# ON CERTAIN VOLTERRA INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS 

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#### Abstract

In this paper we study the existence, uniqueness and other properties of solutions of certain Volterra integral and integrodifferential equations. The well known Banach fixed point theorem coupled with Bielecki type norm and the integral inequalities with explicit estimates are used to establish the results.


## 1. Introduction

Consider the Volterra integral and integrodifferential equations of the forms:

$$
\begin{equation*}
x(t)=f\left(t, x(t), \int_{a}^{t} k(t, \sigma, x(\sigma)) d \sigma\right) \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
x^{\prime}(t)=f\left(t, x(t), \int_{a}^{t} k(t, \sigma, x(\sigma)) d \sigma\right), \quad x(a)=x_{0} \tag{1.2}
\end{equation*}
$$

for $-\infty<a \leq t<+\infty$, where $x, f, k$ are real vectors with $n$ components and $'$ denotes the derivative. Let $\mathbb{R}^{n}$ denotes the real $n$-dimensional Euclidean space with appropriate norm denoted by $|\cdot|$ and $\mathbb{R}$ the set of real numbers. Let $I=$ $[a,+\infty), \mathbb{R}_{+}=[0,+\infty)$ be the given subsets of $\mathbb{R}$ and $C\left(S_{1}, S_{2}\right)$ denotes the class of continuous functions from the set $S_{1}$ to the set $S_{2}$ and assume that $k \in$ $C\left(I^{2} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$ for $a \leq s \leq t<+\infty, f \in C\left(I \times \mathbb{R}^{n} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$.

Integral and integrodifferential equations arise in a variety of applications and their study is of great interest. Many authors have studied the equations of the forms (1.1) and (1.2) and their special and general versions with different view points, see $[1],[3]-[15]$ and the references given therein. The purpose of this paper is to study the existence, uniqueness and other properties of solutions of equations (1.1) and (1.2) under various assumptions on the functions $f$ and $k$. The main tools

[^0]employed in the analysis are based on the applications of the Banach fixed point theorem (see $[6,7]$ ) coupled with Bielecki type norm (see $[2,7]$ ) and the integral inequalities with explicit estimates given in [11] and [12].

## 2. Existence and uniqueness

By following [7] we first construct the appropriate metric space for our analysis. Let $\beta>0$ be a constant and consider the space of continuous functions $C\left(I, \mathbb{R}^{n}\right)$ such that $\sup _{t \in I}|x(t)| / e^{\beta(t-a)}<\infty$, and denote this special space by $C_{\beta}\left(I, \mathbb{R}^{n}\right)$. We couple the linear space $C_{\beta}\left(I, \mathbb{R}^{n}\right)$ with suitable metric, namely

$$
d_{\beta}^{\infty}(x, y)=\sup _{t \in I} \frac{|x(t)-y(t)|}{e^{\beta(t-a)}},
$$

with a norm defined by

$$
|x|_{\beta}^{\infty}=\sup _{t \in I} \frac{|x(t)|}{e^{\beta(t-a)}} .
$$

The above definitions of $d_{\beta}^{\infty}$ and $|\cdot|_{\beta}^{\infty}$ are the variants of Bielecki's metric and norm [2].

The following Lemma proved in [7] deals with some important properties of $d_{\beta}^{\infty}$ and $|\cdot|_{\beta}^{\infty}$.

Lemma 2.1. If $\beta>0$ is a constant, then:
(i) $d_{\beta}^{\infty}$ is a metric,
(ii) $|\cdot|_{\beta}^{\infty}$ is a norm,
(iii) $\left(C_{\beta}\left(I, \mathbb{R}^{n}\right),|\cdot|_{\beta}^{\infty}\right)$ is a Banach space,
(iv) $\left(C_{\beta}\left(I, \mathbb{R}^{n}\right), d_{\beta}^{\infty}\right)$ is a complete metric space.

We are now ready to present the main results concerning the existence and uniqueness of solutions of equations (1.1) and (1.2).

