FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 22, No. 2 (2007), pp. 155–164

POLYNOMIAL APPROXIMATION ON UNBOUNDED INTERVALS BY FOURIER SUMS

Giuseppe Mastroianni and Gradimir V. Milovanović

Abstract. For the generalized Freud weight $w_{\alpha,\beta}(x) = |x|^\alpha e^{-|x|^\beta}$, $\alpha > -1$, $\beta > 1$, on the real line $\mathbb{R}$ and a given function f we study the behaviour of the Fourier sum $S_n(w_{\alpha,\beta}, f) = S_n(w_{\alpha,\beta}, f; x)$ in the weighted space C_u, defined by

$$C_u = \Big\{ f \in C^0(\mathbb{R}) : (fu)(x) = o(1) \text{ for } |x| \to +\infty \text{ or } x \to 0 \Big\}$$

and equipped by the norm $\|f\|_{C_u} = \|fu\| = \sup_{x\in\mathbb{R}} |(fu)(x)|$, where $u(x) = |x|^\gamma e^{-|x|^\beta/2}$, $\gamma \geq 0$, $\beta > 1$. An analogous result is given for the corresponding problem on the half line $\mathbb{R}_+$.

1. Introduction and Preliminaries

Let $w_{\alpha,\beta}(x) = |x|^\alpha e^{-|x|^\beta}$, $\alpha > -1$, $\beta > 1$, $x \in \mathbb{R}$, be a generalized Freud weight and $\{p_n(w_{\alpha,\beta})\}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e., $p_n(w_{\alpha,\beta}; x) = \gamma_n x^n + \cdots$, $\gamma_n = \gamma_n(w_{\alpha,\beta}) > 0$, and

$$\int_{\mathbb{R}} p_n(w_{\alpha,\beta}; x) p_n(w_{\alpha,\beta}; x) w_\alpha dx = \delta_{n,m}.$$

These weights and polynomials were introduced and studied in a complete way in [1]. When $\beta = 2$ we obtain the Sonin-Markov polynomials.

The Fourier sum of a function f can be written as

$$S_n(w_{\alpha,\beta}, f; x) = \sum_{k=0}^{n-1} c_k p_k(w_{\alpha,\beta}; x), \tag{1.1}$$

by assuming

$$c_k = \int_{\mathbb{R}} f(t) p_k(w_{\alpha,\beta}; t) w_{\alpha,\beta}(t)\, dt < +\infty, \quad k = 1, 2, \ldots\ .$$

Received November 8, 2006.
2000 *Mathematics Subject Classification.* Primary 41A10; Secondary 41A17, 42C10.

Using the Christofell-Darboux identity it can be written in the form

$$S_n(w_{\alpha,\beta}, f; x) = \frac{\gamma_{n-1}}{\gamma_n} \int_{\mathbb{R}} \frac{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)}{x-t} f(t) w_{\alpha,\beta}(t)\, dt, \tag{1.2}$$

where $p_k(x) = p_k(w_{\alpha,\beta}; x)$, $k \in \mathbb{N}_0$.

Now, we introduce the following space of functions: Let $u(x) = |x|^\gamma e^{-|x|^\beta/2}$, $\gamma \geq 0$, $\beta > 1$, be another generalized Freud weight and $C^0 = C^0(\mathbb{R})$ the set of all continuous functions in $\mathbb{R}$. We set

$$C_u = \Big\{ f \in C^0(\mathbb{R}) : (fu)(x) = o(1) \text{ for } |x| \to +\infty \text{ or } x \to 0 \Big\}$$

and introduce the norm $\|f\|_{C_u} = \|fu\| = \sup_{x\in\mathbb{R}} |(fu)(x)|$. In C_u the well-known Weierstrass theorem holds and, therefore, we study the behaviour of $S_n(w_{\alpha,\beta}, f) = S_n(w_{\alpha,\beta}, f; x)$ in the weighted space C_u.

