when we apply the theorem of continuity of the convergency in distribution (Serfling [8, p. 122]), we have

$$\sqrt{N}\big(\widetilde{\sigma}^2 - \sigma^2\big) \xrightarrow{d} \mathcal{N}(0, V_3),$$

where $V_3 = 4\psi_1^2 V + \psi_2^2(M_4 - \sigma^4)$ and

$$\begin{aligned} \psi_1 &= \frac{\partial\psi(\widetilde{a}, \bar{D}_N)}{\partial\widetilde{a}}\bigg|_{(\widetilde{a}, \bar{D}_N) = (a, \sigma^2)} = -\frac{3}{2(2a+1)^2} - \frac{6\sigma^2(4a+3)}{(a+1)^2(2a+1)^2}, \\ \psi_2 &= \frac{\partial\psi(\widetilde{a}, \bar{D}_N)}{\partial\bar{D}_N}\bigg|_{(\widetilde{a}, \bar{D}_N) = (a, \sigma^2)} = \frac{6}{(a+1)(2a+1)}. \qquad \square \end{aligned}$$

Beside the stated properties of the estimate $(\widetilde{a}, \widetilde{\sigma}^2)$, it is clear that it is not the most efficient estimate of the parameter (a, σ^2) because the value of the dispersion described by variances V_2 and V_3, can be huge, especially when the values $a \in \mathbb{N}$

and $\sigma^2 > 0$ are great. Thus, it is of interest to find the more efficient estimate of this parameter. In the similar way as it is in Lawrence and Lewis [5], we will define the autoregressive stationary sequence

$$W_k = b\,W_{k-1} + \eta_k, \quad k \in \mathbb{N},$$

where $\eta_k = X_k - E(X_k)$ and $b = \varphi(a) \in (0,1)$ is the continuously differentiable monotonous function of the parameter a. Obviously, the sequence (W_k) is stationary ergodic sequence of the random variables with the mean $E(W_k) = 0$, variance

$$W = \mathrm{Var}(W_k) = E(W_k^2) = (1-b^2)^{-1}\,V$$

and the correlation function

$$\rho_W(h) = \begin{cases} 1, & h = 0; \\ b^{|h|}, & h \neq 0. \end{cases}$$

Now, if we use the standard regression procedure, we can get the estimate of parameter b,

$$\widehat{b} = \left(\sum_{k=0}^{N-1} W_k W_{k+1} \right) \cdot \left(\sum_{k=0}^{N-1} W_k^2 \right)^{-1}. \tag{3.3}$$

From here, we get the estimate of (a, σ^2) in the form

$$(\widehat{a}, \widehat{\sigma}^2) = \Big(\varphi^{-1}(\widehat{b}), \psi(\widehat{a}, \bar{D}_N) \Big)$$

for which, as it was in the previous case, we can show the asymptotic properties.

Theorem 3.2. *The estimate* $(\widehat{a}, \widehat{\sigma}^2)$ *is strictly consistent and asymptotically normal estimate of the parameter* (a, σ^2).

Proof. According to ergodicity and stationarity of the sequence (W_k), we can apply the ergodic theorem on (3.3). Then we have

$$\frac{1}{N} \sum_{k=0}^{N-1} W_k W_{k+1} \xrightarrow{a.s.} bW, \quad N \longrightarrow \infty,$$

where $E(W_k W_{k+1}) = b\,W$ for any k, and, also,

$$\frac{1}{N} \sum_{k=0}^{N-1} W_k^2 \xrightarrow{a.s.} W, \quad N \longrightarrow \infty. \tag{3.4}$$

Therefore,

$$\widehat{b} = \left(\frac{1}{N} \sum_{k=0}^{N-1} W_k W_{k+1} \right) \cdot \left(\frac{1}{N} \sum_{k=0}^{N-1} W_k^2 \right)^{-1} \xrightarrow{a.s.} b, \quad N \longrightarrow \infty,$$

i.e., $\widehat{b}$ is strictly consistent estimate of parameter b. ¿From here, according to the continuity of the almost sure convergence (Serfling [8], pg. 24), we have

$$\widehat{a} - a = \varphi^{-1}(\widehat{b}) - \varphi^{-1}(b) \xrightarrow{a.s.} 0, \quad N \longrightarrow \infty,$$

which shows the strictly consistency of $\widehat{a}$.

