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A K-FUNCTIONAL CHARACTERIZATION OF THE SPLINE APPROXIMATION

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Abstract. The purpose of this paper is to identify the spline operator L_T (see [R. DeVore, G. Lorentz: *Constructive Approximation.* Springer-Verag, Berlin Heidelberg, 1993]) as the Draganov-Ivanov operator $\mathbb{A}$ defined in [B. Draganov and K. Ivanov: *A new characterization of weighted Peetre K-functionals*, Constr. Approx. **21** (2005), 113–148] and to give a new characterization of spline approximation using the K-functionals.

1. Introduction

Let the linear operators be

$$L_T(f) = \sum_{j=1}^{n+r} f(\xi_j) N_j \tag{1.1}$$

which maps $C(I)$ onto $S_r(T, I)$ for each given T. Here the notations I, T, $S_r(T, I)$, N_j, ξ_j, are as follows: $I := [0; 1]$, $T := (t_j)$ consists of n uniform simple knots $0 < t_{r+1} < \cdots < t_{n+r} < 1$ with the step size h and auxiliary knots $t_1 < \cdots < t_{r-1} < t_r := 0$ and $1 := t_{n+r+1} < \cdots < t_{n+2r}$, also uniform. Because of their good support, the normalized B-splines N_j, $j = 1, \ldots, n+r$, for $S_r(T, I)$ - the Schoenberg space on I, which consists of all splines S of order r, with breakpoints contained in T - are particularly useful for constructing good spline approximants. The points ξ_j are selected so as $\xi_j \in I \cap \operatorname{supp} N_j := I \cap (t_j, t_{j+r})$, $j = 1, \ldots, n+r$. The normalized B-spline functions are the linear combinations of the truncated powers $(t_j - x)_+^{(r-1)}$ and therefore $N_j \in S_r(T, I)$.

For later reference we list some known facts about B-splines [1]:

$$N_j(x) \geq 0, \quad \operatorname{supp} N_j = [x_j, x_{j+r}],\ j = 1, \ldots, n+r, \quad \sum_{j=1}^{n+r} N_j(x) = 1$$

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for $x \in [0,1]$.

We will treat the case of continuous functions f with the all $r \geq 0$ derivatives continuous, $f \in C^r(I)$.

On the other hand, the operator introduced by Draganov and Ivanov is defined as follows.

Definition 1.1. (see [4]) Let $(X_1, Y_1, D_1), (X_2, Y_2, D_2)$ be the triples with X_1, X_2 two Banach spaces, and D_1, D_2 two differential operators. The linear operator $\mathbb{A}$ maps continuously (X_1, Y_1, D_1) onto (X_2, Y_2, D_2), and writes

$$\mathbb{A} : (X_1, Y_1, D_1) \longmapsto (X_2, Y_2, D_2)$$

if and only if $\mathbb{A} : X_1 \to X_2$ is invertible and together with its inverse $\mathbb{A}^{-1} : X_2 \to X_1$ satisfies the conditions:

a) $\|\mathbb{A}f\|_{X_2} \leq C \|f\|_{X_1}$, for any $f \in X_1$,

b) $\|D_2 \mathbb{A} f\|_{X_2} \leq C \|D_1 f\|_{X_1}$, for any $f \in Y_1 \cap D_1^{-1}(X_1)$,

c) $\left\|\mathbb{A}^{-1} F\right\|_{X_1} \leq C \|F\|_{X_2}$, for any $F \in X_2$,

d) $\left\|D_1 \mathbb{A}^{-1} F\right\|_{X_1} \leq C \|D_2 F\|_{X_2}$, for any $F \in Y_2 \cap D_2^{-1}(X_2)$,

e) $\mathbb{A}\left(Y_1 \cap D_1^{-1}(X_1)\right) = Y_2 \cap D_2^{-1}(X_2)$, where $D^{-1}(X) := \{g \in X / Dg \in X\} \subset X$ and $C > 0$ is a constant independent on f and t.

