FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 23 (2008), 29–37

CHRISTOFFEL-DARBOUX FORMULA FOR ORTHOGONAL TRIGONOMETRIC POLYNOMIALS OF SEMI-INTEGER DEGREE*

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Abstract. In this paper we introduce orthonormal trigonometric polynomials of semi-integer degree with respect to a weight function on $[-\pi,\pi)$ and prove the Christoffel-Darboux formula for a such orthonormal trigonometric system.

1. Introduction

Let denote by $\mathcal{T}_n^{1/2}$ linear span of the following trigonometric functions

$$\cos x/2, \sin x/2, \cos(1+1/2)x, \sin(1+1/2)x, \ldots, \cos(n+1/2)x, \sin(n+1/2)x.$$

Elements of $\mathcal{T}_n^{1/2}$, i.e., the trigonometric functions of the following form

$$A_{n+1/2}(x) = \sum_{\nu=0}^{n} \left(c_\nu \cos\left(\nu + \frac{1}{2}\right) x + d_\nu \sin\left(\nu + \frac{1}{2}\right) x \right),$$

where $c_\nu, d_\nu \in \mathbb{R}$, $|c_n| + |d_n| \neq 0$, are called *trigonometric polynomials of semi-integer degree.*

For a given nonnegative weight function weight function $w(x)$ on $[-\pi,\pi)$, which equals zero only on a set of the Lebesgue measure zero,

$$(f,g) = \int_0^{2\pi} f(x)g(x)w(x)\,dx, \tag{1.1}$$

denotes the corresponding inner product of the functions f and g. For the given scalar product (1.1), the problem of finding $A_{n+1/2} \in \mathcal{T}_n^{1/2}$, such that

$$\int_{-\pi}^{\pi} A_n(x)t(x)w(x)\,dt = 0, \quad t \in \mathcal{T}_{n-1}^{1/2},$$

Received February 15, 2008.
2000 *Mathematics Subject Classification.* Primary 42C05; Secondary 42A05
*The authors were supported in part by the Serbian Ministry of Science and Technological Development (Project: Orthogonal Systems and Applications, grant number #144004)

was considered at first in [3], and in detail in [1] and [2]. It turns out that this problem has unique solution if the leading coefficients c_n and d_n are fixed in advance (see [3, §3.]).

Those orthogonal trigonometric systems have applications in construction of quadrature formulas with maximal trigonometric degree of exactness.

In cite [1] and [2] the two choices of leading coefficients were considered. For the first choice $c_n = 1$, $d_n = 0$, we denote orthogonal trigonometric polynomial of semi-integer degree by $A^C_{n+1/2}$, and for the second choice $c_n = 0$ and $d_n = 1$ by $A^S_{n+1/2}$. For the expanded forms of $A^C_{n+1/2}$ and $A^S_{n+1/2}$ we use the following notation

$$\text{(1.2)}\quad A^C_{n+1/2}(x) = \cos\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left(c_\nu^{(n)}\cos\left(\nu+\frac{1}{2}\right)x + d_\nu^{(n)}\sin\left(\nu+\frac{1}{2}\right)x\right),$$

$$A^S_{n+1/2}(x) = \sin\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left(f_\nu^{(n)}\cos\left(\nu+\frac{1}{2}\right)x + g_\nu^{(n)}\sin\left(\nu+\frac{1}{2}\right)x\right).$$

In [1] it was proved that orthogonal trigonometric polynomials of semi-integer degree $A^C_{k+1/2}(x)$ and $A^S_{k+1/2}(x)$, $k \in \mathbb{N}$, satisfy the following five-term recurrence relations:

$$\text{(1.3)}\quad \begin{aligned} A^C_{k+1/2}(x) &= (2\cos x - \alpha_k^{(1)})A^C_{k-1/2}(x) - \beta_k^{(1)}A^S_{k-1/2}(x) \\ &\quad -\alpha_k^{(2)}A^C_{k-3/2}(x) - \beta_k^{(2)}A^S_{k-3/2}(x), \end{aligned}$$

