VARIATION DETRACTING PROPERTY OF THE BÉZIER TYPE OPERATORS

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Abstract. The aim of this note is to prove that, under additional assumptions, the Bézier variant of any positive linear operator of discrete type which satisfies the conditions of Bohman-Korovkin theorem possesses the variation detracting property. A special attention is given to Mastroianni operators.

1. Introduction

The first research which deals with the variation detracting property in approximation theory was due to G.G. Lorentz [6] for Bernstein polynomials. Recently, other authors studied this property on Szász-Mirakjan [3] and Baskakov operators [2]. Motivated by this research direction, in Section 2, we will give sufficient conditions under which a general class of Bézier type operators enjoys the variation detracting property. In Section 3, we investigate this property on Mastroianni operators.

We set \( \mathbb{R}_+ := [0, \infty) \) and \( \mathbb{N}_0 := \{0\} \cup \mathbb{N} \). For each integer \( n \geq 1 \) we consider an unbounded sequence \((x_{n,k})_{k \in \mathbb{N}_0}\) of nodes on \( \mathbb{R}_+ \) such that \( 0 = x_{n,0} < x_{n,1} < x_{n,2} < \ldots \). As usually, we denote by \( e_j \) the monomial of \( j \)-degree, \( j \in \mathbb{N}_0 \). Let \((l_n)_{n \geq 1}\) be a sequence of positive linear operators of discrete type, defined by

\[
(l_n f)(x) = \sum_{k=0}^{\infty} u_{n,k}(x) f(x_{n,k}), \quad x \in \mathbb{R}_+, \ f \in C(\mathbb{R}_+),
\]

where \((u_{n,k})_{k \in \mathbb{N}_0}\) is a family of differentiable functions on \( \mathbb{R}_+ \) verifying the following conditions

\[
u_{n,k}(x) \geq 0, \ x \in \mathbb{R}_+,
\]

\[
\sum_{k=0}^{\infty} u_{n,k}(x) = e_0(x), \ x \in \mathbb{R}_+,
\]
\[
\sum_{k=0}^{\infty} u_{n,k} (x) x_n = e_1 (x) + \varphi_n (x), \quad x \in \mathbb{R}_+,
\]

where \(\varphi_n, \psi_n \in C (\mathbb{R}_+)\), \(\lim_{n \to \infty} \varphi_n (x) = 0\) and \(\lim_{n \to \infty} \psi_n (x) = 0\) uniformly on any compact \(K \subset \mathbb{R}_+\).

Further on, we define the Bézier variant of the above operators.

Let \(\alpha\) be a real number, \(\alpha \geq 1\). We consider the operators \(L_{n,\alpha}, n \in \mathbb{N}\), given as follows

\[
(L_{n,\alpha} f) (x) = \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k} (x) f(x_n), \quad x \in \mathbb{R}_+, \ f \in C (\mathbb{R}_+),
\]

where

\[
Q^{(\alpha)}_{n,k} (x) = S^\alpha_{n,k} (x) - S^\alpha_{n,k+1} (x)
\]

and

\[
S_{n,k} (x) = \sum_{j=k}^{\infty} u_{n,j} (x)
\]

for every \(x \in \mathbb{R}_+\) and \(k \in \mathbb{N}_0\).

During the last years many authors studied linear operators of Bézier type. By using probabilistic methods, in [1] has been estimated the rate of pointwise convergence of a class of Bézier type operators for functions of bounded variation. Also was approached the Bézier variants of Baskakov [8], Baskakov-Kantorovich [5], Szász-Durrmeyer [4] operators.

Let \(BV (\mathbb{R}_+)\) be the class of all functions of bounded variation on \(\mathbb{R}_+\). This space can be endowed both with the seminorm \(\| \cdot \|_{BV (\mathbb{R}_+)}\) and with the \(BV\)-norm, \(\| \cdot \|_{BV (\mathbb{R}_+)}\), given respectively by

\[
\| f \|_{BV (\mathbb{R}_+)} := \| f \|_{BV (\mathbb{R}_+)} = \lim_{\mu \to \infty} V_{\mathbb{R}_+} [f],
\]

where \(a\) is a fixed point in \(\mathbb{R}_+\).