Positive constants in this paper are denoted by $\mathcal{C}$, $\mathcal{C}_1$, …, and they can take different values even in subsequent formulae. It will always be clear what indices and variables the constants are independent of. If we use the notation $\mathcal{C}_p$, it means that this constant always depends on a parameter p. Sometimes, we will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ in order to denote that the constant $\mathcal{C}$ is independent only of a, b, …, but it can depend on parameters which are not mentioned in the list $(a, b, \ldots)$. If A and B are two expressions depending on certain indices and variables, then we write

$$A \sim B \quad \textit{if and only if} \quad 0 < \mathcal{C}_1 \leq \left| \frac{A}{B} \right| \leq \mathcal{C}_2$$

uniformly for the indices and variables considered.

Here, we need the so-called *Mhaskar-Rakhmanov-Saff number* (shortly M-R-S number) $a_n = a_n(w)$, which was independently defined by Rakhmanov [6] and Mhaskar and Saff [4, 5] for the weight $w(x) = \exp(-2Q(x))$ on $\mathbb{R}$ as a positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}\, dt.$$

The function $Q\colon \mathbb{R} \to \mathbb{R}$ is even, convex and of smooth polynomial growth at infinity. For example, for the Hermite weight e^{-x^2}, $x \in \mathbb{R}$, this number is $a_n = \sqrt{2n}$. For the general case $w_{\alpha,\beta}$ we have $a_n = a_n(w_{\alpha,\beta}) = C_n(\alpha,\beta) n^{1/\beta}$, i.e., $a_n \cong C n^{1/\beta}$, for a sufficiently large n, which is enough in our investigation.

For a given $\theta \in (0,1)$, by χ_n we denote the characteristic function of the interval $[-\theta a_n, \theta a_n]$, $a_n = a_n(u)$, and we state the following result:

Proposition 1.1. *For all $f \in C_u$ we have*

$$\|(1-\chi_n) fu\| \leq \mathcal{C} \left(E_n(f)_u + e^{-An} \|fu\| \right) \tag{1.3}$$

and, consequently,

$$\|fu\| \leq \mathcal{C} \left(\|\chi_n fu\| + E_n(f)_u \right),$$

where

$$M = \left[n\Big(\frac{\theta}{1+\theta}\Big)^{\beta} \right], \qquad E_n(f)_u = \inf_{P \in \mathcal{P}_n} \|(f-P)u\|_\infty,$$

$\mathcal{P}_n$ is the set of all polynomials of degree at most M and the positive constants $\mathcal{C}$ and A are independent of n and f.

Proof. Setting $f_n := \chi_n f$, for each polynomial $P_n \in \mathcal{P}_n$ we can write

$$\|(f-f_n)u\|_\infty = \max_{|x|\geq \theta a_n(u)} |(fu)(x)| \leq \|(f-P_n)u\|_\infty + \max_{|x|\geq \theta a_n(u)} |P_n(x)u(x)|.$$

By "finite-infinite range inequality"(c.f. [3], [1]) and the assumption on M, we get

$$\max_{|x|\geq \theta a_n(u)} |P_n(x)u(x)| \leq \mathcal{C}e^{-An}\|P_n u\|_\infty \leq \mathcal{C}e^{-An}\Big(\|(f-P_n)u\|_\infty + \|fu\|_\infty\Big),$$

where $A \neq A(n,f)$.

Thus,

$$\|(f-f_n)u\|_\infty \leq \mathcal{C}\Big(\|(f-P_n)u\|_\infty + e^{-An}\|fu\|_\infty\Big), \quad \mathcal{C} \neq C(n,f).$$

Taking the infimum over all $P_n \in \mathcal{P}_n$ the inequality (1.3) follows. After the standard computation we get the second inequality. □

2. Main Results

For a fixed $\theta \in (0,1)$, let χ_n be the characteristic function of the interval $[-\theta a_n, \theta a_n]$, where $a_n = a_n(u)$. We state the following result for the Fourier sum (1.1) on the real line $\mathbb{R}$:

Theorem 2.1. *Let $w_\alpha = |x|^\alpha e^{-|x|^\beta}$, $u(x) = |x|^\gamma e^{-|x|^\beta/2}$, with $\alpha > -1$, $\beta > 1$, $\gamma \geq 0$, and we assume that*

$$\max\Big\{0, \frac{\alpha}{2}\Big\} \leq \gamma < \frac{\alpha}{2} + 1. \tag{2.1}$$

