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NEW REVERSES OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ INTEGRAL INEQUALITY FOR VECTOR-VALUED FUNCTIONS IN HILBERT SPACES AND APPLICATIONS

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Abstract. Some new reverses of the Cauchy-Bunyakovsky-Schwarz integral inequality for vector-valued functions in Hilbert spaces that complement the recent results obtained in [1] are given. Applications for the Heisenberg inequality are provided.

1. Introduction

Let $(K;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho:[a,b]\subset\mathbb{R}\to[0,\infty)$ is a Lebesgue integrable function with the property that $\int_a^b\rho(t)\,dt=1$, then we may consider the space $L^2_\rho([a,b];K)$ of all functions $f:[a,b]\to K$, that are Bochner measurable and $\int_a^b\rho(t)\|f(t)\|^2\,dt<\infty$.

It is well known that $L^2_\rho([a,b];K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f,g\rangle_\rho:=\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt$$

and generating the norm

$$\|f\|_\rho:=\left(\int_a^b\rho(t)\|f(t)\|^2\,dt\right)^{1/2},$$

is a Hilbert space over $\mathbb{K}$.

The following integral inequality is known in the literature as the Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$(1.1)\qquad \int_a^b\rho(t)\|f(t)\|^2\,dt\int_a^b\rho(t)\|g(t)\|^2\,dt\geq\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt\right|^2,$$

provided $f,g\in L^2_\rho([a,b];K)$.

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The case of equality holds in (1.1) iff there exists a constant $\lambda \in \mathbb{K}$ such that $f(t) = \lambda g(t)$ for a.e. $t \in [a,b]$.

Another version of the CBS inequality for one vector-valued function and one scalar function is incorporated in:

$$\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|f(t)\|^2 dt \geq \left\| \int_a^b \rho(t) \alpha(t) f(t) dt \right\|^2, \tag{1.2}$$

provided $\alpha \in L^2_\rho([a,b])$ and $f \in L^2_\rho([a,b];K)$, where $L^2_\rho([a,b])$ denotes the Hilbert space of scalar functions α for which $\int_a^b \rho(t) |\alpha(t)|^2 dt < \infty$. The equality holds in (1.2) iff there exists a vector $e \in K$ such that $f(t) = \overline{\alpha(t)} e$ for a.e. $t \in [a,b]$.

In [1], several reverses of (1.1) and (1.2) were given. Amongst them, we recall here the following ones:

If $f, g \in L^2_\rho([a,b];K)$ and $r > 0$ are such that $\|f(t) - g(t)\| \leq r \leq \|g(t)\|$ for a.e. $t \in [a,b]$, then

$$\begin{aligned}\int_a^b \rho(t) \|f(t)\|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt - \left[\int_a^b \rho(t) \operatorname{Re} \langle f(t), g(t) \rangle dt \right]^2 \\ \leq r^2 \int_a^b \rho(t) \|f(t)\|^2 dt.\end{aligned} \tag{1.3}$$

The multiplicative constant 1 in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

If $f, g \in L^2_\rho([a,b];K)$ and $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$ are such that either $\operatorname{Re} \langle \Gamma g(t) - f(t), f(t) - \gamma g(t) \rangle \geq 0$ for a.e. $t \in [a,b]$, or, equivalently,

$$\left\| f(t) - \frac{\gamma + \Gamma}{2} \cdot g(t) \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|g(t)\|$$

for a.e. $t \in [a,b]$, then

$$\begin{aligned}\left(\int_a^b \rho(t) \|f(t)\|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{1/2} \\ \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma} \right) \int_a^b \rho(t) \langle f(t), g(t) \rangle dt \right]}{[\operatorname{Re}(\Gamma \bar{\gamma})]^{1/2}} \\ \leq \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{[\operatorname{Re}(\Gamma \bar{\gamma})]^{1/2}} \left| \int_a^b \rho(t) \langle f(t), g(t) \rangle dt \right|.\end{aligned} \tag{1.4}$$

The constant $1/2$ in both inequalities is best possible.

The inequality (1.4) implies the following additive reverse of the CBS inequality

$$(1.5)\quad 0 \le \int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2 \le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2.$$

The constant $1/4$ is best possible.

