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VARIATION DETRACTING PROPERTY OF THE BÉZIER TYPE OPERATORS

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Abstract. The aim of this note is to prove that, under additional assumptions, the Bézier variant of any positive linear operator of discrete type which satisfies the conditions of Bohman-Korovkin theorem possesses the variation detracting property. A special attention is given to Mastroianni operators.

1. Introduction

The first research which deals with the variation detracting property in approximation theory was due to G.G. Lorentz [6] for Bernstein polynomials. Recently, other authors studied this property on Szász-Mirakjan [3] and Baskakov operators [2]. Motivated by this research direction, in Section 2. we will give sufficient conditions under which a general class of Bézier type operators enjoys the variation detracting property. In Section 3. we investigate this property on Mastroianni operators.

We set $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For each integer $n \geq 1$ we consider an unbounded sequence $(x_{n,k})_{k\in\mathbb{N}_0}$ of nodes on $\mathbb{R}_+$ such that $0 = x_{n,0} < x_{n,1} < x_{n,2} < \ldots$. As usually, we denote by e_j the monomial of j-degree, $j \in \mathbb{N}_0$. Let $(l_n)_{n\geq 1}$ be a sequence of positive linear operators of discrete type, defined by

$$(1.1)\qquad (l_n f)(x) = \sum_{k=0}^{\infty} u_{n,k}(x) f(x_{n,k}), \quad x \in \mathbb{R}_+,\ f \in C(\mathbb{R}_+),$$

where $(u_{n,k})_{k\in\mathbb{N}_0}$ is a family of differentiable functions on $\mathbb{R}_+$ verifying the following conditions

$$(1.2)\qquad u_{n,k}(x) \geq 0,\ x \in \mathbb{R}_+,$$

$$(1.3)\qquad \sum_{k=0}^{\infty} u_{n,k}(x) = e_0(x),\ x \in \mathbb{R}_+,$$

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$$\sum_{k=0}^{\infty} u_{n,k}(x)\, x_{n,k} = e_1(x) + \varphi_n(x), \; x \in \mathbb{R}_+, \tag{1.4}$$

$$\sum_{k=0}^{\infty} u_{n,k}(x)\, x_{n,k}^2 = e_2(x) + \psi_n(x), \; x \in \mathbb{R}_+, \tag{1.5}$$

where $\varphi_n, \psi_n \in C(\mathbb{R}_+)$, $\lim_{n\to\infty} \varphi_n(x) = 0$ and $\lim_{n\to\infty} \psi_n(x) = 0$ uniformly on any compact $K \subset \mathbb{R}_+$.

Further on, we define the Bézier variant of the above operators.

Let α be a real number, $\alpha \geq 1$. We consider the operators $L_{n,\alpha}$, $n \in \mathbb{N}$, given as follows

$$(L_{n,\alpha} f)(x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) f(x_{n,k}), \quad x \in \mathbb{R}_+, \; f \in C(\mathbb{R}_+), \tag{1.6}$$

where

$$Q_{n,k}^{(\alpha)}(x) = S_{n,k}^{\alpha}(x) - S_{n,k+1}^{\alpha}(x)$$

and

$$S_{n,k}(x) = \sum_{j=k}^{\infty} u_{n,j}(x) \tag{1.7}$$

for every $x \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$.

During the last years many authors studied linear operators of Bézier type. By using probabilistic methods, in [1] has been estimated the rate of pointwise convergence of a class of Bézier type operators for functions of bounded variation. Also was approached the Bézier variants of Baskakov [8], Baskakov-Kantorovich [5], Szász-Durrmeyer [4] operators.

Let $BV(\mathbb{R}_+)$ be the class of all functions of bounded variation on $\mathbb{R}_+$. This space can be endowed both with the seminorm $|\cdot|_{BV(\mathbb{R}_+)}$ and with the BV-norm, $\|\cdot\|_{BV(\mathbb{R}_+)}$, given respectively by

$$|f|_{BV(\mathbb{R}_+)} := V_{\mathbb{R}_+}[f] = \lim_{\mu\to\infty} V_{[0,\mu]}[f],$$

$$\|f\|_{BV(\mathbb{R}_+)} := |f|_{BV(\mathbb{R}_+)} + |f(a)|,$$

where a is a fixed point in $\mathbb{R}_+$.