Then, for each $f \in C_u$, we have

$$\|S_n(w_{\alpha,\beta}, \chi_n f)\chi_n u\| \leq \mathcal{C}\|(\chi_n f)u\| \log n \tag{2.2}$$

and

$$\|[f - \chi_n S_n(w_{\alpha,\beta}, \chi_n f)]u\| \leq \mathcal{C}\left[E_n(f)_u(\log n) + e^{-An}\|fu\|\right], \tag{2.3}$$

where $M = \left[n\left(\dfrac{\theta}{\theta+1}\right)^\beta\right] \sim n$ and the constant $\mathcal{C}$ is independent of n and f.

Proof. Taking the M-R-S number $a_n = a_n(u)$, and denoting the truncated function $\chi_n f$ by f_n, we consider the weighted Fourier sum $u(x)S_n(w, f_n; x)$. According to (1.2) we have

$$u(x)S_n(w, f_n; x) = \frac{\gamma_{n-1}}{\gamma_n} u(x)H\big(p_n(x)p_{n-1}(\,\cdot\,) - p_{n-1}(x)p_n(\,\cdot\,), w, f_n; x\big),$$

where $p_k = p_k(w)$, $k \in \mathbb{N}_0$, are orthonormal polynomials with respect to the weight w and H is the Hilbert transform.

According to a Remez-type inequality we should estimate the previous integrals for

$$x \in [-\theta a_n, \theta a_n] \setminus \Big[-\frac{a_n}{n}, \frac{a_n}{n}\Big],$$

and, because of symmetry, it is enough to consider only the interval $[a_n/n, \theta a_n]$. We note that

$$|\sqrt{w(x)}\, p_n(w; x)| \le \frac{\mathcal{C}}{\sqrt{a_n}}, \qquad |x| \le \theta a_n. \tag{2.4}$$

Thus, let $x \in [a_n/n, \theta a_n]$. Regarding this value of x, we take the following decomposition

$$[-\theta a_n, \theta a_n] = \Big[-\theta a_n, x - \frac{a_n}{n}\Big] \cup \Big[x - \frac{a_n}{n}, x + \frac{a_n}{n}\Big] \cup \Big[x + \frac{a_n}{n}, \theta a_n\Big],$$

in order to estimate the previous mentioned Hilbert transform. In this way, we have to estimate three terms in the weighted sum

$$\begin{aligned} |u(x)S_n(w, f_n; x)| &\le \mathcal{C} a_n u(x)\Big|H\big(p_n(x)p_{n-1}(\,\cdot\,) - p_{n-1}(x)p_n(\,\cdot\,), w, f_n; x\big)\Big| \\ &= |Y_1(x) + Y_2(x) + Y_3(x)| \le |Y_1(x)| + |Y_2(x)| + |Y_3(x)|, \end{aligned}$$

which correspond to the previous decomposition. In this formula we use the fact that $\gamma_n/\gamma_{n-1} \sim a_n$.

First, we give an estimate for $|Y_1(x)|$. Because of linearity in the Hilbert transform,

$$\begin{aligned} H\big(p_n(x)p_{n-1}(\,\cdot\,) - p_{n-1}(x)p_n(\,\cdot\,), w, f_n; x\big) &= p_n(x)H(p_{n-1}, w, f_n; x) \\ &\quad - p_{n-1}(x)H(p_n, w, f_n; x), \end{aligned}$$

we have $|Y_1(x)| \le |A_1(x)| + |B_1(x)|$, where

$$|A_1(x)| = \mathcal{C} a_n u(x)|p_n(x)| \left| \int_{-\theta a_n}^{x - a_n/n} p_{n-1}(t) f_n(t) w(t)\, \frac{dt}{x - t} \right|$$

and

$$|B_1(x)| = \mathcal{C} a_n u(x) |p_{n-1}(x)| \left| \int\limits_{-\theta a_n}^{x-a_n/n} p_n(t) f_n(t) w(t) \, \frac{dt}{x-t} \right|.$$