Now, if we assume that $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b];K)$ and $v \in K$, $r > 0$ such that $\|v\| > r$ and

$$\left\|\frac{g(t)}{\bar{\alpha}(t)} - v\right\| \le r \quad \text{for a.e. } t \in [a,b],$$

then we have the inequality

$$(1.6)\quad \left(\int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|f(t)\|^2 dt\right)^{1/2} \le \frac{1}{\sqrt{\|v\|^2 - r^2}}\operatorname{Re}\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, v\right\rangle \le \frac{\|v\|}{\sqrt{\|v\|^2 - r^2}}\left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|.$$

The multiplicative constant 1 in both inequalities cannot be replaced by a smaller quantity.

Inequality (1.6) implies the following additive reverse of (1.2)

$$(1.7)\quad 0 \le \int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|f(t)\|^2 dt - \left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|^2 \le \frac{r^2}{\|v\|^2\left(\|v\|^2 - r^2\right)}\left[\operatorname{Re}\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, v\right\rangle\right]^2 \le \frac{r^2}{\|v\|^2 - r^2}\left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|^2.$$

Here the inequalities are also sharp.

Finally, if $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b];K)$, $e \in K$, $\|e\| = 1$ and $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and either

$$\operatorname{Re}\left\langle \Gamma e - \frac{g(t)}{\overline{\alpha(t)}}, \frac{g(t)}{\overline{\alpha(t)}} - \gamma e\right\rangle \ge 0 \quad \text{for a.e. } t \in [a,b]$$

or, equivalently

$$\left\|\frac{g(t)}{\overline{\alpha(t)}} - \frac{\Gamma+\gamma}{2}e\right\| \le \frac{1}{2}|\Gamma-\gamma| \quad \text{for a.e. } t \in [a,b],$$

then we have the inequality

$$\left(\int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{1/2} \leq \frac{1}{2}\cdot\frac{\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, e\right\rangle\right]}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{1/2}} \leq \frac{1}{2}\cdot\frac{|\Gamma+\gamma|}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{1/2}}\left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|. \tag{1.8}$$

The constant $1/2$ is best possible in both inequalities.

The inequality (1.8) implies the following additive reverse of (1.2)

$$\begin{aligned} 0 &\leq \int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|f(t)\|^2 dt - \left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|^2 \\ &\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left[\operatorname{Re}\left(\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\left\langle\int_a^b \rho(t)\alpha(t)g(t)\,dt, e\right\rangle\right)\right]^2 \\ &\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|^2. \end{aligned} \tag{1.9}$$

The constant $1/4$ in both inequalities is sharp.

For some classical reverses of CBS inequality, see for instance [3]-[9].

In the present paper some new reverses of (1.1) and (1.2) are given. Applications for the Heisenberg inequality are also provided.

2. Some Reverse Inequalities, The General Case

The following result holds.

Theorem 2.1. *Let $f, g \in L_\rho^2([a,b];K)$ and $r > 0$ be such that*

$$\|f(t)-g(t)\| \leq r \tag{2.1}$$

for a.e. $t \in [a,b]$. Then we have the inequalities:

$$\begin{aligned} 0 &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{1/2} - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right| \\ &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{1/2} - \left|\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt\right| \\ &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{1/2} - \int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt \\ &\leq \frac{1}{2}r^2. \end{aligned} \tag{2.2}$$

The constant $1/2$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. We will use the following result obtained in [2]:

In the inner product space $(H;\langle\cdot,\cdot\rangle)$, if $x,y\in H$ and $r>0$ are such that $\|x-y\|\le r$, then

$$\begin{aligned}0 &\le \|x\|\,\|y\|-|\langle x,y\rangle|\le \|x\|\,\|y\|-|\mathrm{Re}\,\langle x,y\rangle| \\ &\le \|x\|\,\|y\|-\mathrm{Re}\,\langle x,y\rangle\le \frac{1}{2}r^2.\end{aligned}\tag{2.3}$$

The constant $1/2$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller constant.

If (2.1) holds true, then

$$\|f-g\|_\rho^2=\int_a^b\rho(t)\,\|f(t)-g(t)\|^2\,dt\le r^2\int_a^b\rho(t)\,dt=r^2$$

and thus $\|f-g\|_\rho\le r$.

Applying the inequality (2.3) for $\left(L_\rho^2([a,b];K),\langle\cdot,\cdot\rangle_p\right)$, we deduce the desired inequality (2.2).

If we choose $\rho(t)=1/(b-a)$, $f(t)=x$, $g(t)=y$, $x,y\in K$, $t\in[a,b]$, then from (2.2) we recapture (2.3) for which the constant $1/2$ in front of r^2 is best possible. □

We next point out some general reverse inequalities for the second CBS inequality (1.2).