Theorem 2.1. Let $L>0, \beta>0, M \geq 0, \gamma>1$ be constants with $\beta=L \gamma$. Suppose that the functions $f, k$ in equation (1.1) satisfy the conditions

$$
\begin{gather*}
|f(t, u, v)-f(t, \bar{u}, \bar{v})| \leq M[|u-\bar{u}|+|v-\bar{v}|],  \tag{2.1}\\
|k(t, s, u)-k(t, s, \bar{u})| \leq L|u-\bar{u}|, \tag{2.2}
\end{gather*}
$$

and

$$
d_{1}=\sup _{t \in I} \frac{1}{e^{\beta(t-a)}}\left|f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)\right|<\infty
$$

If $M(1+1 / \gamma)<1$, then the integral equation (1.1) has a unique solution $x \in$ $C_{\beta}\left(I, \mathbb{R}^{n}\right)$.

Proof. Consider the following equivalent formulation of equation (1.1), namely

$$
\begin{align*}
x(t)= & f\left(t, x(t), \int_{a}^{t} k(t, \sigma, x(\sigma)) d \sigma\right)-f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)  \tag{2.3}\\
& +f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)
\end{align*}
$$

fot $t \in I$. We will show that (2.3) has a unique solution and thus equation (1.1) must also have a unique solution. Let $x \in C_{\beta}\left(I, \mathbb{R}^{n}\right)$ and define the operator $T$ by

$$
\begin{align*}
(T x)(t)= & f\left(t, x(t), \int_{a}^{t} k(t, \sigma, x(\sigma)) d \sigma\right)-f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)  \tag{2.4}\\
& +f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)
\end{align*}
$$

Now we shall show that $T$ maps $C_{\beta}\left(I, \mathbb{R}^{n}\right)$ into itself. From (2.4) and using the hypotheses we have

$$
\begin{aligned}
|T x|_{\beta}^{\infty}= & \sup _{t \in I} \frac{|(T x)(t)|}{e^{\beta(t-a)}} \\
\leq & \sup _{t \in I} \frac{1}{e^{\beta(t-a)}}\left|f\left(t, x(t), \int_{a}^{t} k(t, \sigma, x(\sigma)) d \sigma\right)-f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)\right| \\
& +\sup _{t \in I} \frac{1}{e^{\beta(t-a)}}\left|f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)\right| \\
\leq & d_{1}+\sup _{t \in I} \frac{1}{e^{\beta(t-a)}} M\left[|x(t)|+\int_{a}^{t} L|x(\sigma)| d \sigma\right] \\
= & d_{1}+M\left[\sup _{t \in I} \frac{|x(t)|}{e^{\beta(t-a)}}+L \sup _{t \in I} \frac{1}{e^{\beta(t-a)}} \int_{a}^{t} e^{\beta(\sigma-a)} \frac{|x(\sigma)|}{e^{\beta(\sigma-a)}} d \sigma\right] \\
\leq & d_{1}+M\left[|x|_{\beta}^{\infty}+L|x|_{\beta}^{\infty} \sup _{t \in I} \frac{1}{e^{\beta(t-a)}} \int_{a}^{t} e^{\beta(\sigma-a)} d \sigma\right] \\
= & d_{1}+M|x|_{\beta}^{\infty}\left[1+L \sup t \in I \frac{1}{e^{\beta(t-a)}}\left(\frac{e^{\beta(t-a)}-1}{\beta}\right)\right] \\
= & d_{1}+M|x|_{\beta}^{\infty}\left[1+\frac{L}{\beta}\right]=d_{1}+|x|_{\beta}^{\infty} M\left(1+\frac{1}{\gamma}\right)<\infty .
\end{aligned}
$$