Using the inequality (2.4), we get

$$\begin{aligned}
|A_1(x)| &= \mathcal{C} \frac{a_n u(x)}{\sqrt{w(x)}} \big|\sqrt{w(x)} p_n(x)\big| \left| \int\limits_{-\theta a_n}^{x-a_n/n} \big(\sqrt{w(t)}\, p_{n-1}(t)\big) \frac{\sqrt{w(t)}}{u(t)} \frac{(f_n(t)u(t))}{x-t} \, dt \right| \\
&\leq \mathcal{C} \sqrt{a_n}\, x^{\gamma-\alpha/2} \int\limits_{-\theta a_n}^{x-a_n/n} \big|\sqrt{w(t)}\, p_{n-1}(t)\big| |t|^{\alpha/2-\gamma} |f_n(t)u(t)| \frac{dt}{x-t} \\
&\leq \mathcal{C} \sqrt{a_n}\, x^{\gamma-\alpha/2} \frac{\mathcal{C}_1}{\sqrt{a_n}} \|f_n u\| \int\limits_{-\theta a_n}^{x-a_n/n} |t|^{\alpha/2-\gamma} \frac{dt}{x-t} \\
&= \mathcal{C} \|f_n u\| \left\{ \int\limits_{-\theta a_n}^{0} + \int\limits_{0}^{x-a_n/n} \right\} |t/x|^{\alpha/2-\gamma} \frac{dt}{x-t} \\
&= \mathcal{C} \|f_n u\| \left\{ \int\limits_{0}^{\theta a_n/x} \frac{\zeta^\nu}{1+\zeta} \, d\zeta + \int\limits_{0}^{1-a_n/(nx)} \frac{\zeta^\nu}{1-\zeta} \, d\zeta \right\},
\end{aligned}$$

where $\nu = \alpha/2 - \gamma \in (-1, 0]$, regarding the conditions (2.1).

Since $x \geq a_n/n$, for the first integral in the last parenthesis $\{\ \cdots\}$ we have

$$I_n^{(1)}(x) = \int_0^{\theta a_n/x} \frac{\zeta^\nu}{1+\zeta} \, d\zeta \leq \int_0^{\theta n} \frac{\zeta^\nu}{1+\zeta} \, d\zeta.$$

Evidently, for $\nu = 0$, $I_n^{(1)}(x) \leq \mathcal{C} \log n$.

For $\nu \in (-1, 0)$, instead of the integral over $(0, \theta n)$, we consider the integral over $(0, +\infty)$, for which we can calculate its value (eg. by using Cauchy's residue theorem),

$$\int_0^{+\infty} \frac{\zeta^\nu}{1+\zeta} \, d\zeta = -\frac{\pi}{\sin(\nu\pi)} < +\infty \qquad (-1 < \nu < 0),$$

such that $I_n^{(1)}(x) \leq \mathcal{C}$.

Since $x \leq \theta a_n$, for the second integral in $\{\ \cdots\}$ we have

$$I_n^{(2)}(x) = \int\limits_{0}^{1-a_n/(nx)} \frac{\zeta^\nu}{1-\zeta} \, d\zeta \leq \int\limits_{0}^{1-1/(n\theta)} \frac{\zeta^\nu}{1-\zeta} \, d\zeta.$$

Evidently, for $\nu = 0$, $I_n^{(2)}(x) \le \mathcal{C} \log n$.

For $\nu \in (-1, 0)$, we have

$$I_n^{(2)}(x) \le \int_0^{1-1/(n\theta)} \frac{d\zeta}{1-\zeta} + \int_0^{1-1/(n\theta)} \frac{\zeta^\nu - 1}{1-\zeta}\, d\zeta \le \log(n\theta) + \int_0^1 \frac{\zeta^\nu(1-\zeta^{-\nu})}{1-\zeta}\, d\zeta.$$

By the inequality $\zeta^{-\nu} + (1-\zeta)^{-\nu} \ge 1$, $0 < -\nu < 1$, we get

$$I_n^{(2)}(x) \le \log(n\theta) + \int_0^1 \frac{\zeta^\nu}{(1-\zeta)^{\nu+1}}\, d\zeta = \log(n\theta) - \frac{\pi}{\sin(\nu\pi)} \le \mathcal{C} \log n.$$