and

$$\text{(1.4)}\quad \begin{aligned} A^S_{k+1/2}(x) &= (2\cos x - \delta_k^{(1)})A^S_{k-1/2}(x) - \gamma_k^{(1)}A^C_{k-1/2}(x) \\ &\quad -\gamma_k^{(2)}A^C_{k-3/2}(x) - \delta_k^{(2)}A^S_{k-3/2}(x), \end{aligned}$$

where recurrence coefficients are given by $\alpha_1^{(2)} = \beta_1^{(2)} = \gamma_1^{(2)} = \delta_1^{(2)} = 0$, and

$$\text{(1.5)}\quad \begin{aligned} \alpha_k^{(1)} &= \frac{I^S_{k-1}J^C_{k-1} - I_{k-1}J_{k-1}}{D_{k-1}}, & \alpha_k^{(2)} &= \frac{I^C_{k-1}I^S_{k-2} - I_{k-1}I_{k-2}}{D_{k-2}}, \\ \beta_k^{(1)} &= \frac{I^C_{k-1}J_{k-1} - I_{k-1}J^C_{k-1}}{D_{k-1}}, & \beta_k^{(2)} &= \frac{I_{k-1}I^C_{k-2} - I^C_{k-1}I_{k-2}}{D_{k-2}}, \\ \gamma_k^{(1)} &= \frac{I^S_{k-1}J_{k-1} - I_{k-1}J^S_{k-1}}{D_{k-1}}, & \gamma_k^{(2)} &= \frac{I_{k-1}I^S_{k-2} - I^S_{k-1}I_{k-2}}{D_{k-2}}, \\ \delta_k^{(1)} &= \frac{I^C_{k-1}J^S_{k-1} - I_{k-1}J_{k-1}}{D_{k-1}}, & \delta_k^{(2)} &= \frac{I^S_{k-1}I^C_{k-2} - I_{k-1}I_{k-2}}{D_{k-2}}, \end{aligned}$$

where $D_{k-j} = I^C_{k-j}I^S_{k-j} - I^2_{k-j}$, $j = 1, 2$, and

$$\text{(1.6)}\quad \begin{aligned} I^C_\nu &= (A^C_{\nu+1/2}, A^C_{\nu+1/2}), & J^C_\nu &= (2\cos x A^C_{\nu+1/2}, A^C_{\nu+1/2}), \\ I^S_\nu &= (A^S_{\nu+1/2}, A^S_{\nu+1/2}), & J^S_\nu &= (2\cos x A^S_{\nu+1/2}, A^S_{\nu+1/2}), \\ I_\nu &= (A^C_{\nu+1/2}, A^S_{\nu+1/2}), & J_\nu &= (2\cos x A^C_{\nu+1/2}, A^S_{\nu+1/2}). \end{aligned}$$

For some special weight functions explicit formulas for five-term recurrence coefficients as well as explicit formulas for coefficients of expanded forms (1.2) were presented in [2].

In this paper, in Section 2, the orthonormal trigonometric polynomials of semi-integer degree are introduced and the Christoffel-Darboux formula for such orthonormal trigonometric system is proved.

2. Main Results

Let us denote

$$m_n = \begin{bmatrix} I_n^C & I_n \\ I_n & I_n^S \end{bmatrix}. \tag{2.1}$$

Lemma 2.1. *The matrix m_n, $n \in \mathbb{N}_0$, given by* (2.1) *is positive definite.*

Proof. Let a_1, a_2 be arbitrary real numbers such that at least one of them differs from zero. Let denote $\mathbf{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}^T$ and $t(x) = a_1 A_{n+1/2}^C(x) + a_2 A_{n+1/2}^S(x)$. Then

$$\mathbf{a}^T m_n \mathbf{a} = \int_{-\pi}^{\pi} t^2(x) w(x)\, dx > 0,$$

since $t(x)$ is nonzero trigonometric polynomial of semi-integer degree $n+1/2$. Therefore, the matrix m_n is positive definite for all $n \in \mathbb{N}_0$. $\square$