This proves that the operator $T$ maps $C_{\beta}\left(I, \mathbb{R}^{n}\right)$ into itself.
Now we verify that the operator $T$ is a contraction map. Let $u, v \in C_{\beta}\left(I, \mathbb{R}^{n}\right)$. From (2.4) and using the hypotheses we have

$$
\begin{aligned}
& d_{\beta}^{\infty}(T u, T v)= \sup _{t \in I} \frac{|(T u)(t)-(T v)(t)|}{e^{\beta(t-a)}} \\
&= \left.\sup _{t \in I} \frac{1}{e^{\beta(t-a)}} \right\rvert\, f\left(t, u(t), \int_{a}^{t} k(t, \sigma, u(\sigma)) d \sigma\right) \\
&-f\left(t, v(t), \int_{a}^{t} k(t, \sigma, v(\sigma)) d \sigma\right) \mid \\
& \leq \sup _{t \in I} \frac{1}{e^{\beta(t-a)}} M\left[|u(t)-v(t)|+\int_{a}^{t} L|u(\sigma)-v(\sigma)|\right] \\
&= M\left[\sup _{t \in I} \frac{|u(t)-v(t)|}{e^{\beta(t-a)}}+\sup _{t \in I} \frac{1}{e^{\beta(t-a)}} L \int_{a}^{t} e^{\beta(\sigma-a)} \underline{|u(\sigma)-v(\sigma)|} \frac{e^{\beta(\sigma-a)}}{} d \sigma\right] \\
& \leq M\left[d_{\beta}^{\infty}(u, v)+L d_{\beta}^{\infty}(u, v) \sup _{t \in I} \frac{1}{e^{\beta(t-a)}} \int_{a}^{t} e^{\beta(\sigma-a)} d \sigma\right] \\
&= M d_{\beta}^{\infty}(u, v)\left[1+L \sup _{t \in I} \frac{1}{e^{\beta(t-a)}}\left(\frac{e^{\beta(t-a)}-1}{\beta}\right)\right] \\
&= M d_{\beta}^{\infty}(u, v)\left[1+\frac{L}{\beta}\right]=M\left(1+\frac{1}{\gamma}\right) d_{\beta}^{\infty}(u, v) .
\end{aligned}
$$

Since $M(1+1 / \gamma)<1$, it follows from the Banach fixed point theorem (see $[6,7]$ ) that $T$ has a unique fixed point in $C_{\beta}\left(I, \mathbb{R}^{n}\right)$. The fixed point of $T$ is however a solution of equation (1.1). The proof is complete.

Theorem 2.2. Let $L, \beta, M, \gamma$ be as in Theorem 2.1. Suppose that the functions $f, k$ in equation (1.2) satisfy the conditions (2.1) and (2.2) and

$$
d_{2}=\sup _{t \in I} \frac{1}{e^{\beta(t-a)}}\left|x_{0}+\int_{a}^{t} f\left(s, 0, \int_{a}^{s} k(s, \sigma, 0) d \sigma\right) d s\right|<\infty
$$

If $M / \beta(1+1 / \gamma)<1$, then the integrodifferential equation (1.2) has a unique solution $x \in C_{\beta}\left(I, \mathbb{R}^{n}\right)$.

Proof. Let $x \in C_{\beta}\left(I, \mathbb{R}^{n}\right)$, and define the operator $S$ by

$$
(S x)(t)=x_{0}+\int_{a}^{t} f\left(s, x(s), \int_{a}^{s} k(s, \sigma, x(\sigma)) d \sigma\right) d s
$$

$$
-\int_{a}^{t} f\left(s, 0, \int_{a}^{s} k(s, \sigma, 0) d \sigma\right) d s+\int_{a}^{t} f\left(s, 0, \int_{a}^{s} k(s, \sigma, 0) d \sigma\right) d s
$$

for $t \in I$. The proof that $S$ maps $C_{\beta}\left(I, \mathbb{R}^{n}\right)$ into itself and is a contraction map, can be completed by closely looking at the proof of Theorem 2.1 given above with suitable modifications. Here we omit the details.

## 3. Estimates on the solutions

In this section we obtain estimates on the solutions of equations (1.1) and (1.2) under some suitable assumptions on the functions involved therein.

We need the following versions of the inequalities given in [11, p. 20], (see also [12, p. 11, Remark 1.2.1], and [12, p. 29]. We shall state them here for completeness.