Thus, $A_1(x) \le \mathcal{C}\|f_n u\| \log n$. Quite the same estimate holds for $|B_1(x)|$, so that we have

$$|Y_1(x)| \le \mathcal{C}\|f_n u\| \log n. \tag{2.5}$$

In a similar way we give the corresponding estimate for $|Y_3(x)| \le |A_3(x)| + |B_3(x)|$, where

$$|A_3(x)| = \mathcal{C} a_n u(x) |p_n(x)| \left| \int_{x+a_n/n}^{\theta a_n} p_{n-1}(t) f_n(t) w(t)\, \frac{dt}{x-t} \right|$$

and

$$|B_3(x)| = \mathcal{C} a_n u(x) |p_{n-1}(x)| \left| \int_{x+a_n/n}^{\theta a_n} p_n(t) f_n(t) w(t)\, \frac{dt}{x-t} \right|.$$

In that case we also obtain

$$|Y_3(x)| \le \mathcal{C}\|f_n u\| \log n. \tag{2.6}$$

In order to estimate $|Y_2(x)|$ we represent it in the form

$$\begin{aligned} |Y_2(x)| \;&\le\; \mathcal{C} a_n u(x) \left| \int_{x-a_n/n}^{x+a_n/n} \frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t}\, f_n(t) w(t)\, dt \right| \\ &\le\; \mathcal{C} \int_{x-a_n/n}^{x+a_n/n} \big|R_n(x,t)\big| \big(f_n(t) w(t)\big) dt \le \mathcal{C}\|f_n u\| \int_{x-a_n/n}^{x+a_n/n} \big|R_n(x,t)\big| dt, \end{aligned}$$

where

$$\big|R_n(x,t)\big| = a_n \frac{u(x)w(t)}{u(t)} \left| \frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t} \right| = U_1 + U_2,$$

with

$$U_1 = a_n \frac{u(x)w(t)}{u(t)} \left| \frac{p_n(x) - p_n(t)}{x - t} \right| \le a_n \frac{u(x)w(t)}{u(t)} |p_n'(\xi)| \, |p_{n-1}(t)|$$

for some ξ such that $|\xi - t| < |x - t|$, and similarly

$$U_2 \le a_n \frac{u(x)w(t)}{u(t)} |p_{n-1}'(\eta)| \, |p_n(t)|$$

for some η such that $|\eta - t| < |x - t|$.

Using (2.4), the Bernstein inequality and $w(\xi) \sim w(x) \sim w(t)$ (cf. [3]) we find

$$U_1 = a_n \frac{u(x)\sqrt{w(t)}}{u(t)\sqrt{w(\xi)}} \left| \sqrt{w(\xi)} p_n'(\xi) \right| \left| \sqrt{w(t)} p_{n-1}(t) \right| \le a_n \left(\frac{\mathcal{C}_1 n}{a_n \sqrt{a_n}} \right) \left(\frac{\mathcal{C}_2}{\sqrt{a_n}} \right) \le \frac{\mathcal{C} n}{a_n} \, ,$$

as well as $U_2 \le \mathcal{C} n / a_n$. Thus,

$$|Y_2(x)| \le \mathcal{C} \| f_n u \|. \tag{2.7}$$

Finally, according to (2.5)–(2.7) we conclude that

$$|u(x) S_n(w, f_n; x)| \le \mathcal{C} \| f_n u \| \log n,$$

i.e., (2.2).

In order to estimate the error we have

$$\| \left[f - \chi_n S_n(w, \chi_n f) \right] u \|_\infty \le \| (f - \chi_n f) u \|_\infty + \| (f - S_n(w, \chi_n f)) \chi_n u \|_\infty .$$

By Proposition 1.1, we get

$$\| (f - \chi_n f) u \|_\infty \le \mathcal{C} \Big(E_n(f)_{u,\infty} + e^{-An} \| f u \|_\infty \Big),$$

where $\mathcal{C}$ and A are independent of f and n and $M = \left[n \left(\frac{\theta}{1+\theta} \right)^\beta \right]$.