We say that a sequence of linear operators $(L_n)_{n\in\mathbb{N}}$ enjoys the variation detracting property if for each $n \in \mathbb{N}$ the following inequality holds

$$|L_n f|_{BV(\mathbb{R}_+)} \leq |f|_{BV(\mathbb{R}_+)},$$

for any $f \in BV(\mathbb{R}_+)$.

2. Main Result

Theorem 2.1. *Let the operator $L_{n,\alpha}$ be defined by (1.6) such that the conditions (1.2)-(1.5) hold true. Also one has*

$$u_{n,0}(0) = 1, \tag{2.1}$$

and

$$S_{n,k},\ \textit{defined by (1.7), is monotone on } \mathbb{R}_+, \tag{2.2}$$

for any $n, k \in \mathbb{N}$. If $f \in BV(\mathbb{R}_+)$, then

$$|L_{n,\alpha} f|_{BV(\mathbb{R}_+)} \le |f|_{BV(\mathbb{R}_+)}, \tag{2.3}$$

and

$$\|L_{n,\alpha} f\|_{BV(\mathbb{R}_+)} \le \|f\|_{BV(\mathbb{R}_+)}. \tag{2.4}$$

Proof. From (1.3) and (1.6) we obtain

$$\begin{aligned} \frac{d}{dx}(L_{n,\alpha} f)(x) &= \frac{d}{dx}(S^{\alpha}_{n,0})(x) f(x_{n,0}) + \sum_{k=1}^{\infty} \frac{d}{dx}(S^{\alpha}_{n,k})(x) f(x_{n,k}) \\ &- \sum_{k=0}^{\infty} \frac{d}{dx}(S^{\alpha}_{n,k+1})(x) f(x_{n,k}), \end{aligned}$$

which yields

$$\frac{d}{dx}(L_{n,\alpha} f)(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(S^{\alpha}_{n,k+1})(x)(f(x_{n,k+1}) - f(x_{n,k})), \tag{2.5}$$

for all $n \in \mathbb{N}$, $f \in BV(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$.

Let $(x_j)_{j \in \mathbb{N}_0}$ be an arbitrary unbounded sequence of nodes on $\mathbb{R}_+$ such that $0 = x_0 < x_1 < x_2 < x_3 < \ldots$.

By using (2.2) and (2.5) we can write

$$\begin{aligned} \sum_{j=1}^{\infty} |(L_{n,\alpha} f)(x_j) - (L_{n,\alpha} f)(x_{j-1})| &= \sum_{j=1}^{\infty} \left| \int_{x_{j-1}}^{x_j} \frac{d}{dx}(L_{n,\alpha} f)(x) dx \right| \\ &= \sum_{j=1}^{\infty} \left| \sum_{k=0}^{\infty} (f(x_{n,k+1}) - f(x_{n,k})) \int_{x_{j-1}}^{x_j} \frac{d}{dx} \left(S^{\alpha}_{n,k+1}\right)(x) dx \right| \\ &\le \sum_{k=0}^{\infty} |f(x_{n,k+1}) - f(x_{n,k})| \sum_{j=1}^{\infty} \left| \int_{x_{j-1}}^{x_j} \frac{d}{dx} \left(S^{\alpha}_{n,k+1}\right)(x) dx \right| \end{aligned}$$

$$= \sum_{k=0}^{\infty} |f(x_{n,k+1}) - f(x_{n,k})| \sum_{j=1}^{\infty} \left(\int_{x_{j-1}}^{x_j} \left| \frac{d}{dx}(S_{n,k+1}^{\alpha})(x) \right| dx \right)$$

$$= \sum_{k=0}^{\infty} |f(x_{n,k+1}) - f(x_{n,k})| \int_{0}^{\infty} \left| \frac{d}{dx}(S_{n,k+1}^{\alpha})(x) \right| dx.$$