Lemma 3.1. Let $u(t) \in C\left(I, \mathbb{R}_{+}\right), r(t, \sigma), \frac{\partial}{\partial t} r(t, \sigma) \in C\left(D, \mathbb{R}_{+}\right)$, where $D=$ $\left\{(t, \sigma) \in I^{2}: a \leq \sigma \leq t<\infty\right\}$ and $c \geq 0$ is a constant. If

$$
u(t) \leq c+\int_{a}^{t} r(t, \sigma) u(\sigma) d \sigma
$$

for $t \in I$, then

$$
u(t) \leq c \exp \left(\int_{a}^{t} A(s) d s\right)
$$

for $t \in I$, where

$$
A(t)=r(t, t)+\int_{a}^{t} \frac{\partial}{\partial t} r(t, \tau) d \tau
$$

Lemma 3.2. Let $u(t), p(t) \in C\left(I, \mathbb{R}_{+}\right), r(t, \sigma), \frac{\partial}{\partial t} r(t, \sigma) \in C\left(D, \mathbb{R}_{+}\right)$, where $D$ is as in Lemma 3.1 and $c \geq 0$ is a constant. If

$$
u(t) \leq c+\int_{a}^{t} p(s)\left[u(s)+\int_{a}^{s} r(s, \sigma) u(\sigma) d \sigma\right] d s
$$

for $t \in I$, then

$$
u(t) \leq c\left[1+\int_{a}^{t} p(s) \exp \left(\int_{a}^{s}[p(\sigma)+A(\sigma)] d \sigma\right) d s\right]
$$

for $t \in I$, where $A(t)$ is as in Lemma 3.1.

First, we shall give the following theorem concerning the estimate on the solution of equation (1.1).

Theorem 3.1. Suppose that the functions $f, k$ in equation (1.1) satisfy the conditions

$$
\begin{gather*}
|f(t, u, v)-f(t, \bar{u}, \bar{v})| \leq N[|u-\bar{u}|+|v-\bar{v}|]  \tag{3.1}\\
|k(t, \sigma, u)-k(t, \sigma, v)| \leq r(t, \sigma)|u-v| \tag{3.2}
\end{gather*}
$$

where $0 \leq N<1$ is a constant and $r(t, \sigma), \frac{\partial}{\partial t} r(t, \sigma) \in C\left(D, \mathbb{R}_{+}\right)$, in which $D$ is as defined in Lemma 3.1. Let

$$
c_{1}=\sup _{t \in I}\left|f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)\right|<\infty
$$

If $x(t), t \in I$, is any solution of equation (1.1), then

$$
\begin{equation*}
|x(t)| \leq\left(\frac{c_{1}}{1-N}\right) \exp \left(\int_{a}^{t} B(s) d s\right) \tag{3.3}
\end{equation*}
$$

for $t \in I$, where

$$
\begin{equation*}
B(t)=\frac{N}{1-N} A(t) \tag{3.4}
\end{equation*}
$$

in which $A(t)$ is as defined in Lemma 3.1.
Proof. By using the fact that the solution $x(t)$ of equation (1.1) satisfies the equivalent equation (2.3) and the hypotheses we have

$$
\begin{align*}
|x(t)| \leq & \left|f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)\right|  \tag{3.5}\\
& +\left|f\left(t, x(t), \int_{a}^{t} k(t, \sigma, x(\sigma)) d \sigma\right)-f\left(t, 0, \int_{a}^{t} k(t, \sigma, 0) d \sigma\right)\right| \\
\leq & c_{1}+N\left[|x(t)|+\int_{a}^{t} r(t, \sigma)|x(\sigma)| d \sigma\right]
\end{align*}
$$

¿From (3.5) and using the assumption $0 \leq N<1$, we observe that

$$
\begin{equation*}
|x(t)| \leq\left(\frac{c_{1}}{1-N}\right)+\frac{N}{1-N} \int_{a}^{t} r(t, \sigma)|x(\sigma)| d \sigma \tag{3.6}
\end{equation*}
$$

Now an application of Lemma 3.1 to (3.6) yields (3.3).
Next, we shall obtain the estimate on the solution of equation (1.2).