2. The Spline Operators as Draganov-Ivanov Operators

In this section we propose the following problem.

Problem 2.1. Find the triples (X_1, Y_1, D_1) and (X_2, Y_2, D_2) knowing that there exists constant C so that the linear operator $L_T : X_1 \to X_2$ from (1.1) maps continuously (X_1, Y_1, D_1) onto (X_2, Y_2, D_2).

Solution: We will follow three steps to solve this problem. First, we choose the Banach spaces X_1, X_2 so that L_T be invertible; second, we find the proper spaces Y_1, Y_2 and the differential operators D_1, D_2. Finally, for the chosen triples (X_1, Y_1, D_1), (X_2, Y_2, D_2) we verify the conditions (a)–(e) from Definition 1.1.

L_T is invertible if and only if it is bijection. In order to ensure the injectivity, we define the equivalence relation $f \sim g$ as follows

$$(\forall f, g \in X_1) \quad f \sim g \Leftrightarrow f(\xi_j) = g(\xi_j), \quad j = 1, \ldots, n+r.$$

Thus, the space X_1 is the equivalence class $C^r(I)/_\sim := \tilde{C}^r(I)$. This space is a Banach space with respect to the norm

$$\|f\|_{X_1} := \max_{j=1,\ldots,n+r} |f(\xi_j)| .$$

Using the surjective characterization $L_T(X_1) = X_2$ it follows $X_2 := \mathrm{Im}(L_T)$, where $\mathrm{Im}(L_T)$ denotes the image space of the L_T operator. It is easy to observe that $\|L_T f\|_p < +\infty$, $1 \le p \le +\infty$, thus $\mathrm{Im}(L_T) \subset L_p(I)$. Here we use the classical notation $L_p(I)$ for the space of all measurable functions with finite norm

$$\|f\|_p := \left(\int_I |f|^p \, dx \right)^{1/p}, \qquad 0 < p < +\infty,$$

$$\|f\|_p := \mathrm{esssup}_{x \in I} |f(x)|, \quad p = +\infty.$$

Again, the space and the L_p-norm form a Banach space.

We complete the triples (X_i, Y_i, D_i), $i \in \{1, 2\}$ by taking Sobolev space $Y_1 = Y_2 := W_p^r(I)$, the set of all functions $f : I \to \mathbb{R}$ with $f^{(r-1)}$ absolutely continuous and $f^{(r)} \in L_p$. Also, we choose

$$D_1 f := \frac{1}{h} \int_{\xi_{j-1}}^{\xi_j} f'(x) \, dx$$

and $D_2 := D^1$, first derivative.

In what follows, there will be analyzed the conditions mentioned in Definition 1.1.

a) The inequality $\|L_T f\|_p \le C \|f\|_{X_1}$ is valid for each $f \in \tilde{C}^r(I)$. Indeed,

$$\begin{aligned} \|L_T f\|_p &\le \left(\int_0^1 \left[\sum_{j=1}^{n+r} |f(\xi_j)| \, |N_j(x)| \right]^p dx \right)^{1/p} \\ &\le \|f\|_{X_1} \left(\int_0^1 \left[\sum_{j=1}^{n+r} |N_j(x)| \right]^p dx \right)^{1/p} = \|f\|_{X_1}, \end{aligned}$$

where we applied the partition of unity property of the normalized B-spline. Furthermore, the value of the constant is $C = 1$.

b) For $1 \le p \le +\infty$, the following inequality holds true

$$(\forall f \in W_p^r(I) \cap D_1^{-1}(\tilde{C}^r(I))) \qquad \|D_2 L_T f\|_p \le C \, \|D_1 f\|_{X_1}.$$

Taking into account the definition of the differential operators D_1, D_2, the inequality which has to be demonstrated becomes $\|(L_T f)'\|_p \le C \, \|D_1 f\|_{X_1}$.