Theorem 2.2. *Let $\alpha\in L_\rho^2([a,b])$, $g\in L_\rho^2([a,b];K)$ and $v\in K$, $r>0$. If*

$$\left\|\frac{g(t)}{\overline{\alpha(t)}}-v\right\|\le r\tag{2.4}$$

for a.e. $t\in[a,b]$, then we have the inequality

$$\begin{aligned}0&\le\left(\int_a^b\rho(t)\,|\alpha(t)|^2\,dt\int_a^b\rho(t)\,\|g(t)\|^2\,dt\right)^{1/2}-\left\|\int_a^b\rho(t)\,\alpha(t)\,g(t)\,dt\right\|\\ &\le\left(\int_a^b\rho(t)\,|\alpha(t)|^2\,dt\int_a^b\rho(t)\,\|g(t)\|^2\,dt\right)^{1/2}\\ &\qquad-\left|\left\langle\int_a^b\rho(t)\,\alpha(t)\,g(t)\,dt,\frac{v}{\|v\|}\right\rangle\right|\\ &\le\left(\int_a^b\rho(t)\,|\alpha(t)|^2\,dt\int_a^b\rho(t)\,\|g(t)\|^2\,dt\right)^{1/2}\\ &\qquad-\left|\mathrm{Re}\left\langle\int_a^b\rho(t)\,\alpha(t)\,g(t)\,dt,\frac{v}{\|v\|}\right\rangle\right|\end{aligned}\tag{2.5}$$

$$\le \left(\int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, \frac{v}{\|v\|}\right\rangle$$
$$\le \frac{1}{2}\cdot\frac{r^2}{\|v\|}\int_a^b \rho(t)|\alpha(t)|^2 dt.$$

The constant $1/2$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

Proof. ¿From (2.4) we deduce

$$\|g(t)\|^2 - 2\operatorname{Re}\langle \alpha(t)g(t), v\rangle + |\alpha(t)|^2\|v\|^2 \le r^2|\alpha(t)|^2$$

which is clearly equivalent to

$$\|g(t)\|^2 + |\alpha(t)|^2\|v\|^2 \le 2\operatorname{Re}\langle \alpha(t)g(t), v\rangle + r^2|\alpha(t)|^2. \tag{2.6}$$

If we multiply (2.6) by $\rho(t)\ge 0$ and integrate over $t\in[a,b]$, then we deduce

$$\int_a^b \rho(t)\|g(t)\|^2 dt + \|v\|^2\int_a^b \rho(t)|\alpha(t)|^2 dt \le 2\operatorname{Re}\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, v\right\rangle + r^2\int_a^b \rho(t)|\alpha(t)|^2 dt. \tag{2.7}$$

Since, obviously

$$2\|v\|\left(\int_a^b \rho(t)|\alpha(t)|^2 dt\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{1/2} \le \int_a^b \rho(t)\|g(t)\|^2 dt + \|v\|^2\int_a^b \rho(t)|\alpha(t)|^2 dt, \tag{2.8}$$

hence, by (2.7) and (2.8), we deduce

$$2\|v\|\left(\int_a^b \rho(t)|\alpha(t)|^2 dt\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{1/2} \le 2\operatorname{Re}\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, v\right\rangle + r^2\int_a^b \rho(t)|\alpha(t)|^2 dt,$$

which is clearly equivalent with the last inequality in (2.5).

The other inequalities are obvious and we omit the details.

Now, if $\rho(t) = 1/(b-a)$, $\alpha(t) = 1$, $g(t) = x$, $x\in K$, then, by the last inequality in (2.5) we get

$$\|x\|\|v\| - \operatorname{Re}\langle x, v\rangle \le \frac{1}{2}r^2,$$

provided $\|x - v\| \le r$, for which we know that (see [2]), the constant $1/2$ is best possible. □

3. Some Particular Cases of Interest

It has been shown in [2] that, for $\gamma, \Gamma \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) with $\Gamma \neq -\gamma$ and $x, y \in H$, $(H; \langle \cdot, \cdot \rangle)$ is an inner product over the real or complex number field $\mathbb{K}$, such that either

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0, \tag{3.1}$$

or, equivalently,

$$\left\| x - \frac{\gamma + \Gamma}{2} \cdot y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|y\| , \tag{3.2}$$

holds, then one has the following reverse of Schwarz's inequality

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y \rangle| \leq \|x\| \|y\| - \left| \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \right] \right| \\ &\leq \|x\| \|y\| - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \right] \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|y\|^2 . \end{aligned} \tag{3.3}$$

The constant $1/4$ is best possible in (3.3) in the sense that it cannot be replaced by a smaller constant.