Theorem 3.2. Suppose that the function $f$ in equation (1.2) satisfies the condition

$$
\begin{equation*}
|f(t, u, v)-f(t, \bar{u}, \bar{v})| \leq p(t)[|u-\bar{u}|+|v-\bar{v}|], \tag{3.7}
\end{equation*}
$$

where $p \in C\left(I, \mathbb{R}_{+}\right)$and the function $k$ in equation (1.2) satisfies the condition (3.2). Let

$$
c_{2}=\sup _{t \in I}\left|x_{0}+\int_{a}^{t} f\left(s, 0, \int_{a}^{s} k(s, \sigma, 0) d \sigma\right) d s\right|<\infty
$$

If $x(t), t \in I$, is any solution of equation (1.2), then

$$
\begin{equation*}
|x(t)| \leq c_{2}\left[1+\int_{a}^{t} p(s) \exp \left(\int_{a}^{s}[p(\sigma)+A(\sigma)] d \sigma\right) d s\right] \tag{3.8}
\end{equation*}
$$

for $t \in I$, where $A(t)$ is as defined in Lemma 3.1.
Proof. Using the fact that $x(t)$ is a solution of equation (1.2) and the hypotheses we have

$$
\begin{align*}
|x(t)| & \leq\left|x_{0}+\int_{a}^{t} f\left(s, 0, \int_{a}^{s} k(s, \sigma, 0) d \sigma\right) d s\right|  \tag{3.9}\\
& +\int_{a}^{t}\left|f\left(s, x(s), \int_{a}^{s} k(s, \sigma, x(\sigma)) d \sigma\right)-f\left(s, 0, \int_{a}^{s} k(s, \sigma, 0) d \sigma\right)\right| d s \\
& \leq c_{2}+\int_{a}^{t} p(s)\left[|x(s)|+\int_{a}^{s} r(s, \sigma)|x(\sigma)| d \sigma\right] d s
\end{align*}
$$

Now an application of Lemma 3.2 to (3.9) yields (3.8).

## 4. Continuous dependence

In this section we shall deal with the continuous dependence of solutions of equations (1.1) and (1.2) on the functions involved therein and also the continuous dependence of solutions of equations of the forms (1.1) and (1.2) on parameters.

Consider the equations (1.1) and (1.2) and the corresponding equations

$$
\begin{equation*}
y(t)=\bar{f}\left(t, y(t), \int_{a}^{t} \bar{k}(t, \sigma, y(\sigma)) d \sigma\right) \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
y^{\prime}(t)=\bar{f}\left(t, y(t), \int_{a}^{t} \bar{k}(t, \sigma, y(\sigma)) d \sigma\right), \quad y(a)=y_{0} \tag{4.2}
\end{equation*}
$$

for $t \in I$, where $\bar{k} \in C\left(I^{2} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$ for $a \leq s \leq t<\infty, \bar{f} \in C\left(I \times \mathbb{R}^{n} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$.
The following theorems deal with the continuous dependence of solutions of equations (1.1) and (1.2) on the functions involved therein.

Theorem 4.1. Suppose that the functions $f, k$ in equation (1.1) satisfy the conditions (3.1) and (3.2). Furthermore suppose that

$$
\left|f\left(t, y(t), \int_{a}^{t} k(t, \sigma, y(\sigma)) d \sigma\right)-\bar{f}\left(t, y(t), \int_{a}^{t} \bar{k}(t, \sigma, y(\sigma)) d \sigma\right)\right| \leq \varepsilon_{1}
$$

where $f, k$ and $\bar{f}, \bar{k}$ are the functions involved in equations (1.1) and (4.1), $\varepsilon_{1}>0$ is an arbitrary small constant and $y(t)$ is a solution of equation (4.1). Then the solution $x(t), t \in I$, of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).

Proof. Let $u(t)=|x(t)-y(t)|, t \in I$. Using the facts that $x(t)$ and $y(t)$ are the solutions of equations (1.1) and (4.1) and the hypotheses we have

$$
\begin{align*}
u(t) \leq & \left|f\left(t, x(t), \int_{a}^{t} k(t, \sigma, x(\sigma)) d \sigma\right)-f\left(t, y(t), \int_{a}^{t} k(t, \sigma, y(\sigma)) d \sigma\right)\right|  \tag{4.3}\\
& +\left|f\left(t, y(t), \int_{a}^{t} k(t, \sigma, y(\sigma)) d \sigma\right)-\bar{f}\left(t, y(t), \int_{a}^{t} \bar{k}(t, \sigma, y(\sigma)) d \sigma\right)\right| \\
& \leq \varepsilon_{1}+N\left[u(t)+\int_{a}^{t} r(t, \sigma) u(\sigma) d \sigma\right]
\end{align*}
$$