In the case of $1 \le p < +\infty$, we apply the result (T.5.9, p. 195) from [5] for $d = 2$: *Let $s(x) = \sum_{i=1}^n c_i N_{i,r}$ be the spline function, and suppose $1 \le d \le r$. Then for all $x \in [x_r, x_n)$,*

$$D_+^{d-1} s(x) = \sum_{i=d}^n c_{i,d} N_{i,r-d+1}(x),$$

where $c_{i,j}$ *are as follows*

$$c_{i,1} = c_i \quad \mathit{for} \;\; i = 1, \ldots, n, \tag{2.1}$$

$$c_{i,j} = \begin{cases} (r-j+1)\dfrac{c_{i,j-1} - c_{i-1,j-1}}{x_{i+r-j+1} - x_i}, & x_{i+r-j+1} - x_i > 0, \\ 0, & \mathit{otherwise}, \end{cases}$$

for $j = 2, \ldots, d$ *and* $i = 1, \ldots, n$.

Thus, denoting $c_{j,1} = c_j := f(\xi_j),\ j = 1, \ldots, n+r$, for $x \in [x_r, x_{n+r})$ the left side of the inequality becomes

$$\|(L_T f)'\|_p = \left\| \sum_{j=2}^{n+r} c_{j,2} N_{j,r-1} \right\|_p .$$

Further, using the recurrent relations (2.1) (we have always the case $t_{j+r-1} - t_j > 0$ because of the increasing knots), and taking into consideration the increasing uniform partition, we have

$$c_{j,2} = \frac{r-1}{t_{j+r-1} - t_j} \left[c_{j,1} - c_{j-1,1}\right] = \frac{r-1}{(r-1)h} \left[c_{j,1} - c_{j-1,1}\right] = \frac{1}{h} \left[f(\xi_j) - f(\xi_{j-1})\right].$$

Thus, for $x \in [x_r, x_{r+n})$

$$\|(L_T f)'\|_p \le \left[\int_0^1 \left(\sum_{j=2}^{n+r} \frac{1}{h} \left| f(\xi_j) - f(\xi_{j-1}) \right| \left| N_{j,r-1}(x) \right| \right)^p dx \right]^{1/p}$$

$$\le C \left\| c_{j,2} \right\|_{X_1} \left[\int_0^1 \left(\sum_{j=2}^{n+r} N_{j,r-1}(x) \right)^p dx \right]^{1/p} = C \left\| D_1 f \right\|_{X_1} .$$

Following the same idea, for the second case $p = +\infty$ we obtain

$$\begin{aligned} \|(L_T f)'\|_\infty &= \left\| \sum_{j=2}^{n+r} c_{j,2} N_{j,r-1} \right\|_\infty \\ &\le \sup_{x \in I} \left(\sum_{j=2}^{n+r} |c_{j,2}| \left| N_{j,r-1}(x) \right| \right) \\ &\le C \left\| c_{j,2} \right\|_{X_1} \sup_{x \in I} \sum_{j=2}^{n+r} \left| N_{j,r-1}(x) \right| \\ &= C \left\| D_1 f \right\|_{X_1} \end{aligned}$$

for each $f \in W_p^r(I) \cap D_1^{-1}(\tilde{C}^r(I))$.

c) The inequality $\left\|L_T^{-1}F\right\|_{X_1} \leq \|F\|_p$, where $F \in \mathrm{Im}(L_T f)$ holds true.

Indeed, taking into accout that L_T^{-1} is the invert operator of the L_T operator, we have the relation $L_T(L_T^{-1}F) = F = L_T f$ for each $f \in \tilde{C}^r(I)$, so that $L_T^{-1}F = f$. The left side of the inequality becomes $\left\|L_T^{-1}F\right\|_{X_1} = \|f\|_{X_1} = \max_{j=1,\dots,n+r} |f(\xi_j)|$ and the right side becomes $\|F\|_p = \|L_T f\|_p \leq C\|f\|_{X_1}$. Because there exists a constant $C \geq 1$ so that $\|f\|_{X_1} \leq C\|f\|_{X_1}$, the required inequality is true, for $1 \leq p \leq +infty$ and for all $F \in \mathrm{Im}(L_T f)$.

d) The inequality $\left\|D_1 L_T^{-1}F\right\|_{X_1} \leq C\|F'\|_p$ holds true for all $F \in W_p^r(I) \cap D_2^{-1}(X_2)$.