Now, we show the asymptotic normality of the estimate $\widehat{b}$. According to (3.3) it follows that

$$\sqrt{N}\,(\widehat{b} - b) = \frac{N^{-1/2} \cdot \mathbf{U}_N}{N^{-1} \cdot \mathbf{V}_N} \tag{3.5}$$

where $\mathbf{U}_N = \sum_{k=0}^{N-1} \eta_{k+1} W_k$, $\mathbf{V}_N = \sum_{k=0}^{N-1} W_k^2$, $N = 1, 2, \ldots$. As the sequence $(\mathbf{U}_N)$ is martingale, applying the central limit theorem for martingales (see for instance Billingsley [1] or Nicholls and Quinn [7]), we have

$$N^{-1/2} \cdot \mathbf{U}_N \xrightarrow{d} \mathcal{N}\big(0, D_0\big), \quad N \longrightarrow \infty,$$

where $D_0 = \operatorname{Var}(\eta_{k+1} W_k) = \operatorname{Var}(X_{k+1}) \operatorname{Var}(W_k) = VW$. Then, according to the almost sure convergence (3.4) and the equation (3.5), we have

$$\sqrt{N}\,(\widehat{b} - b) \xrightarrow{d} \mathcal{N}\big(0, D_1\big), \quad N \longrightarrow \infty,$$

where $D_1 = VW^{-1} = 1 - b^2$. Finally, as $b = \varphi(a)$ is a continuous function of the parameter a, we can apply the continuity of the convergency in distribution (Serfling [8, p. 118]), and then we have

$$\sqrt{N}\,(\widehat{a} - a) \xrightarrow{d} \mathcal{N}\big(0, D_2\big), \quad N \longrightarrow \infty,$$

where $D_2 = \big[\varphi'(a)\big]^{-2} \cdot D_1$.

The strong consistency and asymptotic normality of the estimate $\widehat{\sigma}^2$ can be proved in the same way as in the case of the estimate $\widetilde{\sigma}^2$, i.e., by analogues procedure as in the Theorem 3.1. $\square$

At the end of this section let us notice the very important characteristic of the estimate $\widehat{a}$, as well as of the estimates $\widehat{\sigma}^2$. That is its asymptotic efficacy, expressed by its variance $D_2 = D_2(a)$. In the contrast to the variance $V_2 = V_2(a, \sigma^2)$ of the estimate $\widetilde{a}$, it depends only on the parameter $a \in \mathbb{N}$, and not on the $\sigma^2 > 0$. On the other hand, by choosing the adequate function $\varphi(a)$, it is possible to get strictly consistent and asymptotically normal estimate of the parameter a which will be asymptotically more efficient than the estimate $\widetilde{a}$, that is

$$D_2(a) < V_2(a, \sigma^2).$$

Some of the possibilities of such modelling are shown graphically in Figure 3.1. There, the graphs of variances $V_2(a, \sigma^2)$ of the estimate $\widetilde{a}$ are shown along with the graph of the behavior of the variance of the estimate $\widehat{a}$ in the cases $D_2(a) = a^2 - 1$

and $D_2(a) = (a^2 - 1)/a^2$. (Pictures on the left show the functional dependence of the variances $V_2(a, \sigma^2)$ and $D_2(a, \sigma^2)$, while the special case of that dependency for $\sigma^2 = 1$ is shown on the right). So, the estimate $\widehat{a}$, in the case when it is more efficient than $\widetilde{a}$, enables the more precise estimation of the unknown parameter a, and along with it, the more precise estimation of variance σ^2. That is why it will be payed a special attention during the practical usage of the D-AST model.

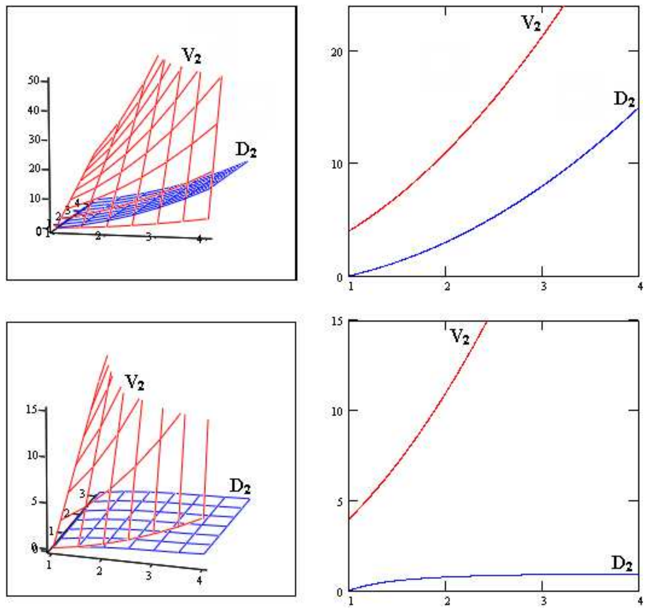

Fig. 3.1: Comparative values of the asymptotic variances of the estimates $\widetilde{a}$ and $\widehat{a}$