By $\widetilde{m}_n$, $n \in \mathbb{N}_0$, we denote the positive definite square root of m_n, i.e., the unique positive definite matrix such that $m_n = \widetilde{m}_n \widetilde{m}_n$ (see [4]). Since m_n is a symmetric matrix, the matrix $\widetilde{m}_n$ is also symmetric, i.e., it has the following form

$$\widetilde{m}_n = \begin{bmatrix} a_n & b_n \\ b_n & c_n \end{bmatrix},$$

where

$$a_n^2 + b_n^2 = I_n^C, \quad a_n b_n + b_n c_n = I_n \quad \text{and} \quad b_n^2 + c_n^2 = I_n^S. \tag{2.2}$$

Let introduce the following trigonometric polynomials of semi-integer degree $n + 1/2$:

$$\begin{aligned} \widetilde{A}_{n+1/2}^C(x) &= \frac{c_n}{a_n c_n - b_n^2} A_{n+1/2}^C(x) - \frac{b_n}{a_n c_n - b_n^2} A_{n+1/2}^S(x), \\ \widetilde{A}_{n+1/2}^S(x) &= \frac{a_n}{a_n c_n - b_n^2} A_{n+1/2}^S(x) - \frac{b_n}{a_n c_n - b_n^2} A_{n+1/2}^C(x). \end{aligned} \tag{2.3}$$

We call the trigonometric polynomials $\widetilde{A}_{n+1/2}^C$ and $\widetilde{A}_{n+1/2}^C$, $n \in \mathbb{N}_0$, *orthonormal trigonometric polynomials of semi-integer degree.* The reason for that name lies in the following simple property.

Theorem 2.1. *If* $\widetilde{A}^C_{n+1/2}(x)$ *and* $\widetilde{A}^C_{n+1/2}(x)$, $n \in \mathbb{N}_0$, *are given by* (2.3), *then the following equalities*

$$(\widetilde{A}^C_{n+1/2}, \widetilde{A}^C_{n+1/2}) = 1, \quad (\widetilde{A}^C_{n+1/2}, \widetilde{A}^S_{n+1/2}) = 0 \quad \text{and} \quad (\widetilde{A}^S_{n+1/2}, \widetilde{A}^S_{n+1/2}) = 1$$

hold.

Proof. By direct calculation we have

$$\begin{aligned}
(\widetilde{A}^C_{n+1/2}, \widetilde{A}^C_{n+1/2}) &= \left(\frac{c_n A^C_{n+1/2} - b_n A^S_{n+1/2}}{a_n c_n - b_n^2}, \frac{c_n A^C_{n+1/2} - b_n A^S_{n+1/2}}{a_n c_n - b_n^2}\right) \\
&= \frac{c_n^2 I_n^C - 2b_n c_n I_n + b_n^2 I_n^S}{(a_n c_n - b_n^2)^2} \\
&= \frac{c_n^2(a_n^2 + b_n^2) - 2b_n c_n(a_n b_n + b_n c_n) + b_n^2(b_n^2 + c_n^2)}{(a_n c_n - b_n^2)^2} \\
&= \frac{a_n^2 c_n^2 - 2a_n b_n^2 c_n + b_n^4}{(a_n c_n - b_n^2)^2} = 1,
\end{aligned}$$

so we get the first equality. The second and the third equalities can be proved analogously. □

It is easy to see that the following equalities

$$\begin{aligned}
A^C_{n+1/2}(x) &= a_n \widetilde{A}^C_{n+1/2}(x) + b_n \widetilde{A}^S_{n+1/2}(x), \\
A^S_{n+1/2}(x) &= b_n \widetilde{A}^C_{n+1/2}(x) + c_n \widetilde{A}^S_{n+1/2}(x)
\end{aligned} \tag{2.4}$$

hold.