We say that a sequence of linear operators \((L_n)_{n \in \mathbb{N}}\) enjoys the variation detracting property if for each \(n \in \mathbb{N}\) the following inequality holds

\[
\| L_n f \|_{BV (\mathbb{R}_+)} \leq \| f \|_{BV (\mathbb{R}_+)}
\]

for any \(f \in BV (\mathbb{R}_+)\).
2. Main Result

**Theorem 2.1.** Let the operator \( L_{n,\alpha} \) be defined by (1.6) such that the conditions (1.2)-(1.5) hold true. Also one has

\[
(2.1) \quad u_{n,0}(0) = 1,
\]
and

\[
(2.2) \quad S_{n,k}, \text{ defined by (1.7), is monotone on } \mathbb{R}_+,
\]
for any \( n, k \in \mathbb{N} \). If \( f \in BV(\mathbb{R}_+) \), then

\[
(2.3) \quad |L_{n,\alpha}f|_{BV(\mathbb{R}_+)} \leq |f|_{BV(\mathbb{R}_+)},
\]
and

\[
(2.4) \quad \|L_{n,\alpha}f\|_{BV(\mathbb{R}_+)} \leq \|f\|_{BV(\mathbb{R}_+)},
\]

**Proof.** From (1.3) and (1.6) we obtain

\[
\frac{d}{dx}(L_{n,\alpha}f)(x) = \frac{d}{dx}(S_{n,0}^\alpha)(x)f(x_n,0) + \sum_{k=1}^{\infty} \frac{d}{dx}(S_{n,k}^\alpha)(x)f(x_n,k) - \sum_{k=0}^{\infty} \frac{d}{dx}(S_{n,k+1}^\alpha)(x)f(x_n,k),
\]
which yields

\[
(2.5) \quad \frac{d}{dx}(L_{n,\alpha}f)(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(S_{n,k+1}^\alpha)(x)(f(x_n,k+1) - f(x_n,k)),
\]
for all \( n \in \mathbb{N}, f \in BV(\mathbb{R}_+) \) and \( x \in \mathbb{R}_+ \).

Let \( (x_j)_{j \in \mathbb{N}_0} \) be an arbitrary unbounded sequence of nodes on \( \mathbb{R}_+ \) such that \( 0 = x_0 < x_1 < x_2 < x_3 < \ldots \).

By using (2.2) and (2.5) we can write

\[
\sum_{j=1}^{\infty} \left| (L_{n,\alpha}f)(x_j) - (L_{n,\alpha}f)(x_{j-1}) \right| = \sum_{j=1}^{\infty} \left| \int_{x_{j-1}}^{x_j} \frac{d}{dx}(L_{n,\alpha}f)(x)dx \right|
\]
\[
= \sum_{j=1}^{\infty} \left| \sum_{k=0}^{\infty} (f(x_n,k+1) - f(x_n,k)) \int_{x_{j-1}}^{x_j} \frac{d}{dx}(S_{n,k+1}^\alpha)(x)dx \right|
\]
\[
\leq \sum_{k=0}^{\infty} |f(x_n,k+1) - f(x_n,k)| \sum_{j=1}^{\infty} \left| \int_{x_{j-1}}^{x_j} \frac{d}{dx}(S_{n,k+1}^\alpha)(x)dx \right|
\]
\[ \sum_{k=0}^{\infty} |f(x_{n,k+1}) - f(x_{n,k})| \sum_{j=1}^{\infty} \left( \int_{x_{j-1}}^{x_j} \left| \frac{d}{dx}(S_{n,k+1}^\alpha(x)) \right| dx \right) \]

\[ = \sum_{k=0}^{\infty} |f(x_{n,k+1}) - f(x_{n,k})| \int_{0}^{\infty} \left| \frac{d}{dx}(S_{n,k+1}^\alpha(x)) \right| dx. \]

Taking into account the relations (1.2), (1.3) and (2.1) we get

\[ \lim_{x \to \infty} S_{n,k+1}^\alpha(x) \leq \lim_{x \to \infty} S_{n,0}^\alpha(x) = 1 \]

and

\[ S_{n,k+1}^\alpha(0) = 0. \]

Consequently, by using (1.7), we obtain

\[ \int_{0}^{\infty} \left| \frac{d}{dx}(S_{n,k+1}^\alpha(x)) \right| dx \leq 1. \]

It follows that

\[ \sum_{j=1}^{\infty} |(L_{n,\alpha}f)(x_j) - (L_{n,\alpha}f)(x_{j-1})| \leq \sum_{k=0}^{\infty} |f(x_{n,k+1}) - f(x_{n,k})| \leq |f|_{BV(R_+)} \]

and (2.3) is proved.