Moreover,

$$\| (f - S_n(w, \chi_n f)) \chi_n u \|_\infty \le \| (f - P_n) \chi_n u \|_\infty$$
$$+ \| S_n(w, (P_n - f) \chi_n) \chi_n u \|_\infty + \| S_n(w, (1 - \chi_n) P_n) \chi_n u \|_\infty .$$

By Proposition 1.1 and (2.2) the first two terms on the right side are dominated by $E_n(f)_u \log n$.

For the last term we observe that for each $F \in C_u$ we have

$$\| S_n(w, F) \chi_n u \|_\infty \le \mathcal{C} n^{1/3} (\log n) \| F u \|_\infty .$$

In order to prove this we can repeat the proof of (2.2) recalling that (2.4) is true for $|x| \le \theta a_n(u)$, but in $[-a_n, a_n]$ the inequality [1]

$$|p_n(w;x)| \le \mathcal{C}\frac{n^{1/3}}{\sqrt{a_n}}$$

holds.

Then, we have

$$\begin{aligned}
\|S_n(w,(1-\chi_n)P_n)\chi_n u\|_\infty &\le \mathcal{C}n^{1/3}(\log n)\max_{[\theta a_n(u),+\infty)}|P_n u|(x)\\
&\le \mathcal{C}n^{1/3}(\log n)e^{-An}\|P_n u\|_\infty\\
&\le \mathcal{C}e^{-An}\|fu\|_\infty,
\end{aligned}$$

using the "finite-infinite range inequality." □

At the end of this section we give an important consequence of the previous theorem. Namely, we consider a generalized Laguerre weight $w_\alpha(x) := w_{\alpha,\beta}(x) = x^\alpha e^{-x^\beta}$, $\alpha > -1$, $\beta > 1/2$, for $x > 0$ and the corresponding sequence of orthonormal polynomials $\{p_n(w_\alpha)\}$ with the positive leading coefficients. For a continuous function f in $(0,+\infty)$ ($f \in C^0(0,+\infty)$) we can write its Fourier sum in the system $p_n(w_\alpha)$ as

$$S_n(w_\alpha, f; x) = \sum_{k=0}^{n-1} c_k p_k(w_\alpha; x), \quad c_k = \int_0^{+\infty} f(t)p_k(w_\alpha;t)w_\alpha(t)\,dt.$$

If $u(x) = x^\gamma e^{-x^\beta/2}$, $\gamma \ge 0$, is another generalized Laguerre weight, we introduce the space of functions

$$C_u = \left\{f \in C^0(0,+\infty) : (fu)(x) = o(1) \text{ for } x \to 0^+ \text{ or } x \to +\infty\right\}$$

equipped with the norm $\|f\|_{C_u} = \|fu\| = \sup_{x\ge 0}|(fu)(x)|$ and we study the behaviour of $S_n(w_\alpha, f)$ in C_u.

First, we observe that with $W(x) = |x|^{2\alpha+1}e^{-x^{2\beta}}$, we have $a_n = a_n(w_\alpha) = a_{2n}^2(W) \sim n^{1/\beta}$ (cf. [3]).

Let χ_n^* be the corresponding characteristic function of the interval $[0, \theta a_n]$, where $\theta \in (0,1)$. Now, we consider the sequence $\{\chi_n^* S_n(w_\alpha, \chi_n^* f)\}$ in C_u.

Denoting by $E_M(f)_u = \inf_{P_M \in \mathcal{P}_M}\|(f-P_M)u\|$, the error of the best approximation, we can prove the following result:

Theorem 2.2. *Let $w_\alpha(x) = x^\alpha e^{-x^\beta}$, $x > 0$, $\alpha > -1$, $\beta > 1/2$, and $u(x) = x^\gamma e^{-x^\beta/2}$, $\gamma \ge 0$, and assume the conditions*

$$\max\left\{0, \frac{\alpha}{2}+\frac{1}{4}\right\} \le \gamma < \frac{\alpha}{2}+\frac{3}{4}.$$