Theorem 2.2. *The orthonormal trigonometric polynomials of semi-integer degree* $\widetilde{A}^C_{n+1/2}(x)$ *and* $\widetilde{A}^S_{n+1/2}(x)$, $n \in \mathbb{N}$ *satisfy the following recurrence relations:*

$$\begin{aligned}
2\cos x \widetilde{A}^C_{n-1/2}(x) &= \frac{a_n c_{n-1} - b_n b_{n-1}}{a_{n-1}c_{n-1} - b_{n-1}^2} \widetilde{A}^C_{n+1/2}(x) + \frac{b_n c_{n-1} - b_{n-1} c_n}{a_{n-1}c_{n-1} - b_{n-1}^2} \widetilde{A}^S_{n+1/2}(x) \\
&+ \frac{c_{n-1}^2 J^C_{n-1} + b_{n-1}^2 J^S_{n-1} - 2b_{n-1}c_{n-1}J_{n-1}}{a_{n-1}c_{n-1} - b_{n-1}^2} \widetilde{A}^C_{n-1/2}(x) \\
&+ \frac{(b_{n-1}^2 + a_{n-1}c_{n-1})J_{n-1} - b_{n-1}c_{n-1}J^C_{n-1} - a_{n-1}b_{n-1}J^S_{n-1}}{a_{n-1}c_{n-1} - b_{n-1}^2} \widetilde{A}^S_{n-1/2}(x) \\
&+ \frac{a_{n-1}c_{n-2} - b_{n-1}b_{n-2}}{a_{n-2}c_{n-2} - b_{n-2}^2} \widetilde{A}^C_{n-3/2}(x) + \frac{a_{n-2}b_{n-1} - a_{n-1}b_{n-2}}{a_{n-2}c_{n-2} - b_{n-2}^2} \widetilde{A}^S_{n-3/2}(x)
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
2\cos x \widetilde{A}^S_{n-1/2}(x) &= \frac{a_{n-1} b_n - a_n b_{n-1}}{a_{n-1}c_{n-1} - b_{n-1}^2} \widetilde{A}^C_{n+1/2}(x) + \frac{a_{n-1} c_n - b_{n-1} b_n}{a_{n-1}c_{n-1} - b_{n-1}^2} \widetilde{A}^S_{n+1/2}(x) \\
&+ \frac{(b_{n-1}^2 + a_{n-1}c_{n-1})J_{n-1} - b_{n-1}c_{n-1}J^C_{n-1} - a_{n-1}b_{n-1}J^S_{n-1}}{a_{n-1}c_{n-1} - b_{n-1}^2} \widetilde{A}^C_{n-1/2}(x)
\end{aligned} \tag{2.6}$$

$$+\frac{b_{n-1}^2 J_{n-1}^C + a_{n-1}^2 J_{n-1}^S - 2a_{n-1}b_{n-1}J_{n-1}}{a_{n-1}c_{n-1} - b_{n-1}^2}\widetilde{A}_{n-1/2}^S(x)$$
$$+\frac{b_{n-1}c_{n-2} - b_{n-2}c_{n-1}}{a_{n-2}c_{n-2} - b_{n-2}^2}\widetilde{A}_{n-3/2}^C(x) + \frac{a_{n-2}c_{n-1} - b_{n-1}b_{n-2}}{a_{n-2}c_{n-2} - b_{n-2}^2}\widetilde{A}_{n-3/2}^S(x).$$

Specially for $n = 1$ coefficients multiplying $\widetilde{A}_{-1/2}^C(x)$ and $\widetilde{A}_{-1/2}^S(x)$ in both recurrence relation are equal to zero.

Proof. Using connections (2.4) from (1.3) and (1.4) as well as (2.2), (1.5) and (1.6), solving obtained linear system for $\cos x\widetilde{A}_{n-1/2}^C$ and $\cos x\widetilde{A}_{n-1/2}^S$ we get what is stated. The statement for $n = 1$ follows directly from the fact that $\alpha_1^{(2)} = \beta_1^{(2)} = \gamma_1^{(2)} = \delta_1^{(2)} = 0$ in (1.3) and (1.4). □