¿From (4.3) and using the assumption that $0 \leq N<1$, we observe that

$$
\begin{equation*}
u(t) \leq \frac{\varepsilon_{1}}{1-N}+\frac{N}{1-N} \int_{a}^{t} r(t, \sigma) u(\sigma) d \sigma \tag{4.4}
\end{equation*}
$$

Now an application of Lemma 3.1 to (4.4) yields

$$
\begin{equation*}
|x(t)-y(t)| \leq\left(\frac{\varepsilon_{1}}{1-N}\right) \exp \left(\int_{a}^{t} B(s) d s\right) \tag{4.5}
\end{equation*}
$$

where $B(t)$ is defined by (3.4). From (4.5) it follows that the solution of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).

Theorem 4.2. Suppose that the functions $f$ and $k$ in equation (1.2) satisfy the conditions (3.7) and (3.2). Furthermore suppose that
$\left|x_{0}-y_{0}\right|+\int_{a}^{t}\left|f\left(s, y(s), \int_{a}^{s} k(s, \sigma, y(\sigma)) d \sigma\right)-\bar{f}\left(s, y(s), \int_{a}^{s} \bar{k}(s, \sigma, y(\sigma)) d \sigma\right)\right| d s \leq \varepsilon_{2}$,
where $f, k$ and $\bar{f}, \bar{k}$ are the functions involved in equations (1.2) and (4.2), $\varepsilon_{2}>0$ is an arbitrary small constant and $y(t)$ is a solution of equation (4.2). Then the solution $x(t), t \in I$, of equation (1.2) depends continuously on the functions involved on the right hand side of equation (1.2).

Proof. Let $u(t)=|x(t)-y(t)|, t \in I$. Using the facts that $x(t)$ and $y(t)$ are the solutions of equations (1.2) and (4.2) and the hypotheses we have
(4.6) $u(t) \leq\left|x_{0}-y_{0}\right|$

$$
\begin{aligned}
& +\int_{a}^{t}\left|f\left(s, x(s), \int_{a}^{s} k(s, \sigma, x(\sigma)) d \sigma\right)-f\left(s, y(s), \int_{a}^{s} k(s, \sigma, y(\sigma)) d \sigma\right)\right| d s \\
& +\int_{a}^{t}\left|f\left(s, y(s), \int_{a}^{s} k(s, \sigma, y(\sigma)) d \sigma\right)-\bar{f}\left(s, y(s), \int_{a}^{s} \bar{k}(s, \sigma, y(\sigma)) d \sigma\right)\right| d s \\
& \leq \varepsilon_{2}+\int_{a}^{t} p(s)\left[u(s)+\int_{a}^{s} r(s, \sigma) u(\sigma) d \sigma\right] d s
\end{aligned}
$$

Now an application of Lemma 3.2 to (4.6) yields

$$
\begin{equation*}
|x(t)-y(t)| \leq \varepsilon_{2}\left[1+\int_{a}^{t} p(s) \exp \left(\int_{a}^{s}[p(\sigma)+A(\sigma)] d \sigma\right) d s\right] \tag{4.7}
\end{equation*}
$$

for $t \in I$, where $A(t)$ is as defined in Lemma 3.1. From (4.7) it follows that the solution of equation (1.2) depends continuously on the functions involved on the right hand side of equation (1.2).