Then, for all $f \in C_u$*, we have*

$$\|\chi_n^* S_n(w_\alpha, \chi_n^* f)u\| \le \|f\chi_n^* u\|(\log n)$$

and

$$\|[f - \chi_n^* S_n(w_\alpha, \chi_n^* f)]u\| \le \mathcal{C}[E_M(f)_u(\log n) + e^{-An}\|fu\|],$$

where $M = \left[n\left(\frac{\theta}{1+\theta}\right)^\beta \right]$*, and* A *and* $\mathcal{C}$ *are positive constants independent of* n *and* f*.*

Proof. We set $F(x) = f(x^2)\ \sigma(x) = |x|^{2\gamma} e^{-|x|^{2\beta}/2}$, $x \in \mathbb{R}$, and denote by $\widetilde{\chi}_n$ the characteristic function of the interval $A_n := [-\widetilde{\theta} a_{2n}(W), \widetilde{\theta} a_{2n}(W)]$ for some $\widetilde{\theta} \in (0,1)$. Then, by Theorem 2.1 with $2\alpha + 1$ instead of α and 2γ instead of γ, we have

$$\|\widetilde{\chi}_n S_{2n}(W, \widetilde{\chi}_n F)\sigma\| \le \mathcal{C}\|\widetilde{\chi}_n F\sigma\|(\log n)$$

if the parameters α and γ satisfy the condition

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} \le \gamma < \frac{\alpha}{2} + \frac{3}{4}.$$

Now we have

$$\sup_{x \in A_n} \left| F(x)|x|^{2\gamma} e^{-|x|^{2\beta}/2} \right| = \sup_{x \in [0, \theta a_{2n}^2(W)]} |f(x) x^\gamma e^{-x^\beta/2}| = \|fu\chi_n^*\|,$$

since $a_n(w_\alpha) = a_{2n}^2(W)$ and $\theta = \widetilde{\theta}^2 \in (0,1)$.

We also have $S_{2n}(W, \widetilde{\chi}_n F; x) = S_n(w_\alpha, \chi_n^* f; x^2)$, for which

$$\|S_{2n}(W, \widetilde{\chi}_n F)\widetilde{\chi}_n U\| = \|S_n(w_\alpha, \chi_n^* f)\chi_n^* u\|$$

and the first part of the theorem follows.

The error estimate follows in a similar way.

Finally, we observe that the conditions on the weights are independent on the parameter β and an interesting special case is $\beta = 1$ (generalized Laguerre systems). □

REFERENCES

1. T. Kasuga and R. Sakai: *Orthonormal polynomials with generalized Freud-type weights.* J. Approx. Theory **121** (2003), 13–53.
2. G. Mastroianni: Polynomial inequalities, functional spaces and best approximation on the real semiaxis with Laguerre weights, In: *Orthogonal Polynomials, Approximation Theory and Harmonic Analysis (Inzel, 2000)*, *Electron. Trans. Numer. Anal.* **14** (2002), 125–134.

3. G. Mastroianni and J. Szabados: *Polynomial approximation on the real semiaxis with generalized Laguerre weights.* Stud. Univ. Babeş-Bolyai Math. **52** (2007), no. 4, (2007), 105–128.
4. H. N. Mhaskar and E. B. Saff: *Extremal problems for polynomials with exponential weights.* Trans. Amer. Math. Soc. **285** (1984), 203-234.
5. H. N. Mhaskar and E. B. Saff: *Where does the sup norm of a weighted polynomial live?.* Constr. Approx. **1** (1985), 71–91.
6. E. A. Rakhmanov: *On asymptotic properties of polynomials orthogonal on the real axis.* Mat. Sbornik **119** (161) (1982), 163–203 (Russian) [Engl. transl. Math. USSR Sb. **47** (1984), 155–193].

Dipartimento di Matematica
Università della Basilicata
Via dell'Ateneo Lucano 10
85100 Potenza, Italy
giuseppe.mastroianni@unibas.it

University of Niš
Faculty of Electronic Engineering
Department of Mathematics
P.O. Box 73
18000 Niš, Serbia
grade@junis.ni.ac.yu