Theorem 2.3. (Christoffel-Darboux formula) *For the orthonormal trigonometric polynomials of semi-integer degree the following formula*

$$(2.7)\qquad 2(\cos x - \cos y)\sum_{k=0}^{n}\left(\widetilde{A}_{k+1/2}^C(x)\widetilde{A}_{k+1/2}^C(y) + \widetilde{A}_{k+1/2}^S(x)\widetilde{A}_{k+1/2}^S(y)\right)$$
$$= \frac{\left(a_{n+1}c_n - b_n b_{n+1}\right)\left(\widetilde{A}_{n+3/2}^C(x)\widetilde{A}_{n+1/2}^C(y) - \widetilde{A}_{n+3/2}^C(y)\widetilde{A}_{n+1/2}^C(x)\right)}{(a_n c_n - b_n^2)}$$
$$+\frac{\left(b_{n+1}c_n - b_n c_{n+1}\right)\left(\widetilde{A}_{n+3/2}^S(x)\widetilde{A}_{n+1/2}^C(y) - \widetilde{A}_{n+3/2}^S(y)\widetilde{A}_{n+1/2}^C(x)\right)}{(a_n c_n - b_n^2)}$$
$$+\frac{\left(a_n b_{n+1} - a_{n+1}b_n\right)\left(\widetilde{A}_{n+3/2}^C(x)\widetilde{A}_{n+1/2}^S(y) - \widetilde{A}_{n+3/2}^C(y)\widetilde{A}_{n+1/2}^S(x)\right)}{(a_n c_n - b_n^2)}$$
$$+\frac{\left(a_n c_{n+1} - b_n b_{n+1}\right)\left(\widetilde{A}_{n+3/2}^S(x)\widetilde{A}_{n+1/2}^S(y) - \widetilde{A}_{n+3/2}^S(y)\widetilde{A}_{n+1/2}^S(x)\right)}{(a_n c_n - b_n^2)}$$

holds.

Proof. Let us introduce some notation

$$\begin{aligned}
e_n &= c_n^2 J_n^C + b_n^2 J_n^S - 2b_n c_n J_n, \quad f_n = (b_n^2 + a_n c_n)J_n - b_n c_n J_n^C - a_n b_n J_n^S,\\
g_n &= b_n^2 J_n^C + a_n^2 J_n^S - 2a_n b_n J_n, \quad h_n = a_n c_n - b_n^2,\\
\Delta_n^a &= a_n b_{n+1} - a_{n+1}b_n, \quad \Delta_n^c = c_n b_{n+1} - c_{n+1}b_n,\\
\Delta_n^1 &= a_{n+1}c_n - b_{n+1}b_n, \quad \Delta_n^2 = a_n c_{n+1} - b_n b_{n+1}.
\end{aligned}$$

We prove theorem using mathematical induction. For $n = 0$, using Theorem 2.2, we have

$$2(\cos x - \cos y)\left(\widetilde{A}_{1/2}^C(x)\widetilde{A}_{1/2}^C(y) + \widetilde{A}_{1/2}^S(x)\widetilde{A}_{1/2}^S(y)\right)$$
$$= \left[\frac{\Delta_0^1}{h_0}\widetilde{A}_{3/2}^C + \frac{\Delta_0^c}{h_0}\widetilde{A}_{3/2}^S + \frac{e_0}{h_0}\widetilde{A}_{1/2}^C + \frac{f_0}{h_0}\widetilde{A}_{1/2}^S\right](x)\widetilde{A}_{1/2}^C(y)$$