We next consider the following systems of Volterra integral and integrodifferential equations

$$
\begin{align*}
& z(t)=h\left(t, z(t), \int_{a}^{t} g(t, \sigma, z(\sigma)) d \sigma, \mu\right)  \tag{4.8}\\
& z(t)=h\left(t, z(t), \int_{a}^{t} g(t, \sigma, z(\sigma)) d \sigma, \mu_{0}\right) \tag{4.9}
\end{align*}
$$

and

$$
\begin{equation*}
z^{\prime}(t)=h\left(t, z(t), \int_{a}^{t} g(t, \sigma, z(\sigma)) d \sigma, \mu\right), \quad z(a)=z_{0} \tag{4.10}
\end{equation*}
$$

$$
\begin{equation*}
z^{\prime}(t)=h\left(t, z(t), \int_{a}^{t} g(t, \sigma, z(\sigma)) d \sigma, \mu_{0}\right), \quad z(a)=z_{0} \tag{4.11}
\end{equation*}
$$

for $t \in I$, where $g \in C\left(I^{2} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right), a \leq s \leq t<\infty, h \in C\left(I \times \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}, \mathbb{R}^{n}\right)$.
The following theorems shows the dependency of solutions of equations (4.8), (4.9) and (4.10), (4.11) on parameters.

Theorem 4.3. Suppose that the functions $h, g$ in equations (4.8), (4.9) satisfy the conditions

$$
\begin{gather*}
|h(t, u, v, \mu)-h(t, \bar{u}, \bar{v}, \mu)| \leq \bar{N}[|u-\bar{u}|+|v-\bar{v}|]  \tag{4.12}\\
\left|h(t, u, v, \mu)-h\left(t, u, v, \mu_{0}\right)\right| \leq q(t)\left|\mu-\mu_{0}\right|  \tag{4.13}\\
|g(t, \sigma, u,)-g(t, \sigma, v)| \leq \bar{r}(t, \sigma)|u-v| \tag{4.14}
\end{gather*}
$$

where $0 \leq \bar{N}<1$ is a constant, $q \in C\left(I, \mathbb{R}_{+}\right)$such that $q(t) \leq Q<\infty, Q$ is a constant and $\bar{r}(t, \sigma), \frac{\partial}{\partial t} \bar{r}(t, \sigma) \in C\left(D, \mathbb{R}_{+}\right)$in which $D$ is defined as in Lemma 3.1. Let $z_{1}(t)$ and $z_{2}(t)$ be the solutions of equations (4.8) and (4.9) respectively. Then

$$
\begin{equation*}
\left|z_{1}(t)-z_{2}(t)\right| \leq \frac{Q\left|\mu-\mu_{0}\right|}{1-\bar{N}} \exp \left(\int_{a}^{t} \bar{B}(s) d s\right) \tag{4.15}
\end{equation*}
$$

for $t \in I$, where

$$
\bar{B}(t)=\frac{\bar{N}}{1-\bar{N}}\left[\bar{r}(t, t)+\int_{a}^{t} \frac{\partial}{\partial t} \bar{r}(t, \tau) d \tau\right]
$$

Proof. Let $z(t)=\left|z_{1}(t)-z_{2}(t)\right|, t \in I$. Using the facts that $z_{1}(t)$ and $z_{2}(t)$ are the solutions of equations (4.8) and (4.9) and hypotheses we have

$$
\begin{align*}
z(t) \leq & \mid h\left(t, z_{1}(t), \int_{a}^{t} g\left(t, \sigma, z_{1}(\sigma)\right) d \sigma, \mu\right)  \tag{4.16}\\
& -h\left(t, z_{2}(t), \int_{a}^{t} g\left(t, \sigma, z_{2}(\sigma)\right) d \sigma, \mu\right) \mid \\
+ & \mid h\left(t, z_{2}(t), \int_{a}^{t} g\left(t, \sigma, z_{2}(\sigma)\right) d \sigma, \mu\right) \\
& -h\left(t, z_{2}(t), \int_{a}^{t} g\left(t, \sigma, z_{2}(\sigma)\right) d \sigma, \mu_{0}\right) \mid \\
\leq & \bar{N}\left[z(t)+\int_{a}^{t} \bar{r}(t, \sigma) z(\sigma) d \sigma\right]+Q\left|\mu-\mu_{0}\right|
\end{align*}
$$

¿From (4.16) and using the assumption $0 \leq \bar{N}<1$, we observe that

$$
\begin{equation*}
z(t) \leq \frac{Q\left|\mu-\mu_{0}\right|}{1-\bar{N}}+\frac{\bar{N}}{1-\bar{N}} \int_{a}^{t} \bar{r}(t, \sigma) z(\sigma) d \sigma \tag{4.17}
\end{equation*}
$$

Now an application of Lemma 3.1 to (4.17) yields (4.15), which shows the dependency of solutions of equations (4.8) and (4.9) on parameters.