Because $L_T^{-1}F = f$ and $F = L_T f$, the above inequality becomes $\|D_1 f\|_{X_1} \leq C\left\|(L_T f)'\right\|_p$ for all $f \in W_p^r(I) \cap D_1^{-1} bigl(\tilde{C}^r(I)\bigr)$. Applying *the stable condition* of the B-spline [2]: *there exists a positive constant C_r which depends only on r so that for all $i = \overline{1, n+r}$,*

$$|c_i| \leq C_r \left\|\sum_j c_j N_{j,r}\right\|_{[t_i, t_{i+r}]},$$

we have: there exists a positive constant C_{r-1} so that for all $i = 1, \dots, n+r$,

$$|c_{i,2}| \leq C_{r-1} \left\|\sum_{j=2}^{n+r} c_{j,2} N_{j,r-1}\right\|_{[t_i, t_{i+r}]}. \tag{2.2}$$

Because for every $i \in \{1, \dots, n+r\}$ the inequality (2.2) holds true, there exists a subscript $i \in \{1, \dots, n+r\}$ so that

$$\|c_{i,2}\|_{X_1} \leq C_{r-1} \left\|\sum_{j=2}^{n+r} c_{j,2} N_{j,r-1}\right\|_{[t_i, t_{i+r}]}.$$

On the other hand

$$\left\|\sum_{j=2}^{n+r} c_{j,2} N_{j,r-1}\right\|_{[t_i, t_{i+r}]} \leq (n{+}r{-}1)\|c_{j,2}\|_{X_1} \left\|\sum_{j=2}^{n+k} N_{j,r-1}\right\|_{[t_i, t_{i+r}]} = (n{+}r{-}1)\|c_{j,2}\|_{X_1}$$

and

$$\left\|\sum_{j=2}^{n+r} c_{j,2} N_{j,r-1}\right\|_{[0,1]} \leq (n+r-1)\|c_{j,2}\|_{X_1} \left\|\sum_{j=2}^{n+k} N_{j,r-1}\right\|_{[0,1]} = (n+r-1)\|c_{j,2}\|_{X_1}.$$

Imposing the condition $rh \le 1$, $(t_{i+r} - t_i = rh)$, we have

$$\left\| \sum_{j=2}^{n+r} c_{j,2} N_{j,r-1} \right\|_{[t_i, t_{i+r}]} \le \left\| \sum_{j=2}^{n+r} c_{j,2} N_{j,r-1} \right\|_{[0,1]} .$$

e) The last condition (of Definition 1.1)

$$L_T \left(W_p^r(I) \cap D^{-1}(\tilde{C}^r(I)) \right) = W_p^r(I) \cap D^{-1}(\mathrm{Im}(L_T f))$$

is true.

We have found a linear operator $\mathbb{A} := L_T f$, so we conclude the notes by the following result:

Conclusion. *Being given the uniform knots* $0 < t_{r+1} < \cdots < t_{r+n} < 1$ *and auxiliary uniform knots* $t_1 < \cdots < t_r := 0$, $1 := t_{r+n+1} < \cdots < t_{n+2r}$, *the linear operator* $(L_T f)(x) = \sum_{j=1}^{n+r} f(\xi_j) N_j(x)$ *maps continuously* $\left(\tilde{C}^r(I), W_p^r(I), D_1\right)$ *onto* $(\mathrm{Im}(L_T f), W_p^r(I), D_2)$, *for all* $f \in \tilde{C}^r(I)$, *where* $\xi_j \in I \cap \operatorname{supp} N_j$, *i.e.,*

$$L_T : \left(\tilde{C}^r(I), W_p^r(I), D_1 \right) \longmapsto \left(\mathrm{Im}(L_T f), W_p^r(I), D_2 \right) . \tag{2.3}$$

3. Spline Error Evaluation

In this section we propose the following problem.