$$
\begin{aligned}
&-\widetilde{A}^C_{1/2}(x)\left[\frac{\Delta_0^1}{h_0}\widetilde{A}^C_{3/2}+\frac{\Delta_0^c}{h_0}\widetilde{A}^S_{3/2}+\frac{e_0}{h_0}\widetilde{A}^C_{1/2}+\frac{f_0}{h_0}\widetilde{A}^S_{3/2}\right](y)\\
&+\left[\frac{\Delta_0^a}{h_0}\widetilde{A}^C_{3/2}+\frac{\Delta_0^2}{h_0}\widetilde{A}^S_{3/2}+\frac{f_0}{h_0}\widetilde{A}^C_{1/2}+\frac{g_0}{h_0}\widetilde{A}^S_{1/2}\right](x)\widetilde{A}^S_{1/2}(y)\\
&-\widetilde{A}^S_{1/2}(x)\left[\frac{\Delta_0^a}{h_0}\widetilde{A}^C_{3/2}+\frac{\Delta_0^2}{h_0}\widetilde{A}^S_{3/2}+\frac{f_0}{h_0}\widetilde{A}^C_{1/2}+\frac{g_0}{h_0}\widetilde{A}^S_{1/2}\right](y)\\
=&\ \frac{\Delta_0^1\big(\widetilde{A}^C_{3/2}(x)\widetilde{A}^C_{1/2}(y)-\widetilde{A}^C_{3/2}(y)\widetilde{A}^C_{1/2}(x)\big)}{h_0}\\
&+\frac{\Delta_0^c\big(\widetilde{A}^S_{3/2}(x)\widetilde{A}^C_{1/2}(y)-\widetilde{A}^S_{3/2}(y)\widetilde{A}^C_{1/2}(x)\big)}{h_0}\\
&+\frac{\Delta_0^a\big(\widetilde{A}^C_{3/2}(x)\widetilde{A}^S_{1/2}(y)-\widetilde{A}^C_{3/2}(y)\widetilde{A}^S_{1/2}(x)\big)}{h_0}\\
&+\frac{\Delta_0^2\big(\widetilde{A}^S_{3/2}(x)\widetilde{A}^S_{1/2}(y)-\widetilde{A}^S_{3/2}(y)\widetilde{A}^S_{1/2}(x)\big)}{h_0}.
\end{aligned}
$$

Suppose it is true for some $n=m$, then we have

$$
\begin{aligned}
&2(\cos x-\cos y)\left(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^C_{m+3/2}(y)+\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^S_{m+3/2}(y)\right)\\
=&\left[\frac{\Delta^1_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+5/2}+\frac{\Delta^c_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+5/2}+\frac{e_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+3/2}+\frac{f_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+3/2}\right.\\
&\left.+\frac{\Delta^1_m}{h_m}\widetilde{A}^C_{m+1/2}+\frac{\Delta^a_m}{h_m}\widetilde{A}^S_{m+1/2}\right](x)\widetilde{A}^C_{m+3/2}(y)\\
&-\widetilde{A}^C_{m+3/2}(x)\left[\frac{\Delta^1_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+5/2}+\frac{\Delta^c_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+5/2}+\frac{e_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+3/2}\right.\\
&\left.+\frac{f_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+3/2}+\frac{\Delta^1_m}{h_m}\widetilde{A}^C_{m+1/2}+\frac{\Delta^a_m}{h_m}\widetilde{A}^S_{m+1/2}\right](y)\\