Theorem 4.4. Suppose that the functions $h, g$ in equations (4.10), (4.11) satisfy the conditions (4.12)-(4.14) with $\bar{p}(t)$ in place of $\bar{N}$ in (4.12), where $\bar{p}(t) \in C\left(I, \mathbb{R}_{+}\right)$ and the function $q(t)$ in (4.13) be such that $\int_{a}^{t} q(s) d s \leq \bar{Q}<\infty$, where $\bar{Q}$ is a constant. Let $z_{1}(t)$ and $z_{2}(t)$ be the solutions of equations (4.10) and (4.11). Then

$$
\left|z_{1}(t)-z_{2}(t)\right| \leq\left(Q\left|\mu-\mu_{0}\right|\right)\left[1+\int_{a}^{t} \bar{p}(s) \exp \left(\int_{a}^{s}[\bar{p}(\sigma)+\bar{A}(\sigma)] d \sigma\right) d s\right]
$$

for $t \in I$, where

$$
\bar{A}(t)=\bar{r}(t, t)+\int_{a}^{t} \frac{\partial}{\partial t} \bar{r}(t, \tau) d \tau
$$

The details of the proof follows by closely looking at the proofs of the theorems given above. We leave it to the reader to fill in where needed.

## REFERENCES

1. A. F. BaChurskaya: Uniqueness and convergence of successive approximations for a class of Volterra equations. Differentsial'nye Uraveniya $\mathbf{1 0 ( 9 )}$ (1974), 17221724.
2. A. Bielecki: Une remarque sur la méethod de Banach-Cacciopoli-Tikhnov dans la théorie des équations differentilles ordinaries. Bull Acad. Polon. Sci. Sér. Sci. Math. Phys. Astr. 4 (1956), 261-264.
3. A. Constantin: Topological transversality: Application to an integrodifferential equation. J. Math. Anal. Appl. 197 (1996), 855-863.
4. C. Corduneanu: Integral Equations and Stability of Feedback Systems. Academic Press, New York, 1993.
5. Kh. Éshmatov: A property of nonlinear Volterra integral equations. Differentsial'nye Uraveniya 10(10) (1974), 1911-1913.
6. M. A. Krasnoselskir: Topological Methods in the Theory of Nonlinear Integral Equations. Pergamon Press, Oxford, 1964.
7. T. Kulik and C. C. Tisdell: Volterra integral equations on time scales: Basic qualitative and quantitative results with applications to initial value problems on unbounded domains. Int. J. Difference Equ. Appl. (to appear).
8. M. Kwapisz: On the existence and uniqueness of solutions of a certain integralfunctional equation. Ann. Polon. Math. 31 (1975), 23-41.
9. R. K. Miller: Nonlinear Volterra Integral Equations. W. A. Benjamin, Menlo Park, CA, 1971.
10. B. G. Pachpatte: On a nonlinear Volterra integral-functional equation. Funkcialaj Ekvacioj 26 (1983), 1-9.
11. B. G. Pachpatte: Inequalities for Differential and Integral Equations. Academic Press, New York, 1998.
12. B. G. Pachpatte: Integral and Finite Difference Inequalities and Applications. North-Holland Mathematics Studies, Vol. 205, Elsevier Science B.V. Amsterdam, 2006.
13. B. G. Pachpatte: On certain applications of Leray-Schauder alternative. In: Fixed Point Theory and Applications, Vol.7, Y.J. Cho,J.K. Kim and S.M. Kang (Editors), Nova Science Publishers, Inc., New York, 2006, pp.147-153.
14. B. G. Pachpatte: On a certain iterated Volterra integrodifferential equation. An. Sti. Univ. Al. I. Cuza Iasi (to appear).
15. P. Talpalaru: Asymptotic behavior and growth properties of Integro-Differential equations. Atti Sem. Mat. Fis. Univ. Modena XLIV (1996), 229-251.
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