Problem 3.1. For a function $f : C^r[0,1] \to \mathbb{R}$, find an equivalence

$$E(f, S_r(T, I))_p := \inf_{S \in S_r(T,I)} \|f - Sf\|_p \sim K(\mathbb{A}f, \eta; X, Y, D^\alpha), \quad 1 \le p \le +\infty,$$

where $\mathbb{A}$ is the Draganov-Ivanov operator, D^α the differential operator of order α, η is a quantity to be determined and X, Y are two spaces (X must to be Banach space). There are supposed to be given the positive integers n, r, $r < n$ and the finite set of uniform knots $T = (t_j)_{j=r}^{n+r+1}$, $0 := t_r < t_{r+1} < \cdots < t_{n+r} < t_{n+r+1} := 1$.

First, we will prove the equivalence between $E(f, S_r(T, I))_p$ and the modulus of the smoothness of f of order r in the L_p-norm

$$\omega_r(f, \eta)_p := \sup_{0 < t \le \eta} \|\Delta_t^r(f, \cdot)\|_p .$$

It is known that a is equivalent with b, $a \sim b$, if there are two constants $c_1, c_2 > 0$ so that $c_1 a \le b \le c_2 a$. The right inequality is known as the direct theorem, and the left one as the inverse theorem. Schumaker proves both of them in [5], [6].

Theorem 3.1. (T 6.27, [5]) *Let* $1 \le p \le +\infty$. *Then there exists a constant* C (*depending only on* r *and* n) *so that for* $f \in L_p[a,b]$, $E(f, S_r(\Delta))_p \le C\omega_r(f; \overline{\Delta})_p$, *where* $S_r(\Delta)$ *is the space of splines of order* r *with simple knots from* $\Delta := (t_j)_{j=0}^{n+1}$ *and* $\overline{\Delta} := \max\limits_{0 \le i \le n} (t_{i+1} - t_i)$.

Identifying $[a,b] := [0,1] = I$, $\Delta := T$ and $S_r(\Delta) := S_r(T, I)$ and taking into account the uniform mesh ($\overline{\Delta} = \max\limits_{r \le i \le n+r} (t_{i+1} - t_i) = \frac{1}{n+1}$), we have

$$E(f, S_r(T,I))_p \le \omega_r(f, (n+1)^{-1})_p. \tag{3.1}$$

Theorem 3.2. ([5], [6]) *Let* $0 \le m \le r$, $1 \le p \le +\infty$, $1 \le q \le +\infty$. *There is a constant* $C > 0$ (*depending on* r, m, p) *so that for any partition* Δ *of* $[a,b]$ *there exists a function* $f \in W_q^m[a,b]$ *with* $\|f\|_{W_q^m[a,b]} = 1$ *and*

$$E(f, PP_r(\Delta))_p \ge C(\overline{\Delta})^{m+1/p-1/q}\omega_{r-m}(f^{(m)}, \overline{\Delta})_q,$$

where $PP_r(\Delta)$ *is the space of piecewise polynomials of order* r *associated to the partition* Δ.

We consider the particular case $p = q$. Because $S_r(T,I) \subset PP_r(T)$ ($\Delta := T$), the lower bounds for $PP_r(T)$ will automatically produce lower bounds for all spline spaces $S_r(T,I)$ ([5], p. 210). It remains to prove the inequality

$$\omega_{r-m}(f^{(m)}, (n+1)^{-1})_p \ge \omega_r(f, (n+1)^{-1}),$$

for $f \in W_p^m(I)$.