&+\left[\frac{\Delta^a_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+5/2}+\frac{\Delta^2_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+5/2}+\frac{f_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+3/2}\right.\\
&\left.+\frac{g_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+3/2}+\frac{\Delta^c_m}{h_m}\widetilde{A}^C_{m+1/2}+\frac{\Delta^2_m}{h_m}\widetilde{A}^S_{m+1/2}\right](x)\widetilde{A}^S_{m+3/2}(y)\\
&-\widetilde{A}^S_{m+3/2}(x)\left[\frac{\Delta^a_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+5/2}+\frac{\Delta^2_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+5/2}+\frac{f_{m+1}}{h_{m+1}}\widetilde{A}^C_{m+3/2}\right.\\
&\left.+\frac{g_{m+1}}{h_{m+1}}\widetilde{A}^S_{m+3/2}+\frac{\Delta^c_m}{h_m}\widetilde{A}^C_{m+1/2}+\frac{\Delta^2_m}{h_m}\widetilde{A}^S_{m+1/2}\right](y)\\
=&\ \frac{\Delta^1_{m+1}\big(\widetilde{A}^C_{m+5/2}(x)\widetilde{A}^C_{m+3/2}(y)-\widetilde{A}^C_{m+5/2}(y)\widetilde{A}^C_{m+3/2}(x)\big)}{h_{m+1}}\\
&+\frac{\Delta^c_{m+1}\big(\widetilde{A}^S_{m+5/2}(x)\widetilde{A}^C_{m+3/2}(y)-\widetilde{A}^S_{m+5/2}(y)\widetilde{A}^C_{m+3/2}(x)\big)}{h_{m+1}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Delta^a_{m+1}\big(\widetilde{A}^C_{m+5/2}(x)\widetilde{A}^S_{m+3/2}(y)-\widetilde{A}^C_{m+5/2}(y)\widetilde{A}^S_{m+3/2}(x)\big)}{h_{m+1}}\\
&+\frac{\Delta^2_{m+1}\big(\widetilde{A}^S_{m+5/2}(x)\widetilde{A}^S_{m+3/2}(y)-\widetilde{A}^S_{m+5/2}(y)\widetilde{A}^S_{m+3/2}(x)\big)}{h_{m+1}}\\
&-\frac{\Delta^1_{m}\big(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^C_{m+1/2}(y)-\widetilde{A}^C_{m+3/2}(y)\widetilde{A}^C_{m+1/2}(x)\big)}{h_{m}}\\
&-\frac{\Delta^c_{m}\big(\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^C_{m+1/2}(y)-\widetilde{A}^S_{m+3/2}(y)\widetilde{A}^C_{m+1/2}(x)\big)}{h_{m}}\\
&-\frac{\Delta^a_{m}\big(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^S_{m+1/2}(y)-\widetilde{A}^C_{m+3/2}(y)\widetilde{A}^S_{m+1/2}(x)\big)}{h_{m}}\\
&-\frac{\Delta^2_{m}\big(\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^S_{m+1/2}(y)-\widetilde{A}^S_{m+3/2}(y)\widetilde{A}^S_{m+1/2}(x)\big)}{h_{m}}.
\end{aligned}
$$