Further on, using (1.3) and (2.1) we have \( S_{n,k+1}(0) = 0 \) for all \( k \in \mathbb{N} \) and

\[ (L_{n,\alpha}f)(0) = S_{n,0}(0)f(x_{n,0}) = f(0). \]

The above relation and (2.3) complete the proof. \( \square \)

**Remark 2.1.** Choosing \( \alpha = 1 \) in (1.6) the operator \( L_{n,1} \) becomes \( l_n \). If the conditions (2.1) and (2.2) are satisfied then the operator \( l_n \) has the variation detracting property, too.

### 3. On Mastroianni operators

We refer here at a general class of operators introduced and studied by G. Mastroianni [7].

Let \( x_{n,k} = \frac{k}{n} \) and \( u_{n,k} = (-1)^k e_1 \phi_n^{(k)} \) for all \( (n,k) \in \mathbb{N} \times \mathbb{N}_0 \), where \( (\phi_n)_{n \in \mathbb{N}} \) is a sequence of real functions on \( \mathbb{R}_+ \) which are infinitely differentiable and completely monotone on \( \mathbb{R}_+ \) verifying the following conditions

1. \( \phi_n(0) = 1 \) for all \( n \in \mathbb{N} \);
2. for each \((n, k) \in \mathbb{N} \times \mathbb{N}_0\), there exists a number \(p(n, k) \in \mathbb{N}\) and a function 
\(\beta_{n,k} : \mathbb{R}_+ \to \mathbb{R}\) such that 
\[
\phi_n^{(i+k)}(x) = (-1)^k \phi_p^{(i)}(x) \beta_{n,k}(x)
\]
and 
\[
\lim_{n \to \infty} \frac{n}{p(n, k)} = \lim_{n \to \infty} \frac{\beta_{n,k}(0)}{n^k} = 1.
\]
In this case the operator \(l_n\) defined by (1.1) becomes the Mastroianni operator, 
\[
(M_n f) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \phi_n^{(k)}(x) f \left( \frac{k}{n} \right).
\]
It is known that these operators satisfy (1.2)-(1.5) and (2.1), (see [7]). Since 
\[
dx S_{n,k}(x) = \sum_{j=k}^{\infty} \frac{(-1)^j}{j!} x^j \phi_n^{(j)}(x) = \frac{(-1)^k}{(k-1)!} x^{k-1} \phi_n^{(k)}(x) \geq 0,
\]
for all \(x \in \mathbb{R}_+\) and \(n, k \in \mathbb{N}\), (2.2) also holds true.

On the basis of Theorem 2.1 we can assert that the Bézier type of Mastroianni operators enjoy the variation detracting property.

### 3.1. Particular cases

Mastroianni operators include some well-known classical sequences of linear positive operators.

1. Choosing \(\phi_n(x) = e^{-nx}\), \(p(n, k) = n\) and \(\beta_{n,k}(x) = n^k\), the operators (3.1) reduce to the Szász-Favard-Mirakjan operators. In this case 
\[
dx S_{n,k}(x) = nu_{n,k-1}(x), \quad k \in \mathbb{N}.
\]

2. Choosing \(\phi_n(x) = (1+x)^{-n}\), \(p(n, k) = n + k\) and 
\[
\beta_{n,k}(x) = n(n+1) \ldots (n+k-1)(1+x)^{-k},
\]
the operators (3.1) reduce to the Baskakov operators. In this case 
\[
dx S_{n,k}(x) = nu_{n+1,k-1}(x), \quad k \in \mathbb{N}.
\]
REFERENCES


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