Taking into account the relations (1.2), (1.3) and (2.1) we get

$$\lim_{x\to\infty} S_{n,k+1}^{\alpha}(x) \leq \lim_{x\to\infty} S_{n,0}^{\alpha}(x) = 1$$

and

$$S_{n,k+1}^{\alpha}(0) = 0.$$

Consequently, by using (1.7), we obtain

$$\int_{0}^{\infty} \left| \frac{d}{dx}(S_{n,k+1}^{\alpha})(x) \right| dx \leq 1.$$

It follows that

$$\begin{aligned} \sum_{j=1}^{\infty} |(L_{n,\alpha}f)(x_j) - (L_{n,\alpha}f)(x_{j-1})| &\leq \sum_{k=0}^{\infty} |f(x_{n,k+1}) - f(x_{n,k})| \\ &\leq |f|_{BV(\mathbb{R}_+)} \end{aligned}$$

and (2.3) is proved.

Further on, using (1.3) and (2.1) we have $S_{n,k+1}(0) = 0$ for all $k \in \mathbb{N}$ and

$$(L_{n,\alpha}f)(0) = S_{n,0}(0)f(x_{n,0}) = f(0).$$

The above relation and (2.3) complete the proof. □

Remark 2.1. Choosing $\alpha = 1$ in (1.6) the operator $L_{n,1}$ becomes l_n. If the conditions (2.1) and (2.2) are satisfied then the operator l_n has the variation detracting property, too.

3. On Mastroianni operators

We refer here at a general class of operators introduced and studied by G. Mastroianni [7].

Let $x_{n,k} = \frac{k}{n}$ and $u_{n,k} = \frac{(-1)^k}{k!} e_1^k \phi_n^{(k)}$ for all $(n,k) \in \mathbb{N} \times \mathbb{N}_0$, where $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of real functions on $\mathbb{R}_+$ which are infinitely differentiable and completely monotone on $\mathbb{R}_+$ verifying the following conditions

1. $\phi_n(0) = 1$ for all $n \in \mathbb{N}$;

2. for each $(n,k) \in \mathbb{N} \times \mathbb{N}_0$, there exists a number $p(n,k) \in \mathbb{N}$ and a function $\beta_{n,k} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\phi_n^{(i+k)}(x) = (-1)^k \phi_{p(n,k)}^{(i)} \beta_{n,k}(x)$$

and

$$\lim_{n\to\infty} \frac{n}{p(n,k)} = \lim_{n\to\infty} \frac{\beta_{n,k}(0)}{n^k} = 1.$$

In this case the operator l_n defined by (1.1) becomes the Mastroianni operator,

$$(M_n f) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \phi_n^{(k)}(x) f\left(\frac{k}{n}\right). \tag{3.1}$$

It is known that these operators satisfy (1.2)-(1.5) and (2.1), (see [7]). Since

$$\frac{d}{dx} S_{n,k}(x) = \frac{d}{dx} \sum_{j=k}^{\infty} \frac{(-1)^j}{j!} x^j \phi_n^{(j)}(x) = \frac{(-1)^k}{(k-1)!} x^{k-1} \phi_n^{(k)}(x) \geq 0,$$

for all $x \in \mathbb{R}_+$ and $n, k \in \mathbb{N}$, (2.2) also holds true.

On the basis of Theorem 2.1 we can assert that the Bézier type of Mastroianni operators enjoy the variation detracting property.

3.1. Particular cases

Mastroianni operators include some well-known classical sequences of linear positive operators.

1. Choosing $\phi_n(x) = e^{-nx}$, $p(n,k) = n$ and $\beta_{n,k}(x) = n^k$, the operators (3.1) reduce to the Szász-Favard-Mirakjan operators. In this case

$$\frac{d}{dx} S_{n,k}(x) = n u_{n,k-1}(x), \quad k \in \mathbb{N}.$$

2. Choosing $\phi_n(x) = (1+x)^{-n}$, $p(n,k) = n+k$ and

$$\beta_{n,k}(x) = n(n+1)\dots(n+k-1)(1+x)^{-k},$$

the operators (3.1) reduce to the Baskakov operators. In this case

$$\frac{d}{dx} S_{n,k}(x) = n u_{n+1,k-1}(x), \quad k \in \mathbb{N}.$$

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