Now statement readily follows, since we have

$$
\begin{aligned}
&2(\cos x-\cos y)\sum_{k=0}^{m+1}\Big(\widetilde{A}^C_{k+1/2}(x)\widetilde{A}^C_{m+1/2}(y)+\widetilde{A}^S_{m+1/2}(x)\widetilde{A}^S_{m+1/2}(y)\Big)\\
&=2(\cos x-\cos y)\sum_{k=0}^{m}\Big(\widetilde{A}^C_{k+1/2}(x)\widetilde{A}^C_{m+1/2}(y)+\widetilde{A}^S_{m+1/2}(x)\widetilde{A}^S_{m+1/2}(y)\Big)\\
&\quad+2(\cos x-\cos y)\Big(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^C_{m+3/2}(y)+\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^S_{m+3/2}(y)\Big)\\
&=\frac{\Delta^1_{m}\big(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^C_{m+1/2}(y)-\widetilde{A}^C_{m+3/2}(y)\widetilde{A}^C_{m+1/2}(x)\big)}{h_{m}}\\
&\quad+\frac{\Delta^c_{m}\big(\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^C_{m+1/2}(y)-\widetilde{A}^S_{m+3/2}(y)\widetilde{A}^C_{m+1/2}(x)\big)}{h_{m}}\\
&\quad+\frac{\Delta^a_{m}\big(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^S_{m+1/2}(y)-\widetilde{A}^C_{m+3/2}(y)\widetilde{A}^S_{m+1/2}(x)\big)}{h_{m}}\\
&\quad+\frac{\Delta^2_{m}\big(\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^S_{m+1/2}(y)-\widetilde{A}^S_{m+3/2}(y)\widetilde{A}^S_{m+1/2}(x)\big)}{h_{m}}\\
&\quad+\frac{\Delta^1_{m+1}\big(\widetilde{A}^C_{m+5/2}(x)\widetilde{A}^C_{m+3/2}(y)-\widetilde{A}^C_{m+5/2}(y)\widetilde{A}^C_{m+3/2}(x)\big)}{h_{m+1}}\\
&\quad+\frac{\Delta^c_{m+1}\big(\widetilde{A}^S_{m+5/2}(x)\widetilde{A}^C_{m+3/2}(y)-\widetilde{A}^S_{m+5/2}(y)\widetilde{A}^C_{m+3/2}(x)\big)}{h_{m+1}}\\
&\quad+\frac{\Delta^a_{m+1}\big(\widetilde{A}^C_{m+5/2}(x)\widetilde{A}^S_{m+3/2}(y)-\widetilde{A}^C_{m+5/2}(y)\widetilde{A}^S_{m+3/2}(x)\big)}{h_{m+1}}\\
&\quad+\frac{\Delta^2_{m+1}\big(\widetilde{A}^S_{m+5/2}(x)\widetilde{A}^S_{m+3/2}(y)-\widetilde{A}^S_{m+5/2}(y)\widetilde{A}^S_{m+3/2}(x)\big)}{h_{m+1}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Delta_m^1\big(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^C_{m+1/2}(y)-\widetilde{A}^C_{m+3/2}(y)\widetilde{A}^C_{m+1/2}(x)\big)}{h_m}\\
&-\frac{\Delta_m^c\big(\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^C_{m+1/2}(y)-\widetilde{A}^S_{m+3/2}(y)\widetilde{A}^C_{m+1/2}(x)\big)}{h_m}\\
&-\frac{\Delta_m^a\big(\widetilde{A}^C_{m+3/2}(x)\widetilde{A}^S_{m+1/2}(y)-\widetilde{A}^C_{m+3/2}(y)\widetilde{A}^S_{m+1/2}(x)\big)}{h_m}\\
&-\frac{\Delta_m^2\big(\widetilde{A}^S_{m+3/2}(x)\widetilde{A}^S_{m+1/2}(y)-\widetilde{A}^S_{m+3/2}(y)\widetilde{A}^S_{m+1/2}(x)\big)}{h_m}\\
=&\frac{\Delta_{m+1}^1\big(\widetilde{A}^C_{m+5/2}(x)\widetilde{A}^C_{m+3/2}(y)-\widetilde{A}^C_{m+5/2}(y)\widetilde{A}^C_{m+3/2}(x)\big)}{h_{m+1}}\\
&+\frac{\Delta_{m+1}^c\big(\widetilde{A}^S_{m+5/2}(x)\widetilde{A}^C_{m+3/2}(y)-\widetilde{A}^S_{m+5/2}(y)\widetilde{A}^C_{m+3/2}(x)\big)}{h_{m+1}}\\
&+\frac{\Delta_{m+1}^a\big(\widetilde{A}^C_{m+5/2}(x)\widetilde{A}^S_{m+3/2}(y)-\widetilde{A}^C_{m+5/2}(y)\widetilde{A}^S_{m+3/2}(x)\big)}{h_{m+1}}\\
&+\frac{\Delta_{m+1}^2\big(\widetilde{A}^S_{m+5/2}(x)\widetilde{A}^S_{m+3/2}(y)-\widetilde{A}^S_{m+5/2}(y)\widetilde{A}^S_{m+3/2}(x)\big)}{h_{m+1}}. \qquad \square
\end{aligned}
$$

REFERENCES

1. G. V. Milovanović, A. S. Cvetković and M. P. Stanić: *Trigonometric orthogonal systems and quadrature formulae.* Comput. Math. Appl. **56** (11) (2008), 2915–2931.
2. G. V. Milovanović, A. S. Cvetković and M. P. Stanić: *Explicit formulas for five-term recurrence coefficients of orthogonal trigonometric polynomials of semi-integer degree.* Appl. Math. Comput. **198** (2008), 559–573.
3. A. H. Turetzkii: *On quadrature formulae that are exact for trigonometric polynomials.* East J. Approx. **11** (2005), 337–359 (translation in English from Uchenye Zapiski, Vypusk **1**(149), Seria math. Theory of Functions, Collection of papers, Izdatel'stvo Belgosuniversiteta imeni V.I. Lenina, Minsk, (1959), 31–54).
4. E. W. Weisstein: *Positive Definite Matrix.* From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/PositiveDefiniteMatrix.html`

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