We can conclude that the procedure of the construction of such estimates depends on the choice of the proper function $\varphi(a)$, but also it depends on the estimates $\widetilde{a}$ which may be used as the initial estimates. In that way a sequence of the estimates may be formed

$$\widehat{a}_{j+1} = \varphi^{-1}\big(\widehat{b}_{j+1}\big), \quad j = 1, 2, \ldots \tag{3.6}$$

where $\widehat{a}_0 = \widetilde{a}$, and $\widehat{b}_{j+1}$ is the estimate of the parameter $b = \varphi(a)$ obtained by the regressive procedure described above. The sequence (W_k) is generated by the estimate $\widehat{a}_j$. Of course, the important problem which appears here is the obtaining of the necessary conditions for the convergence of the iterative method (3.6) in some of the familiar stochastic forms.

4. Monte Carlo Simulation. The Application of the Model

In this section we will describe some of the concrete applications of the D-AST model in the analysis of the empirical time series, i.e., real data. In order to do that we shall consider the possibilities of the estimation of the parameters a and σ^2 of the D-AST model, i.e., the stopping sequence (τ_k) which is described by the equation (2.1) and the Monte Carlo simulations of the sequences (ε_k) and (λ_k). In this purpose, we will assume that the sequence (ε_k) has Poisson's normalized distribution described by the equation (2.4). In order to simulate this sequence and the sequence (λ_k) we can form the suitable simulations of the sequence $X_k = \lambda_k \varepsilon_k$, and in that way, we can determine the estimates $(\widetilde{a}, \widetilde{\sigma}^2)$ and $(\widehat{a}, \widehat{\sigma}^2)$ by the procedure described in the section above.

In Table 4.1 the average values of these estimates are shown, together with the correspondent estimating errors (the values shown in the brackets), based on the 80 independent Monte Carlo simulations of (X_k) of the length $N = 500$. As the real values of the parameter a, $a = 2$ and $a = 5$ are taken, while the real value of the variance of the noise (ε_k), based on the assumption of its distribution stated above, is $\sigma^2 = 1$. The initial value for the sequence (W_k) is stated to be $W_0 = 0$.

Table 4.1: Estimated values of Monte Carlo simulations

Parameters	**Estimates**	
	$(a, \sigma^2) = (2, 1)$	$(a, \sigma^2) = (5, 1)$
$\widetilde{a}$	$1,990\ (0,0143)$	$5,049\ (0,0514)$
$\widetilde{\sigma}^2$	$1,009\ (0,0917)$	$1,003\ (0,1231)$
$\widehat{a}$	$2,001\ (0,0876)$	$5,009\ (0,0581)$
$\widehat{\sigma}^2$	$0,998\ (0,0385)$	$1,007\ (0,0682)$

In Figure 4.1, the histograms of the empirical distributions of the stated estimates in the case $a = 5$, $\sigma^2 = 1$ are shown. The grouping of the received values around the true values of the parameters a and σ^2 and their asymptotic tendency to the normal distribution is clearly noticeable.

On the other hand, the same method of the estimation of the parameter is applied to the real data. To clarify, the changing of the price of shares of some companies which where trading on the Belgrade Stock Exchange in the given moment are considered as the empirical sets. As all the changes of the stocks' prices until recently were registered daily, the realization of the sequence $X_1, \ldots, X_N$ on which the unknown limits of reactions a and variance σ^2 is based, can be determined in a very simple way. Of course, the estimated values are obtained by the procedure described above in the text. Using the official data from the Belgrade

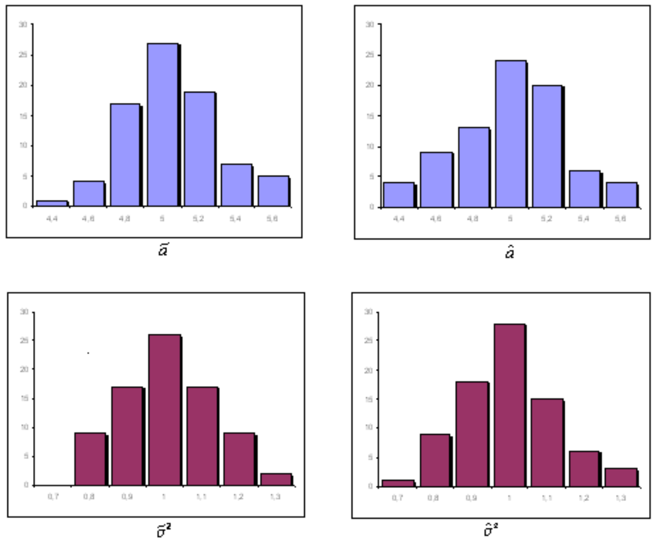

FIG. 4.1: Histograms of the empirical distributions of estimates $(\widetilde{a}, \widetilde{\sigma}^2)$ and $(\widehat{a}, \widehat{\sigma}^2)$.