Indeed, by using the relation (7.13), Chap. 2, from [3] we have

$$\omega_k(f^{(r)}, t)_p \ge t^{-r}\omega_{r+k}(f,t)_p,$$

for $t \ge 0$, $1 \le p \le +\infty$, $f \in L_p(I)$, with our notation we have

$$\omega_{r-m}(f^{(m)}, (n+1)^{-1})_p \ge ((n+1)^{-1})^{-m}\omega_{m+r-m}(f, (n+1)^{-1})_p$$

or

$$\omega_{r-m}(f^{(m)}, (n+1)^{-1})_p \ge (n+1)^m\omega_r(f, (n+1)^{-1})_p.$$

Thus, the inverse theorem is

$$\begin{aligned} E(f, S_r(T,I))_p &\ge C((n+1)^{-1})^m\omega_{r-m}(f^{(m)}, (n+1)^{-1}) \\ &\ge C(n+1)^{-m}(n+1)^m\omega_r(f, (n+1)^{-1})_p = C\omega_r(f, (n+1)^{-1})_p. \end{aligned} \tag{3.2}$$

Because $W_p^m(I)$ is a subspace of the space $L_p(I)$, from (3.1) and (3.2) we can conclude the next proposition.

Proposition 3.1. *There exists a function $f \in W_p^m(I)$ with $0 \leq m \leq r$, $1 \leq p \leq +\infty$ and the constant $C > 0$ so that the equivalence holds true*

$$E(f, S_r(T, I))_p \sim \omega_r(f, (n+1)^{-1})_p. \tag{3.3}$$

The next step is to apply the Johnson result to the equivalence (3.3).

Theorem 3.3. (Johnson) *For $m \geq 0$, $1 \leq p \leq +\infty$ and Banach spaces $L_p(I), W_p^m(I)$ the following equivalence holds true*

$$K(f, t^m; L_p, W_p^m) \sim \omega_m(f, t)_p,$$

where

$$K(f, t^m; L_p, W_p^m) = \inf \left\{ \|f - g\|_p + t^m \left\| g^{(m)} \right\|_p : g \in W_p^m \right\}$$

is the Peetre K-functional

Thus, we have:

Proposition 3.2. *There exists a function $f \in W_p^m(I)$ so that*

$$E(f, S_r(T, I))_p \sim K(f, (n+1)^{-m}; L_p, W_p^m). \tag{3.4}$$

Now, from (2.3) and the following Draganov-Ivanov proposition:

Proposition 3.3. (Proposition 2.1, [4]) *Let the linear operator $\mathbb{A}$ maps continuously (X_1, Y_1, D_1) onto (X_2, Y_2, D_2). Then, for every $f \in X_1$ and $t > 0$, we have $K(f, t; X_1, Y_1, D_1) \sim K(\mathbb{A}f, t; X_2, Y_2, D_2)$.*

we obtain the following result:

Proposition 3.4. *For every $f \in \tilde{C}^r(I)$, $I = [0, 1]$ and uniform partition T, for $t > 0$ and $1 \leq p \leq +\infty$ the equivalence holds true*

$$K(f, t; \tilde{C}^r, W_p^r, D_1) \sim K(L_T f, t; \mathrm{Im}(L_t f), W_p^r, D_2), \tag{3.5}$$

with $D_1 f := \frac{1}{h} \int_{\xi_{j-1}}^{\xi_j} f'(x)\, dx$ and $D_2 := D^1$.

Finally, by combining (3.4) (in which we take the particular case $m := r$) and (3.5), by denoting the space of all classes of equivalence of the functions $f \in W_p^r(I)$ with $\tilde{W}_p^r(I)$ and knowing that $\tilde{C}^r(I) \subset C^r(I)$, $\mathrm{Im}(L_t f) \subset L_p(I)$, $W_p^r(I) \subset L_p(I)$ we can give the final result.

Theorem 3.4. *For $1 \leq p \leq +\infty$, $r \in \mathbb{N}$ there exists a function $f \in \tilde{W}_p^r(I)$ so that the equivalence takes place*

$$E(f, S_r(T, I))_p \sim K(L_T f, (n+1)^{-r}; \mathrm{Im}(L_t f), W_p^r, D_2), \tag{3.6}$$

with D_1, D_2 defined in Proposition 3.4.

The above result (3.6) shows a new characterization of spline approximation from Peettre K-functional's point of view because the last one is applied not to the function f, but to the spline operator L_T.

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