If we assume that $\Gamma = M$, $\gamma = m$ with $M \geq m > 0$, then from (3.3) we deduce the much simpler and more useful result

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y \rangle| \leq \|x\| \|y\| - |\operatorname{Re} \langle x, y \rangle| \\ &\leq \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{1}{4} \cdot \frac{(M - m)^2}{Mm} \|y\|^2 , \end{aligned} \tag{3.4}$$

provided (3.1) or (3.2) holds true with M and m instead of Γ and γ.

Using the above inequalities for vectors in inner product spaces, we are able to state the following theorem concerning reverses of the CBS integral inequality for vector-valued functions in Hilbert spaces.

Theorem 3.1. *Let $f, g \in L^2_\rho([a, b]; K)$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq -\gamma$. If*

$$\operatorname{Re} \langle \Gamma g(t) - f(t), f(t) - \gamma g(t) \rangle \geq 0 \tag{3.5}$$

for a.e. $t \in [a, b]$, or, equivalently,

$$\left\| f(t) - \frac{\gamma + \Gamma}{2} \cdot g(t) \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|g(t)\| \tag{3.6}$$

for a.e. $t \in [a, b]$, then we have the inequalities

$$\begin{aligned} 0 &\leq \left(\int_a^b \rho(t) \|f(t)\|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{1/2} - \left| \int_a^b \rho(t) \langle f(t), g(t) \rangle dt \right| \\ &\leq \left(\int_a^b \rho(t) \|f(t)\|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{1/2} \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&\qquad -\left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\int_a^b \rho(t)\langle f(t), g(t)\rangle\, dt\right]\right| \\
&\leq \left(\int_a^b \rho(t)\|f(t)\|^2\, dt \int_a^b \rho(t)\|g(t)\|^2\, dt\right)^{1/2} \\
&\qquad - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\int_a^b \rho(t)\langle f(t), g(t)\rangle\, dt\right] \\
&\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\int_a^b \rho(t)\|g(t)\|^2\, dt.
\end{aligned}
$$

The constant $1/4$ *is best possible in* (3.7).

Proof. Since, by (3.5),

$$
\operatorname{Re}\langle \Gamma g - f, f - \gamma g\rangle_\rho = \int_a^b \rho(t)\operatorname{Re}\langle \Gamma g(t) - f(t), f(t) - \gamma g(t)\rangle\, dt \geq 0,
$$

hence, by (3.3) applied for the Hilbert space $\left(L^2_\rho([a,b];K);\langle\cdot,\cdot\rangle_\rho\right)$, we deduce the desired inequality (3.7).

The best constant follows by the fact that $1/4$ is a best constant in (3.7) and we omit the details. □

Corollary 3.1. *Let* $f, g \in L^2_\rho([a,b];K)$ *and* $M \geq m > 0$. *If*

$$
\operatorname{Re}\langle Mg(t) - f(t), f(t) - mg(t)\rangle \geq 0 \tag{3.8}
$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$
\left\| f(t) - \frac{m+M}{2}\cdot g(t)\right\| \leq \frac{1}{2}(M-m)\|g(t)\| \tag{3.9}
$$

for a.e. $t \in [a,b]$, *then*

$$
\begin{aligned}
0 &\leq \left(\int_a^b \rho(t)\|f(t)\|^2\, dt \int_a^b \rho(t)\|g(t)\|^2\, dt\right)^{1/2} - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle\, dt\right| \\
&\leq \left(\int_a^b \rho(t)\|f(t)\|^2\, dt \int_a^b \rho(t)\|g(t)\|^2\, dt\right)^{1/2} - \left|\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle\, dt\right| \\
&\leq \left(\int_a^b \rho(t)\|f(t)\|^2\, dt \int_a^b \rho(t)\|g(t)\|^2\, dt\right)^{1/2} - \int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle\, dt \\
&\leq \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\int_a^b \rho(t)\|g(t)\|^2\, dt.
\end{aligned} \tag{3.10}
$$

The constant $1/4$ *is best possible.*

The case when a function is scalar is incorporated in the following theorem.