Stock Exchange, the estimated values of the parameter (a, σ^2) for some of the leading Serbian companies are shown in the Table 4.2. It is clear that all the companies have the nontrivial estimated values of the parameter $a > 1$. Thus, the frequency of their changes is slower than the regular, i.e., it happens rarely than daily changes in price.

Of course, the greater the values of the limit a indicate the slower dynamics in price changing, while, on the other hand, the greater the frequency in price changing corresponds to the smaller estimate of the given parameter. Finally, the analysis of the estimated values of the parameter σ^2 enables the further research in the degree of the regularity in the dynamics of sequence changing. The cases of the two companies, Hemofarm and Sunce, are very interesting, where the estimated values $\widetilde{\sigma}^2$ and $\widehat{\sigma}^2$ give a reason for the presumption that the real value of this parameter is $\sigma^2 = 1$. On this way, the distribution of the noise (ε_k) can be interpreted whit Poisson's normalized distribution which we have already mentioned above.

Table 4.2: Estimated values of real data

Parameters	Companies			
	Alfa Plam Vranje	**Hemofarm** Vršac	**Metalac** G. Milanovac	**Sunce** Sombor
$\widetilde{a}$	4,826	5,000	3,928	6,171
$\widetilde{\sigma}^2$	1,156	1,117	0,694	0,935
$\widehat{a}$	4,228	5,387	3,824	6,218
$\widehat{\sigma}^2$	1,505	0,953	0,587	1,037

5. The Conclusion

The D–AST model defined here represents stopping time sequence (τ_k) with discrete innovations of special type, shows its efficiency specially when the dynamics in price changing is low. We can specially remark that it is convenient to estimate prices whose changes are not regular, i.e., there are no changes in daily prices in a several sequent days. Specially, models selection depends of the first correlation of increments (X_k). If empirical investigations indicate low dependence of X_k and X_{k+1}, D-AST model is adequate stochastic model. On the other hand, presence of emphatic correlation indicate to different behavior and, at same time, different stochastic modelling. In our case, some of the possibilitie is usage another, so called D-ACD model, introduced by Popović and Stojanović [10].

6. Acknowledgment

The authors are grateful to Belgrade Stock Exchange for consigning the data sets and giving the constructive information about them.

REFERENCES

1. P. Billingsley: *The Lindeberg–Levy theorem for martingales.* Proceedings of the American Mathematical Society **12**, No. 5 (1961), 788–792.
2. L. Bauwens and P. Giot: *The logarithmic ACD model: An application to the Bid-Ask quote process of three NYSE stocks.* Annales d'Économie et de Statistique **60** (2000), 117–149.
3. P. Embrechts, C. Klüppelberg and T. Mikosch: *Modeling Extremal Events for Insurance and Finance.* Application of Mathematics 33, Springer-Verlag, New York, 2001.

4. R. F. Engle and J. R. Russel: *Autoregressive conditional duration: A new model for irregularly spaced transaction data.* Econometrica **66** (1998), 1127–1162.

5. A. J. Lawrence and P. A. W. Lewis: *Reversed residuals in autoregressive time series analysis.* Journal of Time Series Analysis (1992), 253–266.

6. M. Meitz and T. Teräsvirta: *Evaluating models of autoregressive conditional duration.* SSE/EFI Working Papers Series in Economics and Finance, Technical Report No. 557, Stockholm, 2004.

7. D. Nicholls and B. Quinn: *Random coefficient autoregressive models: An introduction.* Lecture Notes in Statistics, Springer-Verlag, New York, 1982.

8. R. J. Serfling: *Approximation Theorems of Mathematical Statistics.* John Wiley & Sons, New York, 1980.

9. V. Stojanović: *P-ACD model of the financial sequences dynamics in stopping time.* Proceedings of the Conference SYM-OP-IS (2005), 501–504. (in Serbian)

10. B. Popović and V. Stojanović: *Stochastic modeling of the irregular time dynamics on the Serbian stock market.* Proceedings of the VIII Balkan Conference on Operational Research, 2007.

Faculty of Economics
Department of Mathematics and Informatics
Kolašinska 160
28000 Kosovska Mitrovica, Serbia
vlada70@verat.net

Faculty of Sciences and Mathematics
Department of Mathematics and Informatics
P. O. Box 224, Višegradska 33
18000 Niš, Serbia
biljanap@ban.junis.ni.ac.yu