Theorem 3.2. *Let* $\alpha \in L^2_\rho([a,b])$, $g \in L^2_\rho([a,b];K)$, *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\Gamma \neq -\gamma$. *If* $e \in K$, $\|e\| = 1$ *and*

$$\left\| \frac{g(t)}{\alpha(t)} - \frac{\Gamma + \gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{3.11}$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\operatorname{Re}\left\langle \Gamma e - \frac{g(t)}{\alpha(t)}, \frac{g(t)}{\alpha(t)} - \gamma e \right\rangle \geq 0 \tag{3.12}$$

for a.e. $t \in [a,b]$, *then we have the inequalities*

$$\begin{aligned} 0 &\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{1/2} - \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\| \\ &\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{1/2} - \left| \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right| \\ &\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{1/2} \\ &\qquad - \left| \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right] \right| \\ &\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{1/2} \\ &\qquad - \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right] \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \int_a^b \rho(t) |\alpha(t)|^2 dt. \end{aligned} \tag{3.13}$$

The constant $1/4$ *is best possible in* (3.13).

Proof. Follows by Theorem 2.2 on choosing

$$v := \frac{\Gamma + \gamma}{2} e \quad \text{and} \quad r := \frac{1}{2} |\Gamma - \gamma|.$$

We omit the details. □

Corollary 3.2. *Let* $\alpha \in L^2_\rho([a,b])$, $g \in L^2_\rho([a,b];K)$, *and* $M \geq m > 0$. *If* $e \in K$, $\|e\| = 1$ *and*

$$\left\| \frac{g(t)}{\alpha(t)} - \frac{M + m}{2} \cdot e \right\| \leq \frac{1}{2} (M - m)$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\operatorname{Re}\left\langle Me - \frac{g(t)}{\alpha(t)}, \frac{g(t)}{\alpha(t)} - me \right\rangle \geq 0$$

for a.e. $t \in [a,b]$*, then we have the inequalities:*

$$
\begin{aligned}
(3.14)\ 0 &\leq \left(\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt\right)^{1/2} - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\| \\
&\leq \left(\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt\right)^{1/2} - \left|\left\langle\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right| \\
&\leq \left(\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt\right)^{1/2} - \left|\mathrm{Re}\left\langle\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right| \\
&\leq \left(\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt\right)^{1/2} - \mathrm{Re}\left\langle\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle \\
&\leq \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt.
\end{aligned}
$$

The constant $1/4$ *is best possible in* (3.14).

4. Applications for the Heisenberg Inequality

It is well known that if $(H;\langle\cdot,\cdot\rangle)$ is a real or complex Hilbert space and $f:[a,b]\subset\mathbb{R}\to H$ is an *absolutely continuous vector-valued* function, then f is differentiable almost everywhere on $[a,b]$, the derivative $f':[a,b]\to H$ is Bochner integrable on $[a,b]$ and

$$
f(t) = \int_a^t f'(s)\,ds \quad \text{for any} \quad t\in[a,b]. \tag{4.1}
$$

The following theorem (see also [1]) provides a version of the Heisenberg inequalities in the general setting of Hilbert spaces. For details on the classical Heisenberg inequality, see, for instance, [4].

For the sake of completeness we give here a simple proof of this result.

Theorem 4.1. *Let* $\varphi:[a,b]\to H$ *be an absolutely continuous function with the property that* $b\left\|\varphi(b)\right\|^2 = a\left\|\varphi(a)\right\|^2$*. Then we have the inequality:*

$$
\int_a^b \left\|\varphi(t)\right\|^2 dt \leq 2\left(\int_a^b t^2\left\|\varphi(t)\right\|^2 dt\right)^{1/2}\cdot\left(\int_a^b \left\|\varphi'(t)\right\|^2 dt\right)^{1/2}. \tag{4.2}
$$

The constant 2 *is best possible in* (4.2) *in the sense that it cannot be replaced by a smaller quantity.*

Proof. Integrating by parts, we have successively

$$
\begin{aligned}
\int_a^b \left\|\varphi(t)\right\|^2 dt &= t\left\|\varphi(t)\right\|^2\Big|_a^b - \int_a^b t\frac{d}{dt}\left(\left\|\varphi(t)\right\|^2\right)dt \\
&= b\left\|\varphi(b)\right\|^2 - a\left\|\varphi(a)\right\|^2 - \int_a^b t\frac{d}{dt}\left\langle\varphi(t),\varphi(t)\right\rangle dt
\end{aligned}
$$

$$= -\int_a^b t\left[\langle \varphi'(t), \varphi(t)\rangle + \langle \varphi(t), \varphi'(t)\rangle\right] dt$$
$$= -2\int_a^b t \operatorname{Re}\langle \varphi'(t), \varphi(t)\rangle dt = 2\int_a^b \operatorname{Re}\langle \varphi'(t), (-t)\varphi(t)\rangle dt.$$

If we apply the CBS inequality

$$\int_a^b \operatorname{Re}\langle g(t), h(t)\rangle dt \le \left(\int_a^b \|g(t)\|^2 dt \int_a^b \|h(t)\|^2 dt\right)^{1/2}$$

for the choices $g(t) = \varphi'(t)$ and $h(t) = -t\varphi(t)$, $t \in [a,b]$, then we deduce the desired inequality (4.2).

The sharpness of the constant follows by the sharpness of the CBS inequality, and we omit the details. □

The following reverse of the Heisenberg type inequality (4.2) holds.

Theorem 4.2. *Assume that $\varphi : [a,b] \to H$ is as in the hypothesis of Theorem 4.1. In addition, if there exists a $r > 0$ such that*

$$\|\varphi'(t) + t\varphi(t)\| \le r \tag{4.3}$$

for a.e. $t \in [a,b]$, then we have the inequalities

$$\begin{aligned} 0 &\le \left(\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt\right)^{1/2} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt \\ &\le \frac{1}{2} r^2 (b-a). \end{aligned} \tag{4.4}$$

Proof. We observe, by the identity (4.3), that

$$\int_a^b \operatorname{Re}\langle \varphi'(t), (-t)\varphi(t)\rangle dt = \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt. \tag{4.5}$$

Now, if we apply Theorem 2.1 for the choices $f(t) = t\varphi(t)$, $g(t) = -t\varphi'(t)$, $\rho(t) = 1/(b-a)$, $t \in [a,b]$, then we deduce the desired inequality (4.4). □

Remark 4.1. It is interesting to remark that, from (4.5), we obviously have

$$\frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt = \left|\int_a^b \operatorname{Re}\left\langle \varphi'(t), t\varphi(t)\right\rangle dt\right|. \tag{4.6}$$

Now, if we apply the inequality (see (2.2))

$$\int_a^b \|f(t)\|^2 dt \int_a^b \|g(t)\|^2 dt - \left|\int_a^b \operatorname{Re}\langle f(t), g(t)\rangle dt\right| \le \frac{1}{2} r^2 (b-a),$$

for the choices $f(t) = \varphi'(t)$, $g(t) = t\varphi(t)$, $t \in [a,b]$, then we get the same inequality (4.4), but under the condition

$$\left\|\varphi'(t) - t\varphi(t)\right\| \le r \tag{4.7}$$

for a.e. $t \in [a,b]$.

The following result holds as well.

Theorem 4.3. *Assume that* $\varphi : [a, b] \to H$ *is as in the hypothesis of Theorem* 4.2*. In addition, if there exists* $M \geq m > 0$ *such that*

$$\operatorname{Re} \langle Mt\varphi(t) - \varphi'(t), \varphi'(t) - mt\varphi(t)\rangle \geq 0 \tag{4.8}$$

for a.e. $t \in [a, b]$*, or, equivalently,*

$$\left\| \varphi'(t) - \frac{M+m}{2} t\varphi(t) \right\| \leq \frac{1}{2}(M - m)|t| \|\varphi(t)\| \tag{4.9}$$

for a.e. $t \in [a, b]$*, then we have the inequalities*

$$\begin{aligned} 0 &\leq \left(\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt \right)^{1/2} - \frac{1}{2} \int_a^b \|\varphi(t)\|^2 dt \\ &\leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \int_a^b t^2 \|\varphi(t)\|^2 dt. \end{aligned} \tag{4.10}$$

Proof. The proof follows by Corollary 3.1 applied for the function $g(t) = t\varphi(t)$ and $f(t) = \varphi'(t)$, and on making use of the identity (4.6). We omit the details. □

Remark 4.2. If one is interested in reverses for the Heisenberg inequality for real or complex valued functions, then all the other inequalities obtained above for one scalar and one vectorial function may be applied as well. For the sake of brevity, we do not list them here.

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